Quantum resonances for classically chaotic systems

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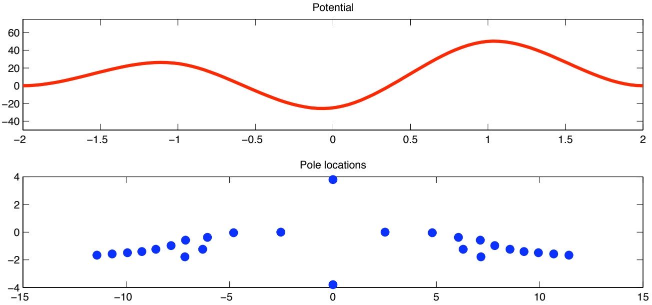

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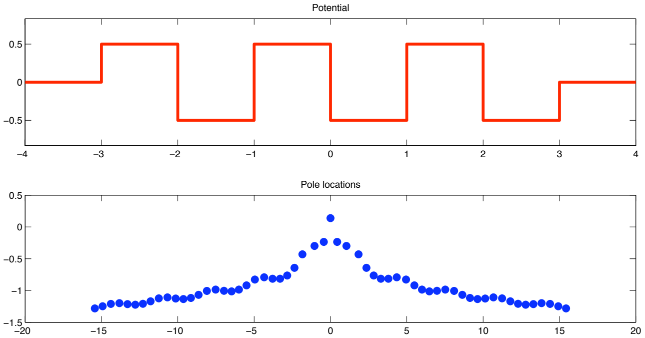

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Similar results for

$$H = -h^2\Delta_g + V(x)$$

for large classes of potentials V and metrics g.

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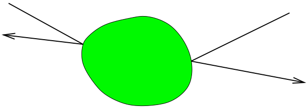
Then for any $M > 0$ *there exists a constant* C *such that there no resonances in*

$$\{\lambda \; : \; \mathrm{Im}\,\lambda > -M \log|\lambda| \,, \quad |\lambda| > C\} \,.$$

In particular, there exists a resonance free strip, $\mathrm{Im}\,\lambda > -C_0$.



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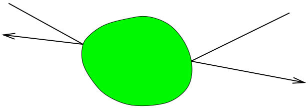

For $H = -h^2\Delta + V(x)$ the non-trapping condition means that the flow of $\dot{x} = 2\xi$, $\dot{\xi} = -\nabla V(x)$, on $|\xi|^2 + V(x) = E > 0$ is non-trapping. Then near E we have

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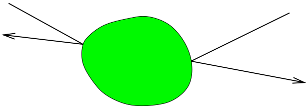

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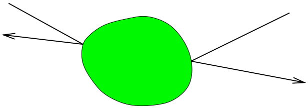

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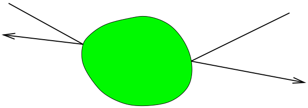

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The last condition is the exact analogue of the condition in the theorem.

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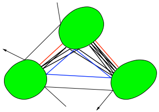

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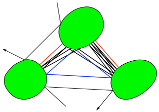

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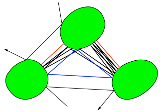

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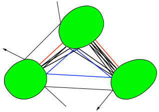

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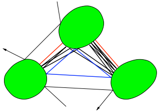

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$$P(f) = \lim_{T \to \infty} \frac{1}{T} \log \sum_{T_\gamma < T} \exp \left(\int_0^{T_\gamma} \Phi_t^* f|_\gamma dt \right) ,$$

where Φ_t is the flow, γ are closed orbits with period T_γ.

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Following the work of Dolgopyat and Naud, Petkov-Stoyanov 2007 prove much more: there exists $\delta > 0$ such that there are no resonances in

$$\mathrm{Im}\,\lambda > P(-\Lambda_+/2) - \delta\,, \quad \mathrm{Re}\,\lambda > C\,.$$

For operators $H = -h^2\Delta_g + V(x)$ the results similar to Ikawa's result became known only recently. We consider the pressure of the flow on the (non-degenerate) energy surface $|\xi|_g^2 + V(x) = E$ and resonances in $D(E, Ch)$, $E > 0$.

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Vodev 1994: similar results for n even.

