SMT: quantifiers, and future prospects

Pascal Fontaine (University of Liège)

Based on joint works with many, including
Haniel Barbosa, Jasmin Blanchette, Daniel El Ouraoui, Mathias Fleury, Mikolás Janota, Cezary Kaliszyk, Andrew Reynolds, Hans-Jörg Schurr, Sophie Tourret...

...and built on the work of many others (see citations)

IPAM Workshop, UCLA, February 2023

Outline

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Motivation

- Formal proofs should not be mostly about proving easy things
- Automated theorem provers (ATPs) should prove the easy things for you
- ATP proofs can be replayed: confidence is not compromised
- E.g. Sledgehammer
- Proof obligations often use quantifiers

SMT = SAT + expressiveness

- SAT solvers
$$\neg\big[\,(p \Rightarrow q) \Rightarrow \big[\,(\neg p \Rightarrow q) \Rightarrow q\big]\big]$$
- Congruence closure (uninterpreted symbols + equality)
$$a = b \wedge \big[f(a) \neq f(b) \vee (q(a) \wedge \neg q(b))\big]$$
- and with arithmetic
$$a \leq b \wedge b \leq a + x \wedge x = 0 \wedge \big[f(a) \neq f(b) \vee (q(a) \wedge \neg q(b + x))\big]$$
- ...
- What about quantifiers?

Quantifiers in SMT

- ☹ Full first-order logic is undecidable
- ☺ First-order logic is semi-decidable
  refutationally complete procedures terminate on UNSAT
- ☺ If finite model property, then decidable
- ☹ Presburger with even one unary predicate is not even semi-decidable [Halper91]
- ☺ Pragmatic approaches are quite successful

Why does the pragmatic SMT approach work?

- Verification problems are big and shallow
- SMT appropriate for long, mostly ground, uninterpreted function reasoning

Working hypothesis

Quantifier handling for pure FOL will work most of the time sufficiently for SMT

Outline

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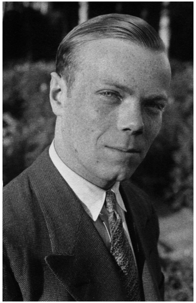
Herbrand

Unlike superposition-based FOL provers, SMT solvers essentially based on instantiation



1908-1931

Herbrand instance of a Skolem formula $\forall \bar{x} \varphi(\bar{x})$: any ground formula $\varphi(\bar{t})$, where $\bar{t}$ are terms in the language

Theorem (Herbrand)

A finite set of Skolem formulas is unsatisfiable if and only if there exists a finite unsatisfiable set of Herbrand instances

Caveats

- there should be at least one constant available for every sort
- holds for pure FOL, might not in presence of theories

Example

Is this syllogism correct?

$$\frac{\text{All humans are mortal} \quad \text{All Greeks are humans}}{\text{Then all Greeks are mortal}}$$

Artistotle
384–322 BC

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- Checking the validity of this formula
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- Checking the unsatisfiability of
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- Skolemize
  $\forall x.\, H(x) \Rightarrow M(x), \forall x.\, G(x) \Rightarrow H(x), \neg(G(s) \Rightarrow M(s))$

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Translate to FOL

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- Instantiate: add the two formulas (Herbrand instances)
  $H(s) \Rightarrow M(s), G(s) \Rightarrow H(s)$

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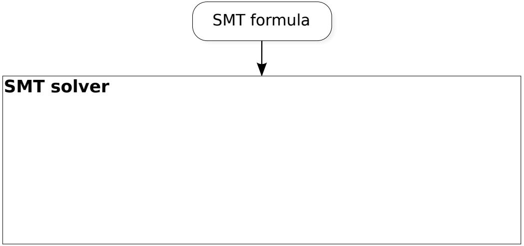
- Instantiate: add the two formulas (Herbrand instances)
  $H(s) \Rightarrow M(s), G(s) \Rightarrow H(s)$
- A ground (SAT/SMT) solver will deduce unsatisfiability.

From SAT to SMT,...



Input: $a \leq b \wedge b \leq a + x \wedge x = 0 \wedge \left[f(a) \neq f(b) \vee (q(a) \wedge \neg q(b + x))\right]$

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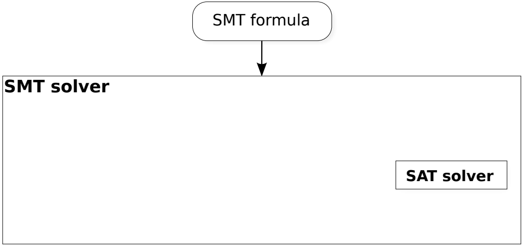

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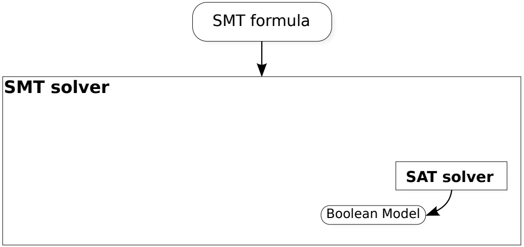

