

Communicating mathematics using a proof assistant

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Machine assisted proof in IPAM

A simplified history of mathematics communication

- 550 BC: listen to Pythagoras
- 250 BC: get papyrus scrolls in Alexandria
- 1993: get TeX files from arXiv

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What if you need more details about the text?

Theorem

Let X be a topological space, A a dense subset of X , $f : X \rightarrow Y$ a mapping of X into a regular space Y . If, for each $x \in X$, $f(y)$ tends to $f(x)$ when y tends to x while remaining in A then f is continuous.

Proof.

We have to show that f is continuous at each point $x \in X$. Let then V' be a closed neighbourhood of $f(x)$ in Y ; then by hypothesis there is an open neighbourhood V of x in X such that $f(V \cap A) \subset V'$. Since V is a neighbourhood of each of its points, we have

$$f(z) = \lim_{y \rightarrow z, y \in V \cap A} f(y)$$

for each z in V , and from this it follows that $f(z) \in \overline{f(V \cap A)} \subset V'$, since V' is closed. The result now follows from the fact that the closed neighbourhoods of $f(x)$ form a fundamental system of neighbourhoods of $f(x)$ in Y . □

Observations

Nobody doubts this theorem is true. That proof is fine.

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Can you say precisely where the assumption that A is dense is used?

Let's ask OpenAI

We now give a detailed version of this proof, explaining every single line. In this version, we make sure every theorem hypothesis that was mentioned is actually used.

Answer:

We have to show that f is continuous at each point $x \in X$. To do this, let V' be a closed neighbourhood of $f(x)$ in Y , and consider the hypothesis that, for each $x \in X$, $f(y)$ tends to $f(x)$ when y tends to x while remaining in A . From this hypothesis, there is an open neighbourhood V of x in X such that $f(V \cap A) \subset V'$.

We now use the fact that V is a neighbourhood of each of its points. This means that for each z in V , the function $f(y)$ tends to $f(z)$ when y tends to z while remaining in $V \cap A$. This is expressed by the equation

$$f(z) = \lim_{y \rightarrow z, y \in V \cap A} f(y).$$

Answer continued

From this equation, it follows that $f(z) \in \overline{f(V \cap A)} \subset V'$, since V' is closed. This is because the limit of the sequence of values $f(y)$ for $y \in V \cap A$ is contained in the closure of the set of values $f(y)$ for $y \in V \cap A$.

Finally, the fact that f is continuous at x follows from the fact that the closed neighbourhoods of $f(x)$ form a fundamental system of neighbourhoods of $f(x)$ in Y .

Using a proof assistant

Let's see a formalized version in Lean (any modern proof assistant would be suitable).

What did we gain?

The formalization author learned the details.

Lean experts can read the Lean code on a computer running Lean (and same with Coq, Isabelle, Agda...).

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Experience suggests it is hopeless to get everybody to read that.

Readable proofs from formal proofs

Claim: trying to make proof scripts readable is hopeless and counter-productive in general.

Readable proofs should be the output, not the input.

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One exception is teaching how to write proofs

Using OpenAI Da Vinci 3 language model

The following is a Lean proof. Your task is to explain in plain informal English (1) the theorem statement and (2) the proof. You must explain this in a precise way, just like any mathematical proof that appears in a mathematics textbook. You should be writing in a way that is understandable for a first-year graduate student.

```
theorem continuous_of_dense ...
```

Da Vinci's explanation

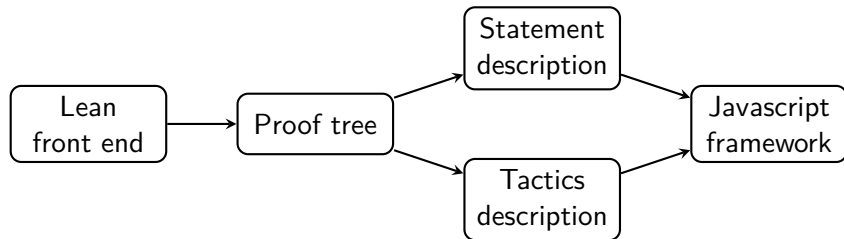
First, assume that V' is an open neighborhood of $f(x)$. Then by the Regular Space property, there exists an open neighborhood W of x such that the closure of W is contained in V' . Then, by the assumption that f is continuous within A at x , there exists an open neighborhood U of x such that the image of U under f is contained in W . Thus, the image of U intersection A under f is contained in V' .

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Can we do better?

What if we use programming instead?

General pipeline



Key pieces

- \LaTeX printer for math expressions.
- English (partial) sentences printer for statements
- Proof term describer
- Tactic describer

Each item uses the previous ones.

Future steps

- Support more tactics
- Handle definitions
- Render multiple files with links
- Handle discussions, pictures, animations
- Render mathlib (probably changing some proofs)
- Write whole textbooks (elementary or advanced)

What is it doing?

Let's take a quick glance at how English sentences are constructed.

Lean is a functional programming language

Everything is based on functions, including logic.

A proof of $P \implies Q$ is a function from proofs of P to proofs of Q .

A proof of $\forall x, P(x)$ is a function mapping x to a proof of $P(x)$.

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Let's see an example.

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Programming is also all about types and functions.