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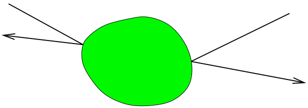

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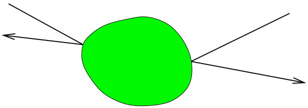
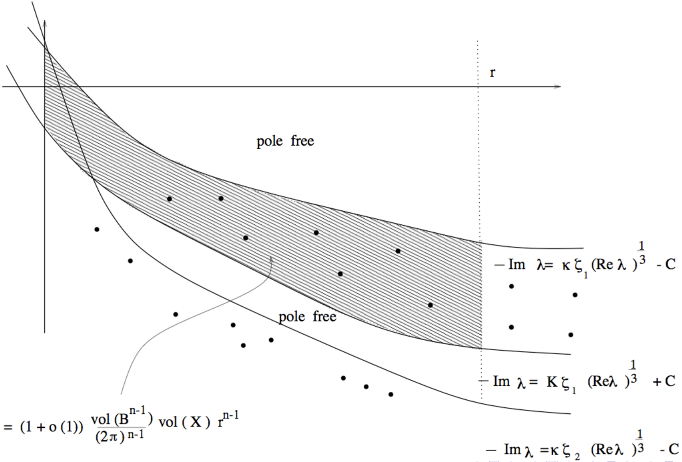


Sjöstrand-Zworski 1999



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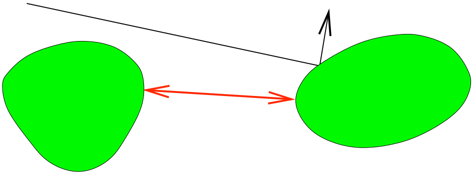

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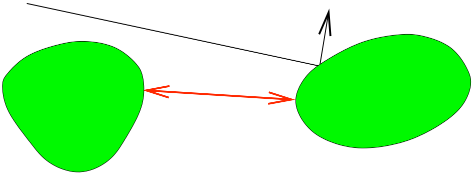

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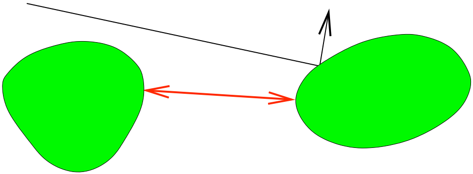

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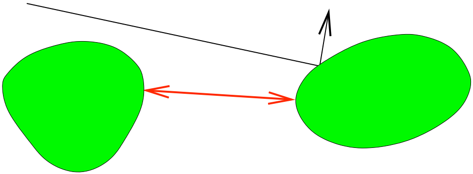

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$$\sum_{\operatorname{Im} z > -\alpha, |z| \leq r} m_R(z) \sim C(\alpha) r \, .$$

Note that for one convex obstacle this sum would be $O(1)$.

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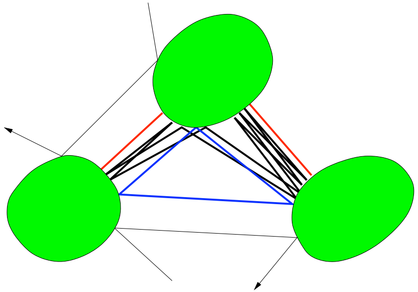

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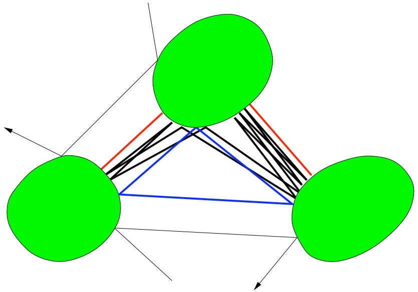

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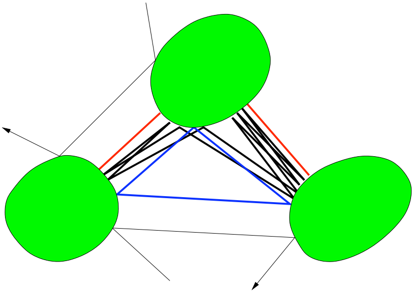

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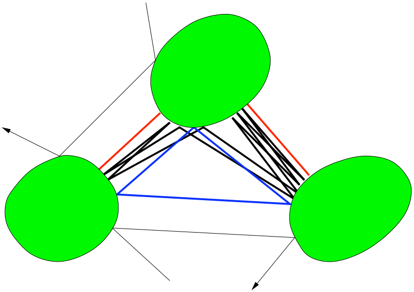

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but no counting results better than Melrose's theorem...

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This theorem is part of a larger project on open hyperbolic systems with topologically one dimensional trapped sets (always satisfied for several convex bodies satisfying Ikawa's condition).

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If $E \mapsto \mathcal{L}(\{x \ : \ V(x) \geq E\})$ has an *analytic singularity* at E_0 then

$$\sum_{|z-E_0|\leq C_0} m_R(z) \geq h^{-n}/C_1\,.$$

Fractal Weyl laws:

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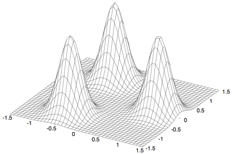
Analytic potential with hyperbolic dynamics 

$$\sum_{|z-E|\leq\delta\,,\mathrm{Im}\,z>-Ch} m_R(z) = \mathcal{O}(h^{-\mu-1-0})\,,$$

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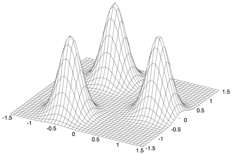

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Zworski 1999, Guillopé-Lin-Zworski 2004

More precise bounds in the case of convex-cocompact Schottky quotients $\Gamma\backslash\mathbf{H}^n$, $\mu=\delta(\Gamma)$, dimension of the limit set.