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Boolean model: $p_{a \leq b}, p_{b \leq a+x}, p_{x=0}, \neg p_{f(a)=f(b)}$

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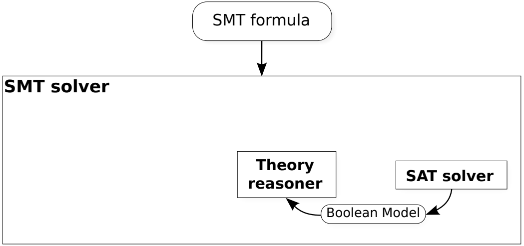

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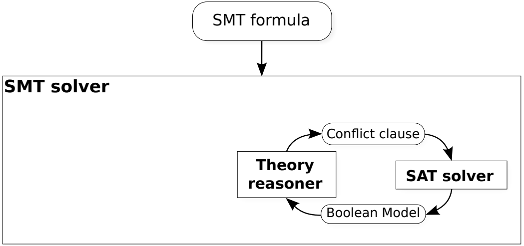

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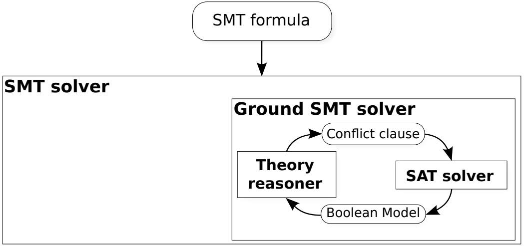
New clause: $\neg p_{a \leq b} \vee \neg p_{b \leq a+x} \vee \neg p_{x=0} \vee p_{f(a)=f(b)}$

Conflict clauses are negation of unsatisfiable conjunctive sets of literals

From SAT to SMT,... and then to quantified SMT



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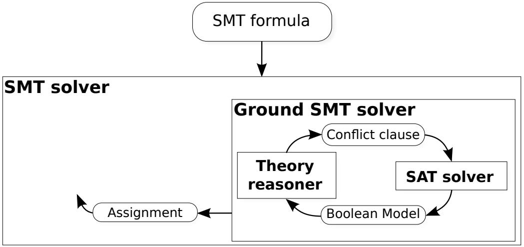

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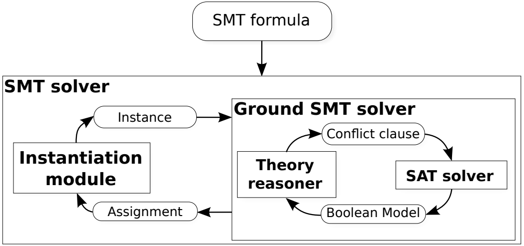

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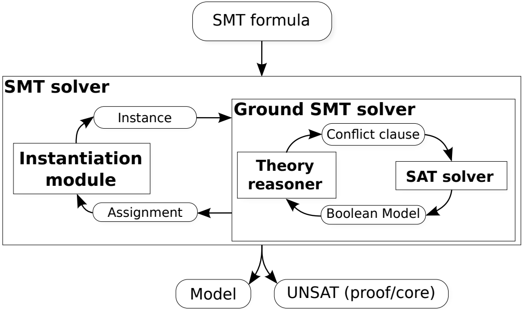

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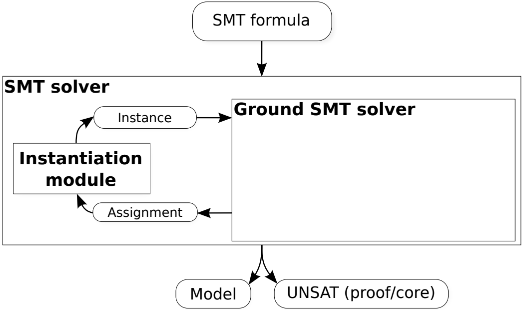

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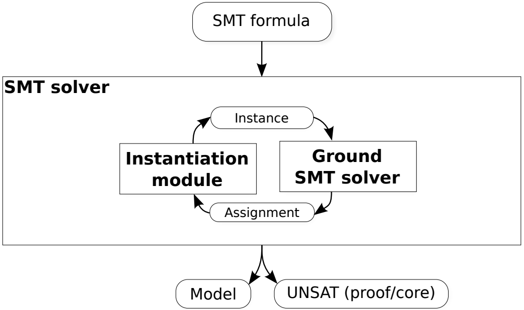

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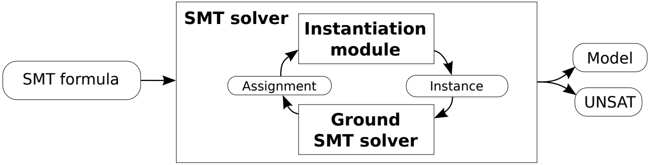
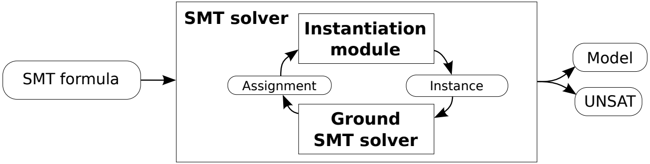
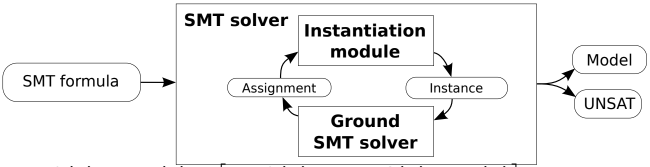


Instance?

