Mode localization on chaotic manifolds: an entropy approach

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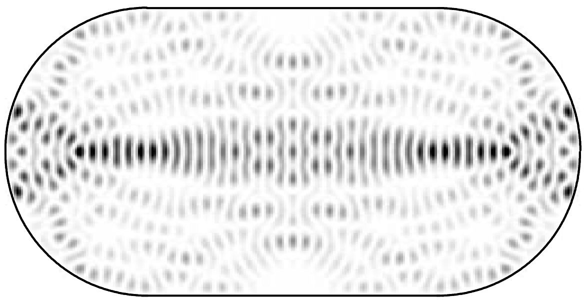

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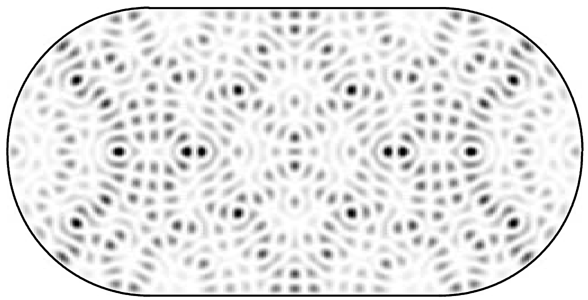

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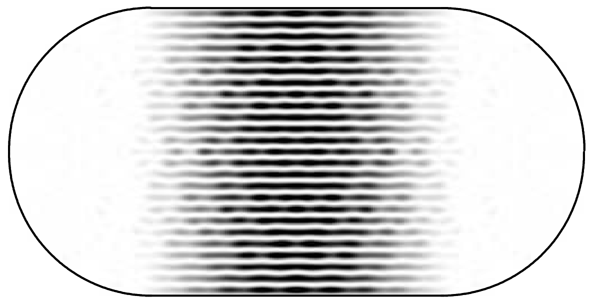

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Outline

- Objective: describe high-frequency eigenmodes of Δ on a smooth, compact Riemannian manifold. High-frequency $\rightsquigarrow$ classical (geodesic) flow

- Rough tool to describe the ψ_n: **semiclassical measures** μ_{sc} associated with subsequences $(\psi_{n_j})_{j\geq 1}$. $\mu_{sc} \subset$ flow-invariant measures.

- Main question: which invariant measures can semiclassically occur? Quantum (Unique?) Ergodicity.

- Choice of specific manifolds: strongly chaotic (Anosov) geodesic flows. Both **complex** (non-integrable) and **treatable** (well-understood).
  Many examples (negative sectional curvature).

- Our approach: compute the **Kolmogorov-Sinai entropy** of μ_{sc}, which partially characterizes its localization properties. $\rightsquigarrow$ μ_{sc} is at least half-delocalized.

- (?) sketch of proof

High-frequency $\equiv$ semiclassical

(X, g) compact smooth Riemannian manifold (with/out boundary). We want to analyze the eigenmodes $(\psi_n)_{n\geq 0}$ of the Laplace-Beltrami operator $\Delta = \Delta_g$:

$$\Delta\psi_n + k_n^2\psi_n = 0$$

This Helmholtz equation can be rewritten as a stationary Schrödinger equation

$$\frac{-\hbar_n^2\Delta}{2}\,\psi_n = \frac{1}{2}\,\psi_n\,,$$

with "Planck's constant" $\hbar = \hbar_n = k_n^{-1}$. In this setting, the eigenmode $\psi_n = \psi_\hbar$ is associated with the *classical energy* $E = \frac{1}{2}$.

high-frequency $(k_n \to \infty) \equiv$ semiclassical $(\hbar \ll 1)$.

$\hbar_n =$ **wavelength** of the state ψ_n.