Sjöstrand-Zworski 2006

For C^∞ potentials with hyperbolic dynamics at energy E,

$$\sum_{|z-E|\leq Ch} m_R(z) = \mathcal{O}(h^{-\mu_E -}) \, ,$$

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The only lower bound showing "optimality" comes from an open quantum map "toy mode", Nonnenmacher-Zworski 2005.

The interest in physics is picking up:

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Lu et al, Phys. Rev. Lett. 91, 154101 (2003)

Schomerus-Tworzydło, Phys. Rev. Lett. 93, 154102 (2004)

Schomerus-Jacquod, J. Phys. A: Math. Gen, (2005)

Vaa et al Phys. Rev. E 72, 056211 (2005)

Keating et al Phys. Rev. Lett. 97, 150406 (2006)

Nonnenmacher-Rubin Nonlinearity (2007)

Wisniacki-Carlo Phys. Rev. E 77, 045201(R) (2008)

Wiersig-Main Phys. Rev. E 77, 036205 (2008)

Shepelyansky Phys. Rev. E 77, 015202(R) (2008)

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Lu-Sridhar-Zworski 2003

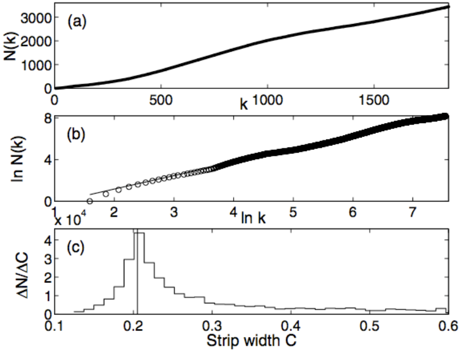

FIG. 2. (a) The counting function, $N(k)$, for width $C = 0.28$ for the resonances in Fig. 1. (b) The plot of $\ln N(k)$ against $\ln k$. The least square approximation slope is equal to 1.288. (c) Dependence of density of resonances $\Delta N/\Delta C$ on strip width C. The vertical line is $\frac{1}{2}\gamma_0$.

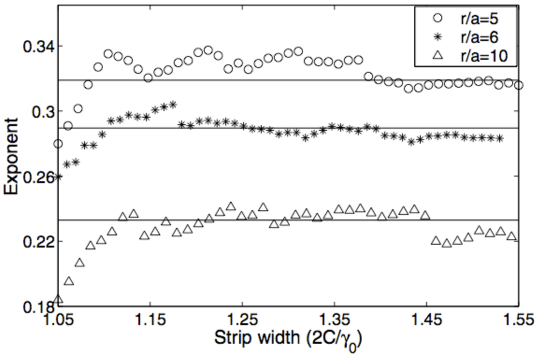

FIG. 3. Dependence of exponent on the rescaled strip width, $2C/\gamma_0$, for the 3-disk system in three cases with $r/a = 5$, 6, and 10. $\gamma_0 = 0.4703$, 0.4103, and 0.2802 is the corresponding classical escape rate. The solid lines are the corresponding Hausdorff dimensions $d_H = 0.3189$, 0.2895, and 0.2330. The values of γ_0 and d_H are calculated following Ref. [3] and references therein.

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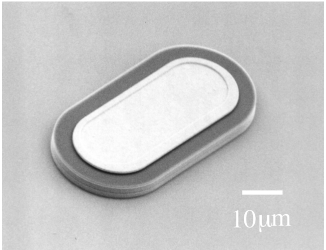

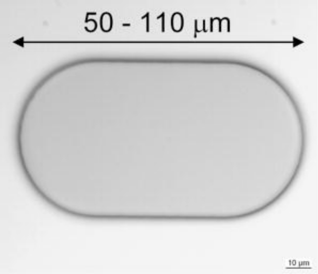

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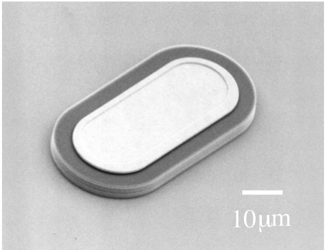

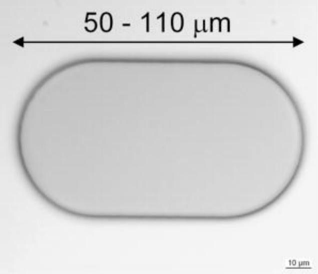

On the left a weakly opened semiconductor (GaAs), on the right a strongly open polymer.