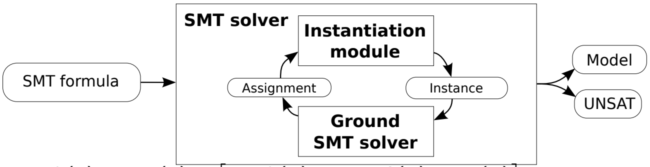

SMT formula
SMT solver
Instantiation module
Instance
Ground SMT solver
Assignment
Model
UNSAT

Instance?



Input: $a = b \wedge S(b) \wedge \neg Q(a) \wedge \neg R(a) \wedge \left[\forall x\, Q(x) \vee \forall x\,.\, S(x) \equiv R(x)\right]$

Instance?



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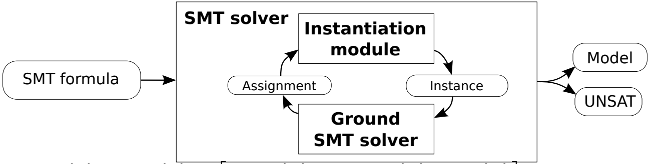

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Boolean model: $p_{a=b}, p_{S(b)}, \neg p_{Q(a)}, \neg p_{R(a)}, p_{\forall x\, .\, S(x) \equiv R(x)}$

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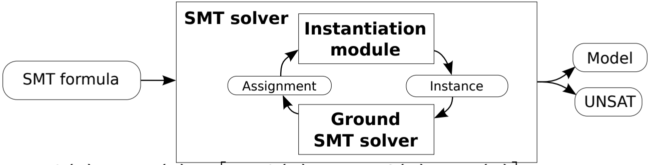

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Theory reasoner: fine! ...but does not understand $\forall x\,.\,S(x) \equiv R(x)$

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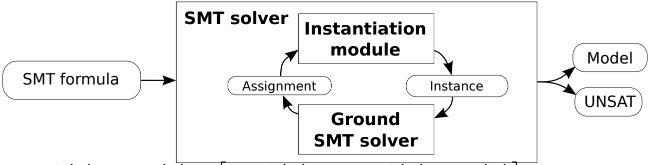

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Instantiation module: there is something to do with $\forall x\,.\, S(x) \equiv R(x)$

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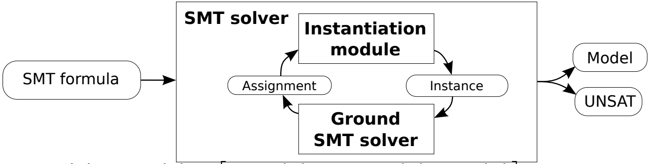

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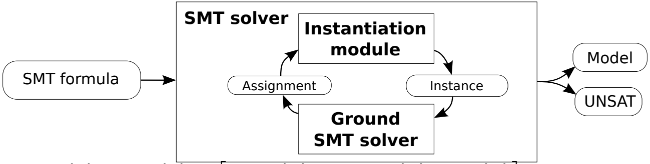
Boolean model: $p_{a=b}, p_{S(b)}, \neg p_{Q(a)}, \neg p_{R(a)}, p_{\forall x\,.\,S(x)\equiv R(x)}$

Theory reasoner: fine! ... but does not understand $\forall x\,.\,S(x) \equiv R(x)$

Instantiation module: there is something to do with $\forall x\,.\,S(x) \equiv R(x)$

New clause: $\neg p_{a=b}, \neg p_{S(b)} \vee p_{R(a)} \vee \neg p_{\forall x\,.\,S(x)\equiv R(x)}$

Instance?



Input: $a = b \land S(b) \land \neg Q(a) \land \neg R(a) \land \left[\forall x\, Q(x) \lor \forall x\,.\, S(x) \equiv R(x)\right]$

To SAT solver: $p_{a=b} \land p_{S(b)} \land \neg p_{Q(a)} \land \neg p_{R(a)} \land \left[p_{\forall x\, Q(x)} \lor p_{\forall x\,.\, S(x) \equiv R(x)}\right]$

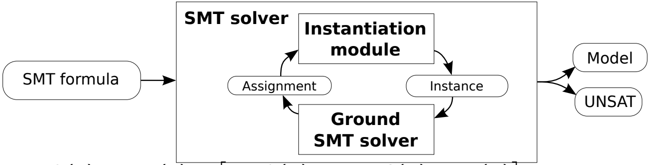
Boolean model: $p_{a=b}, p_{S(b)}, \neg p_{Q(a)}, \neg p_{R(a)}, p_{\forall x\,.\, S(x) \equiv R(x)}$

Theory reasoner: fine! . . . but does not understand $\forall x\,.\, S(x) \equiv R(x)$

Instantiation module: there is something to do with $\forall x\,.\, S(x) \equiv R(x)$

New clause:$\neg p_{a=b}, \neg p_{S(b)} \lor p_{R(a)} \lor \neg p_{\forall x\,.\, S(x) \equiv R(x)}$
. . . too complicated to find/generate

Instance?



