Setting the Stage with Networks and Network Dynamics

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Abstract

- To help set the mathematical stage for the workshop, I will briefly present some fundamental ideas in networks and network dynamics. I will discuss various types of networks (such as directed networks, multilayer networks, and so on), with an aim towards trying to convey what we do and don't know how to do. In this vein, I will also comment about dynamics on networks, dynamics of networks, and networks that coevolve with dynamics.

- I plan to use roughly half of the allotted time for my presentation slides, and I expect active discussions and audience questions for the other half.

- Springboarding from these discussions, we should try to set the stage for formulating interesting and concrete projects at the conclusion of the workshop.
Interrupt me with questions!

This is supposed to be a discussion to help set the stage for the workshop.
Two Flavors of Issues when Generalizing Network Representations and Data Structures

1. There are many different ways to generalize some approach or quantity (betweenness centrality, clustering coefficient, etc.) and many different choices to make in the process, so we need to think about which way to do it and what is most appropriate for a given problem.

   - Example: Given a weighted network (with larger weights encoding stronger relationships), how do we want to measure the distances of a walk on the network?

2. When generalizing, we lose some convenient feature (e.g., a important theorem no longer applies) and then we need to figure out — through development of theory and/or methods — how to get the generalization to work.

   - Example: The adjacency matrix of a directed network is generically asymmetric, so the leading eigenvalue is no longer guaranteed to be real, and spectral methods that rely on having a real leading eigenvalue or leading eigenvector need to be adjusted.
Graphs: The Simplest Networks

- **Remark:** In much networks literature, it is common to use the word “networks” in a way that does not automatically include dynamics. This is different in some other traditions.

- **Adjacency matrix** $A$
  - $A_{ij} = 1$ if there is a connection between nodes $i$ and $j$
  - $A_{ij} = 0$ if no connection

- We can generalize these representations to account for edge directions, edge weights, multiple relationships, changes with time, and other complications.

- **Some things that we like:**
  - Generative models of random graphs, local clustering coefficient, centrality measures (to measure importance of nodes and edges), community structure and other mesoscale properties, etc.
  - How do we generalize them for more complicated network representations?
Configuration-Model Random Graphs


- Fix degree sequence and connect stubs (i.e., ends of edges) to each other uniformly at random
  - Or fix a degree distribution and draw a degree sequence from it
- Self-edges and multi-edges?
- Null model in community detection

Graph Laplacian Matrices

  - Combinatorial graph Laplacian: \( L = D - A \)
  - Random-walk normalized Laplacian: \( D^{-1}L \)
- Spectral clustering: Use eigenvalues and eigenvectors of a Laplacian or related matrix (e.g., a modularity matrix)
Weighted Networks

- Note: Technically, these are edge-weighted networks, which is what people usually mean when they say “weighted networks”. One can also think about node weights.

- Weights versus “distances”
  - In a weight matrix (i.e., weighted adjacency matrix), larger values indicate stronger connections
  - In a distance matrix, larger values indicate weaker connections
  - Data may arise most naturally as costs (i.e., “distances”) or as weights

- If you are given weights, how do you measure distances, such as for walks on networks and anything (e.g., centrality measures like betweenness) that use them?

- Weighted generalizations of ideas like local clustering coefficients: How should you normalize?

- Conceptual trickiness in generalizations of random-graph models: shuffling edges versus redistributing weights?
  - For example, in generalizations of configuration models and stochastic block models
  - Conceptually harder than some other generalizations (e.g., multilayer networks) that are often harder for other things
Directed Networks


- Graph Laplacians: out-degree versus in-degree

\[ L' = D^{-1}L = I - T. \]

That is, \((L')_{ij} = \delta_{ij} - (A_{ij}/s_{i}^{\text{out}})\).

- The leading eigenvalue and leading eigenvector are no longer typically guaranteed to be real, so spectral approaches need to be generalized to account for that.

- Generalizing ideas like local clustering coefficients and network “motifs” to directed networks?

- Null models for community detection through directed generalization of configuration-model networks

  - Or use statistical-inference methods, such as with degree-corrected stochastic block models

  - Or use local methods, such as via personalized PageRank (although many of these methods want a real leading eigenvalue)
Signed Networks


- Positive and negative edges

- Examples: Correlations, excitatory versus inhibitory dynamics, etc.

- Community structure and random-graph null models
  - A signed generalization of configuration models
  - Spectral properties of adjacency and Laplacian matrices?
    - Difficulties in generalizing centrality measures (including not usually having a guarantee of a real, positive leading eigenvalue)

  - For imaging data: Threshold weights to obtain an unweighted network, making all weights positive somehow, or using signed weighted networks?
  - This paper does not consider signed-network methods
Networks with Complex-Valued Weights


- Applications include quantum walks on networks, graph signal processing, graph neural networks, transmission-line networks, coupled-oscillator networks, etc.