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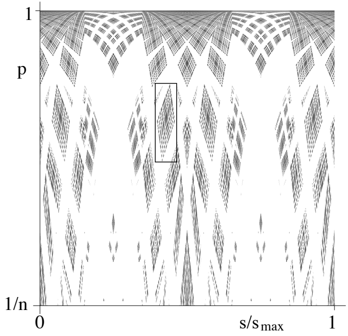

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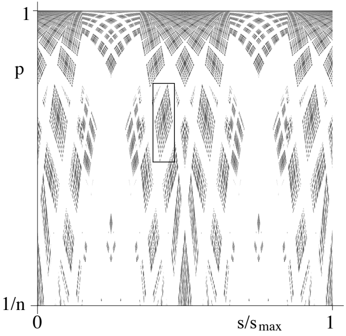

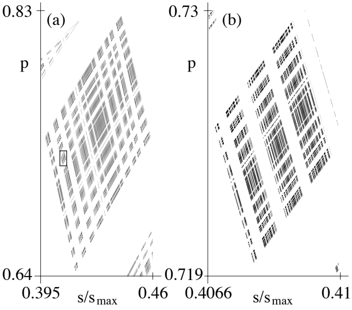

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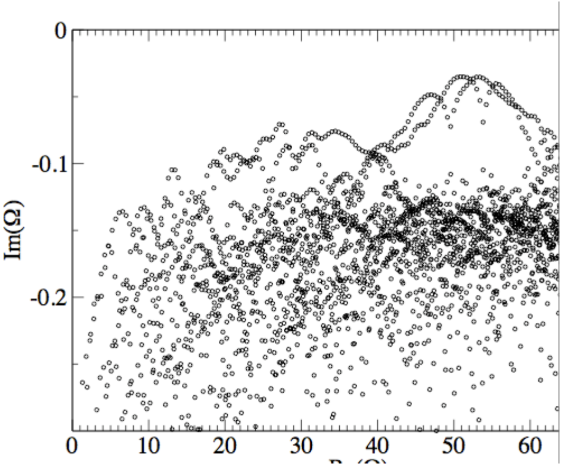

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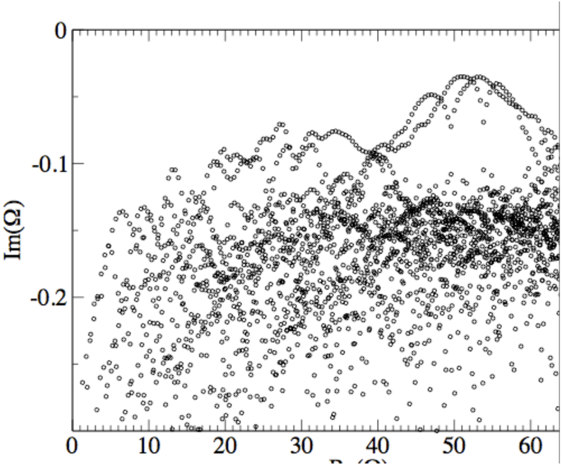

A suitably modified (due to partial opennes of the system) is claimed to hold in this case (Wiersig et al Phys. Rev. 2008).

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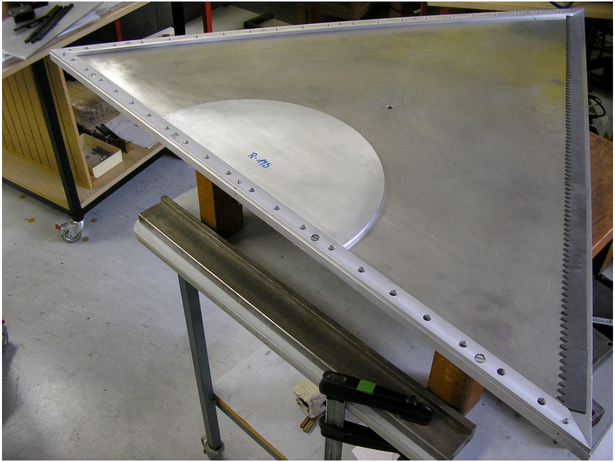

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with $M(z)$ a quantum map, that is

$$M(z) = \Pi\mathcal{M}(z)\Pi + \mathcal{O}(h^M)\,,$$

where Π is a finite rank ($\sim h^{-n+1}$) projection and $\mathcal{M}(z)$ is an h-Fourier integral operator associated to a Poincaré map on a Markov partition of the flow.