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To SAT solver: $p_{a=b} \wedge p_{S(b)} \wedge \neg p_{Q(a)} \wedge \neg p_{R(a)} \wedge \left[p_{\forall x\, Q(x)} \vee p_{\forall x \,.\, S(x)\equiv R(x)}\right]$

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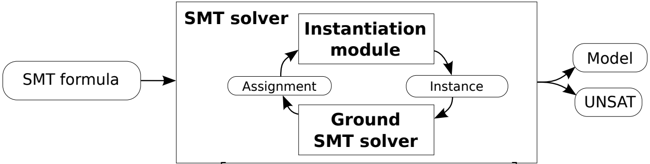
What is the right formula to generate?

Instance?



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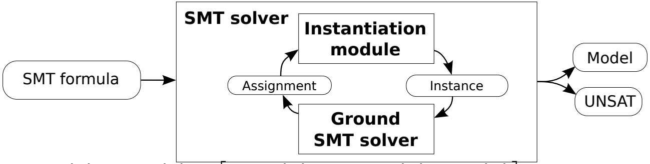

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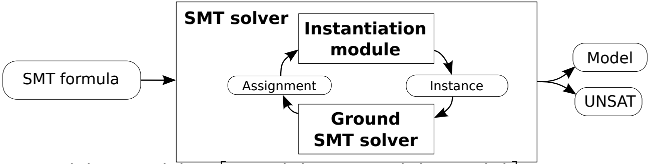

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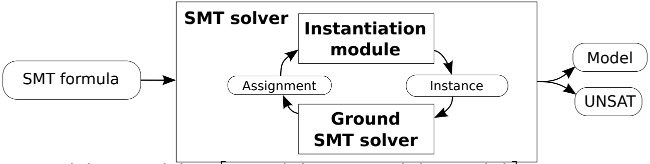

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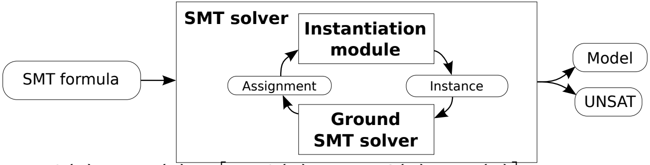

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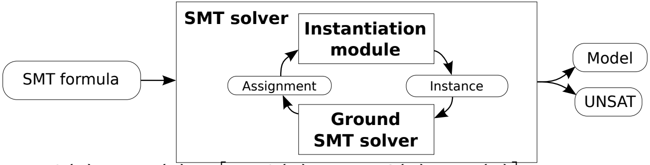

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Together with $\forall x\, Q(x) \Rightarrow Q(a)$, this grounds the problem

Instance in an SMT context

$$\forall \bar{x}\, \varphi(\bar{x}) \Rightarrow \varphi\sigma$$

where σ is a ground substitution for variables $\bar{x}$

E.g. $\forall \bar{x}\, \varphi(\bar{x})$ is $\forall x\,.\, S(x) \equiv R(x)$, σ is $x \mapsto a$, $\varphi\sigma$ is $S(a) \equiv R(a)$

Remarks

- Above formula is a FOL tautology. E.g. $(\forall x\,.\, S(x) \equiv R(x)) \Rightarrow (S(a) \equiv R(a))$
- $\forall \bar{x}\, \varphi(\bar{x})$ gets abstracted as a propositional variable in the SAT solver, that has a meaning only for the instantiation module
- $\varphi\sigma$ gets abstracted as a Boolean combination of propositional variables. . .
- . . . that have meaning at the level of the *ground* theory reasoner
- $\varphi\sigma$ gets "activated"/relevant only in the models where $p_{\forall \bar{x}\, \varphi(\bar{x})}$ is true.

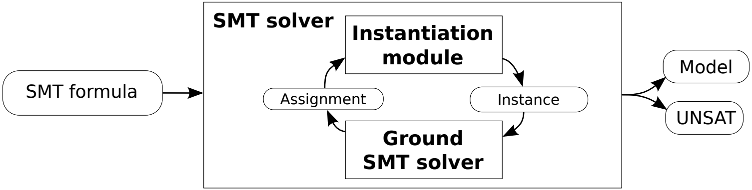
We might refer to $\varphi\sigma$ as the instance, but remember: all is fine at the level of the SAT solver/ground SMT solver