- Hermitian versus non-Hermitian matrices

- Need much more network analysis to generalize our favorite ideas to this setting

- Further studies using ideas like pseudospectra and non-normal matrices should be useful (e.g., for generalizing ideas like eigenvector centrality)
  - Lucas wanted me to ask if any folks at the workshop have some ideas to share
Multilayer Networks

(note: different use of the term than in machine learning)

- Manlio De Domenico [2022], *Multilayer Networks: Analysis and Visualization*, Springer International Publishing, Cham, Switzerland
- Adjacency tensor versus “flattening” into a supra-adjacency matrix
- Intralayer edges versus interlayer edges: Often less reliable data and more conceptual difficulties with the latter (what do the $\omega$ mean?)

Multilayer Networks

Tradeoffs here and also for the other more complicated representations:

- Generalizing methods versus losing information?
- Trustworthy data that jibes with these more complicated representations?
  - E.g., interlayer edge weights
- Multilayer representation of time-dependent networks
- Much state-of-the-art research is in trying to generalize familiar network concepts
  - Random-graph models, Laplacians, walks, centralities, clustering coefficients, etc.
  - E.g., eigenvector-based centralities (such as multilayer PageRank) where you can construct things in a way to still use Perron–Frobenius theorem and guarantee a real and positive leading eigenvalue
- How do more complicated network structures affect dynamics on networks?

Polyadic (i.e., Higher-Order) Relationships


- Hypergraphs and simplicial complexes
  - Generalizing structural measurements and analysis
    - E.g., generalizations of Laplacians (e.g., Hodge Laplacian)
  - How do polyadic interactions affect dynamics?
    - Modeling choice: Simplicial complexes (which require downward closure) versus hypergraphs?

- Topological data analysis (TDA)
  - See Moo Chung’s talk this afternoon for tutorial on TDA
Networks and Dynamics
Dynamics on Networks


  - Includes models of neuronal dynamics on networks

- How does network structure affect dynamics on a network?
  - Faster synchronization for some types of networks? Slower flow of something on some networks?

- Coupled phase oscillators (e.g., Kuramoto model):
  \[
  \dot{\theta}_i := \frac{d\theta_i}{dt} = \omega_i + \sum_{j=1}^{N} b_{ij}A_{ij}\varphi(\theta_j - \theta_i), \quad i \in \{1, \ldots, N\}.
  \]

- Better-developed theory for autonomous systems than for nonautonomous systems
  - E.g., One needs to generalize ideas like stable and unstable manifolds for nonautonomous systems

- Deterministic or stochastic dynamical processes

- Dynamics on combinatorial networks versus dynamics on ‘metric networks’
Dynamics of Networks


- Discrete time versus continuous time
  - Chopping up time series?!? (oy vey)

- Multilayer representation of temporal networks, with e.g. generalizing eigenvector-based centralities and community-detection methods
Adaptive (i.e., Coevolving) Networks


- Coupling between dynamics on networks and dynamics of networks

- Adaptive network weights:

\[
\dot{x}_i = f_i(x_i, t) + \sum_{j=1}^{N} \kappa_{ij} g(x_i, x_j, t),
\]

\[
\dot{\kappa}_{ij} = h(x_i, x_j, t),
\]

- Adaptive time delays (less well-studied?):

\[
\dot{x}_i(t) = f_i(x_i(t), t) + \sum_{j=1}^{N} \kappa_{ij} g(x_i(t), x_j(t - \tau_{ij}(t)), t),
\]

\[
\dot{\tau}_{ij} = h(x_i(t), x_j(t), t).
\]

- Adaptive natural frequencies \(f_i\)

- With dynamical systems or with agent-based models

- Example: ‘Hebbian’ learning in coupled oscillators on networks

- It’s often hard to make the models simple enough to do math.
Conclusions

- There are many different ways to generalize some approach or quantity (betweenness centrality, clustering coefficient, etc.) and many different choices to make in the process, so we need to think about which way to do it and what is most appropriate for a given problem.

- When generalizing, we lose some convenient feature (e.g., such as losing the guarantee that the leading eigenvalue and leading eigenvector are real) and then we need to figure out — through development of theory and/or methods — how to get the generalization to work.

- “Springboarding from these discussions, we should try to set the stage for formulating interesting and concrete projects at the conclusion of the workshop.”