Outline

Outline

Instantiation techniques

The framework



Ground SMT solver enumerates assignments $E \cup Q$

- E set of ground literals
- Q set of quantified clauses

Instantiation module generates instances of Q that will further feed E

classic Herbrand Theorem: instantiate with all possible terms in language

Outline

E-matching/Trigger-based instantiation (**e**) [Detlefs05, deMoura07]

Search for relevant instances according to a set of triggers and E-matching

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Search for relevant instances according to a set of triggers and E-matching

- $E = \{\neg P(a), \neg P(b), P(c), \neg R(b)\}$ and $Q = \{\forall x.\, P(x) \vee R(x)\}$
- Assume trigger $P(x)$
- Find substitution σ for x such $P(x)$ is a know term (in E)
- Three suitable substitutions: $x \mapsto a$, $x \mapsto b$, or $x \mapsto c$
  E.g. $E \models P(x)[x/a] = P(a)$ and $P(a) \in E$
- Formally

$\mathbf{e}(E, \forall \bar{x}.\, \varphi)$

1. Select a set of triggers $\{\bar{t}_1, \ldots \bar{t}_n\}$ for $\forall \bar{x}.\, \varphi$
2. For each $i = 1, \ldots, n$, select a set of substitutions S_i s.t for each $\sigma \in S_i$, $E \models \bar{t}_i\sigma = \bar{g}_i$ for some tuple $\bar{g}_i \in \mathcal{T}_E$.
3. Return $\bigcup_{i=1}^{n} S_i$

E-matching/Trigger-based instantiation

Ideal for expanding definitions/rewriting rules

- Example

  $$\forall x \forall y \,.\, \mathrm{sister}(x, y) \equiv \\ (\mathrm{female}(x) \wedge \mathrm{mother}(x) = \mathrm{mother}(y) \wedge \mathrm{father}(x) = \mathrm{father}(y))$$

  $\mathrm{sister}(\mathrm{Eliane}, \mathrm{Eloïse})$

  $\mathrm{sister}(\mathrm{Eloïse}, \mathrm{Elisabeth})$

  $\neg\mathrm{sister}(\mathrm{Eliane}, \mathrm{Elisabeth})$

- Choosing instantiation trigger $\mathrm{sister}(x, y)$ suffices for SMT solver to prove unsatisfiability

Remarks

- Decision procedure for, e.g., expressive arrays, lists [Dross16]
- Mostly efficient (see later evaluation)
- But can easily blow or miss the right instances
- Requires triggers (human or auto-generated)

E-matching/Trigger-based instantiation, prospects

Machine learning for instance filtering

- Instantiation method issue: number of useless generated instances
- It often occurs that $> 99\%$ of 100k generated instances are useless

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separate the wheat from the chaff: select the useful instances

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- We investigated XGBoost to filter instances [Blanchette19]

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An opportunity for machine learning

separate the wheat from the chaff: select the useful instances

- We investigated XGBoost to filter instances [Blanchette19]
- Trained on successful proofs (good instance $\leftarrow$ survives pruning of proof)

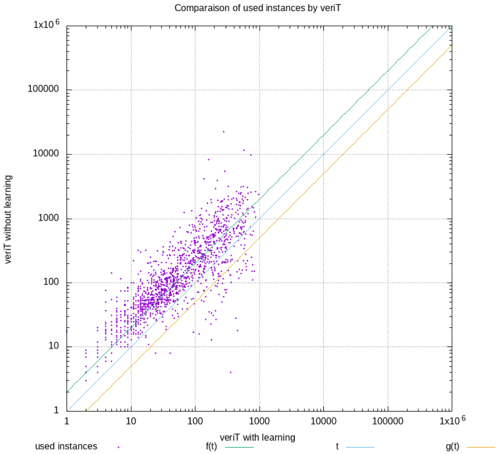
ML for instance filtering: experimental results

	30 s	60 s	120 s	180 s
veriT	2896	2913	2923	2929
veriT($\mathcal{M}$)	2907	2917	2925	2936
veriT($\mathcal{M}^2$)	2916	2927	2935	2944
veriT($\mathcal{M} + \mathcal{M}^2$)	2936	2959	2969	2975
veriT + portfolio	3181	3215	3228	3234
veriT($\mathcal{M} + \mathcal{M}^2$) + portfolio	3190	3247	3312	3322
Vampire smtcomp mode	3154	3165	3175	3197
CVC4 portfolio	3311	3345	3393	3404

Results on the benchmarks in the UF category of the SMT-LIB

- veriT: vanilla
- veriT($\mathcal{M}$): veriT with instance selection trained with veriT successes
- veriT($\mathcal{M}^2$): veriT with instance selection trained with veriT($\mathcal{M}$) successes
- veriT($\mathcal{M} + \mathcal{M}^2$): portfolio of above two
- veriT($\mathcal{M} + \mathcal{M}^2$) + portfolio of several strategies, with instance selection

ML for instance filtering: number of instances on test + training set



veriT on UF SMT-LIB benchmarks (with vs. without filtering)

ML for instance filtering: number of instances on test set only

veriT on UF SMT-LIB benchmarks (with vs. without filtering)

Outline

Conflict-based instantiation ($\mathbf{c}$)

[Reynolds14]

Search for *one* instance of one quantified formula in Q that is unsatisfiable together with E

Conflict-based instantiation ($\mathbf{c}$)

[Reynolds14]

Search for *one* instance of one quantified formula in Q that is unsatisfiable together with E

- $E = \{\neg P(a), \neg P(b), P(c), \neg R(b)\}$ and $Q = \{\forall x.\, P(x) \vee R(x)\}$
- Since E, $P(b) \vee R(b) \models \bot$, this strategy returns $x \mapsto b$
- Formally

  $\mathbf{c}(E, \forall \bar{x}.\, \varphi)$ Either return σ where $E \models \neg\varphi\sigma$, or return $\emptyset$

$\mathbf{c}$: solving the problem

$$E \wedge \psi\sigma \models \bot, \text{ for some } \forall \bar{x}\, \psi \in Q$$

c: solving the problem

$$E \models \neg\psi\sigma, \text{ for some } \forall\bar{x}\,\psi \in Q$$

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$$E = \{f(a) = f(b),\, g(b) \neq h(c)\},\; Q = \{\forall xyz.\; f(x) = f(z) \rightarrow h(y) = g(z)\}$$

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$$f(a) = f(b) \wedge g(b) \neq h(c) \models (f(x) = f(z) \wedge h(y) \neq g(z))\,\sigma$$

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Given conjunctive sets of equality literals E and L, with E ground, find substitution σ s.t. $E \models L\sigma$

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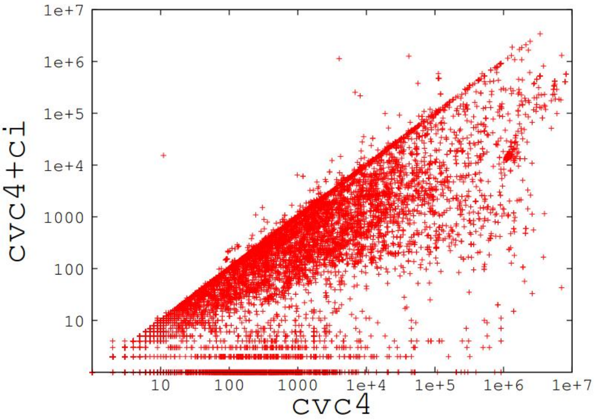
c: solving the problem with E-ground (dis)unification

Given conjunctive sets of equality literals E and L, with E ground, find substitution σ s.t. $E \models L\sigma$

- Variant of classic (non-simultaneous) rigid E-unification
- NP-complete
  - NP: solutions can be restricted to ground terms in $E \cup L$
  - NP-hard: reduction of 3-SAT
- CCFV: congruence closure with free variables [Barbosa17]
  - sound, complete and terminating calculus for solving E-ground (dis)unification
  - goal oriented
  - efficient in practice
- Still, 60% of time in veriT

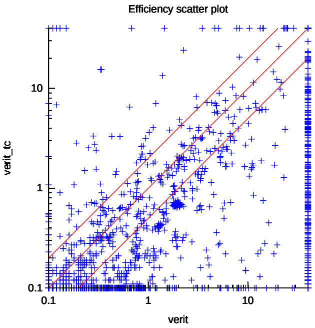
c evaluation (1/2) [Reynolds14]

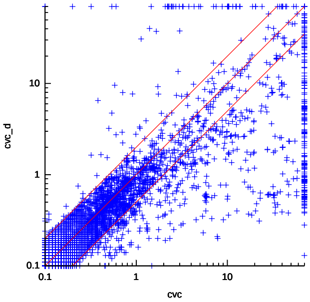


Reported number of instances.

- Evaluation on SMT-LIB, TPTP, Isabelle benchmarks
- Using conflict-based instantiation (cvc4+ci), require an order of magnitude fewer instances to prove unsatisfiability w.r.t. E-matching alone

c evaluation (2/2) [Barbosa17]





veriT: + 800 out of 1 785 unsolved problems

CVC4: + 200 out of 745 unsolved problems

* experiments in the "UF", "UFLIA", "UFLRA" and "UFIDL" categories of SMT-LIB, which have 10 495 benchmarks annotated as *unsatisfiable*, with 30s timeout.

Conflicting instances, prospects

- Still, 60% of time in veriT
- CCFV is an NP-complete problem
- It can be encoded into SAT
- We expect careful encoding of CCFV into SAT will provide efficient procedure
- We are investigating a SAT-based algorithm for higher-order CCFV

- Conflicting instances only work for one instance
- Finding out a pair of instances that contradict a model?
- Maybe use superposition? Extend algorithm to find conflicts with several clauses?

Outline

Model-based instantiation/MBQI (**m**) [Ge09]

Build a candidate model for $E \cup Q$ and instantiate with counter-examples from model checking

Model-based instantiation/MBQI ($\mathbf{m}$) [Ge09]

Build a candidate model for $E \cup Q$ and instantiate with counter-examples from model checking

- $E = \{\neg P(a), \neg P(b), P(c), \neg R(b)\}$ and $Q = \{\forall x.\, P(x) \vee R(x)\}$
- Ground solver provides a partial model...
- ...extended to a full model s.t. $P^{\mathcal{M}} = \lambda x.\, \text{ite}(x = c, \top, \bot)$ and $R^{\mathcal{M}} = \lambda x.\, \bot$
- Since $\mathcal{M} \models \neg\,(P(a) \vee R(a))$, this strategy may return $x \mapsto a$
- Formally

$\mathbf{m}(E, \forall \bar{x}.\, \varphi)$ 1. Construct a model $\mathcal{M}$ for E

2. Return $\bar{x} \mapsto \bar{t}$ where $\bar{t} \in \mathcal{T}(E)$ and $\mathcal{M} \models \neg\varphi[\bar{x}/\bar{t}]$, or $\emptyset$ if none exists

Outline

Why can't we directly use Herbrand instantiation?

Theorem (Herbrand)

A finite set of Skolem formulas is unsatisfiable if and only if there exists a finite unsatisfiable set of Herbrand instances

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  - Instantiate with all possible terms in the language
- Enumerating all instances is unfeasible in practice!
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Revisiting enumerative instantiation with benefits:

- strengthening of Herbrand theorem
- efficient implementation techniques

Theorem (Strengthened Herbrand)

If R is a (possibly infinite) set of instances of Q closed under Q-instantiation w.r.t. itself and if $E \cup R$ is satisfiable, then $E \cup Q$ is satisfiable.

Theorem (Strengthened Herbrand)

If there exists an infinite sequence of finite satisfiable sets of ground literals E_i and of finite sets of ground instances Q_i of Q such that

- $Q_i = \left\{ \varphi\sigma \;\middle|\; \forall \bar{x}.\, \varphi \in Q,\, \mathrm{dom}(\sigma) = \{\bar{x}\} \wedge \mathrm{ran}(\sigma) \subseteq \mathcal{T}(E_i) \right\}$;
- $E_0 = E,\ E_{i+1} \models E_i \cup Q_i$;

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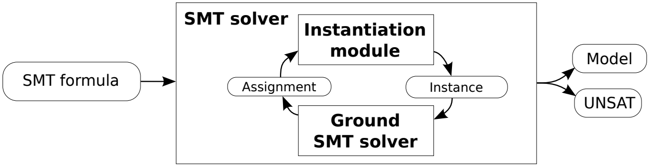
Direct application to



- Ground solver enumerates assignments $E \cup Q$
- Instantiation module generates instances of Q

Enumerative instantiation ($\mathbf{u}$)

[Reynolds18]

$\mathbf{u}(E, \forall \bar{x}.\, \varphi)$

1. Choose an ordering $\preceq$ on tuples of ground terms
2. Return $\bar{x} \mapsto \bar{t}$ where $\bar{t}$ is a minimal tuple of terms w.r.t $\preceq$, such that $\bar{t} \in \mathcal{T}(E)$ and $E \not\models \varphi[\bar{x}/\bar{t}]$, or $\emptyset$ if none exist

- $E = \{\neg P(a), \neg P(b), P(c), \neg R(b)\}$ and $Q = \{\forall x.\, P(x) \vee R(x)\}$
- $\mathbf{u}$ chooses an ordering on tuples of terms, e.g. $a \prec b \prec c$
- Since $E \not\models P(a) \vee R(a)$, enumerative instantiation returns $x \mapsto a$

u as an alternative for **m**

- Enumerative instantiation plays a similar role to **m**
- It can also serve as a "completeness fallback" to **c** and **e**
- However, **u** has advantages over **m** for UNSAT problems
- And it is significantly simpler to implement
  - no model building
  - no model checking

Example

$$
\begin{array}{rcl}
E & = & \{\neg P(a),\ R(b),\ S(c)\} \\
Q & = & \{\forall x.\ R(x) \lor S(x),\ \forall x.\ \neg R(x) \lor P(x),\ \forall x.\ \neg S(x) \lor P(x)\} \\
M & = & \left\{ \begin{array}{rcl} P^{\mathcal{M}} & = & \lambda x.\ \bot, \\ R^{\mathcal{M}} & = & \lambda x.\ \mathrm{ite}(x = b,\ \top,\ \bot), \\ S^{\mathcal{M}} & = & \lambda x.\ \mathrm{ite}(x = c,\ \top,\ \bot) \end{array} \right\}, \qquad a \prec b \prec c
\end{array}
$$

φ	x s.t. $\mathcal{M} \models \neg\varphi$	x s.t. $E \not\models \varphi$	$\mathbf{m}(E, \forall x.\ \varphi)$	$\mathbf{u}(E, \forall x.\ \varphi)$
$R(x) \lor S(x)$	a	a	$x \mapsto a$	$x \mapsto a$
$\neg R(x) \lor P(x)$	b	a, b, c	$x \mapsto b$	$x \mapsto a$
$\neg S(x) \lor P(x)$	c	a, b, c	$x \mapsto c$	$x \mapsto a$

- **u** instantiates uniformly so that less new terms are introduced
- **m** instantiates depending on how model was built
- **u** directly leads to $E \land Q[x/a] \models \bot$

Advanced **u**: restricting enumeration space

- Strengthened Herbrand Theorem allows restriction to $\mathcal{T}(E)$

- *Sort inference* reduces instantiation space by computing more precise sort information
  - $E \cup Q = \{a \neq b,\ f(a) = c\} \cup \{P(f(x))\}$
    - $a, b, c, x : \tau$
    - $f : \tau \to \tau$ and $P : \tau \to \mathrm{Bool}$
  - This is equivalent to $E^s \cup Q^s = \{a_1 \neq b_1,\ f_{12}(a_1) = c_2\} \cup \{P_2(f_{12}(x_1))\}$
    - $a_1, b_1, x_1 : \tau_1$
    - $c_2 : \tau_2$
    - $f_{12} : \tau_1 \to \tau_2$ and $P : \tau_2 \to \mathrm{Bool}$

  - **u** would derive e.g. $x \mapsto c$ for $E \cup Q$, while for $E^s \cup Q^s$ the instantiation $x_1 \mapsto c_2$ is not well-sorted

Advanced $\mathbf{u}$: entailment checks

Two-layered method for checking whether $E \models \varphi[\bar{x}/\bar{t}]$ holds

- cache of instances already derived
- on-the-fly rewriting of $\varphi[\bar{x}/\bar{t}]$ modulo E
  with extension to other theories through theory-specific rewriting

Advanced $\mathbf{u}$: term ordering

Instances are enumerated according to the order

$$(t_1, \dots, t_n) \prec (s_1, \dots, s_n) \quad \text{if} \quad \begin{cases} \max_{i=1}^n t_i \prec \max_{i=1}^n s_i, \text{ or} \\ \max_{i=1}^n t_i = \max_{i=1}^n s_i \text{ and} \\ \qquad\qquad (t_1, \dots, t_n) \prec_{\text{lex}} (s_1, \dots, s_n) \end{cases}$$

for a given order $\preceq$ on ground terms.

If $a \prec b \prec c$, then

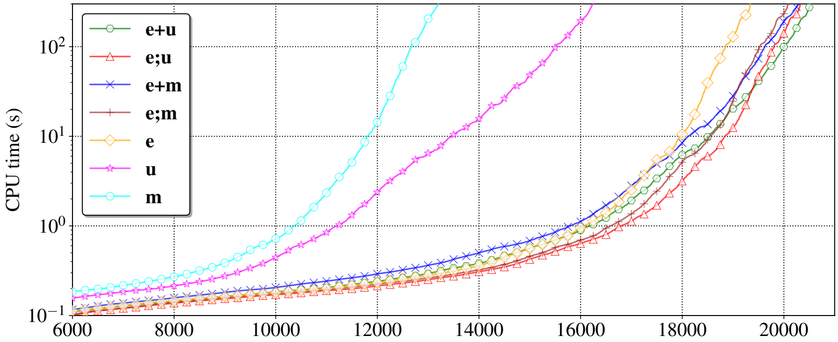
$$(a, a) \prec (a, b) \prec (b, a) \prec (b, b) \prec (a, c) \prec (c, b) \prec (c, c)$$

- instances with c considered only after considering all cases with a and b
- goal is to introduce new terms less often
- order on $\mathcal{T}(E)$ fixed for finite set of terms $t_1 \prec \dots \prec t_n$
  - instantiate in order with $t_1, \dots, t_n$
  - then choose new non-congruent term $t \in \mathcal{T}(E)$ and have $t_n \prec t$
- Still a lot of room for improvement (and ML?) [Janota21]

Outline

Experimental evaluation (UNSAT)

CVC4 configurations on unsatisfiable benchmarks



- 42 065 benchmarks: 14 731 TPTP + 27 334 SMT-LIB
- **e**+**u**: interleave **e** and **u**
- **e**;**u**: apply **e** first, then **u** if it fails
- All CVC4 configurations have **c**; as prefix

Experimental evaluation (SAT)

Library	#	**u**	**e;u**	**e+u**	**e**	**m**	**e;m**	**e+m**
TPTP	14731	471	492	464	17	930	808	829
UF	7293	39	42	42	0	70	69	65
Theories	20041	3	3	3	3	350	267	267
Total	42065	513	537	509	20	1350	1144	1161

Outline

Outline

Conclusion

- Quantifiers in SMT: handled in an ad hoc manner
- Techniques presented here are pure FOL with equality
  (i.e. not "Modulo Theories")
- Reasonably effective nonetheless

Future works and perspectives

- New instantiation techniques (Vampire-like attitude, in SMT?)
- Machine learning
- More convergence with state-of-the-art FOL techniques from saturation theorem proving
- Symbiosis with quantifier elimination for theory reasoning
- Convergence with FOL provers?
- Higher-order logic

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Outline

Outline

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