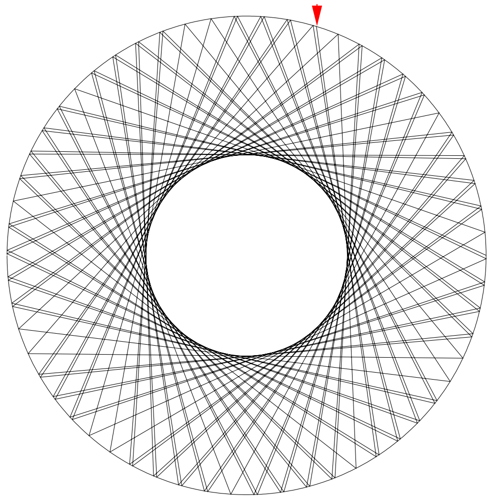
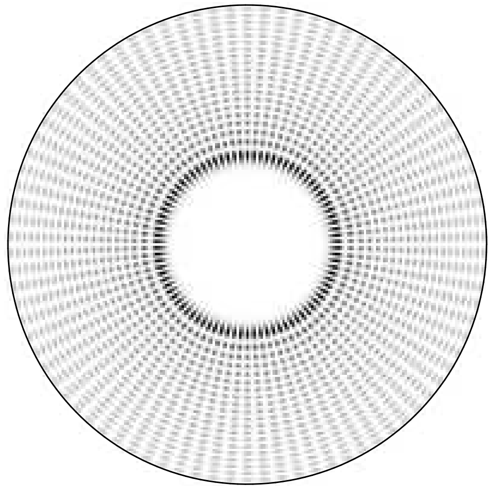
Advantage: use the tools of *semiclassical analysis*. Connection with the **classical dynamics**: Hamiltonian flow on T^*X generated by the Hamiltonian

$$p(x,\xi) = \frac{|\xi|_g^2}{2}$$

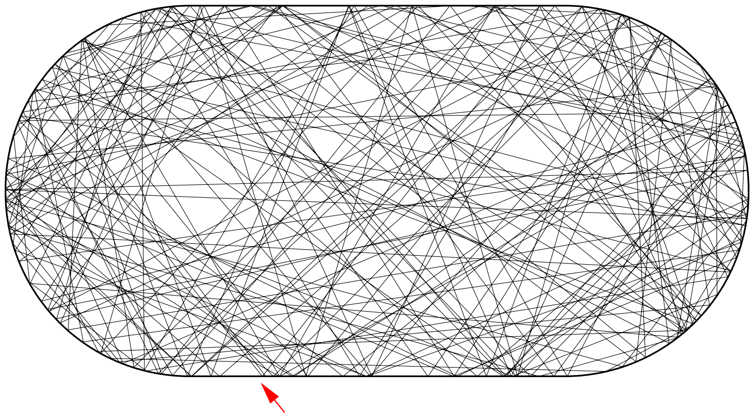
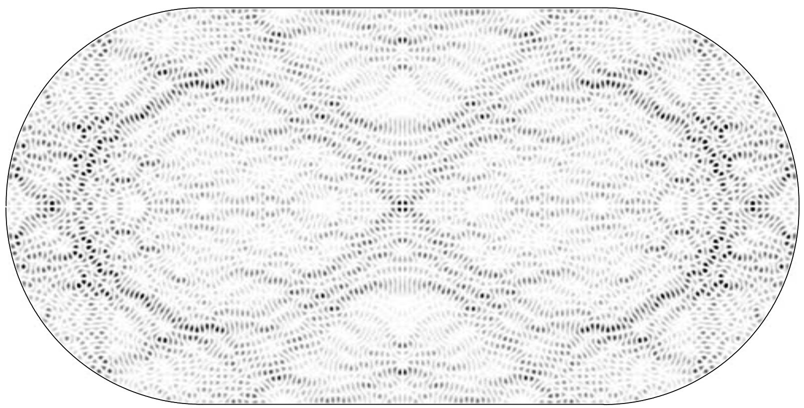
($\equiv$ geodesic flow on X.)

Regular vs. chaotic flows

For some exceptional mfolds X (Liouville-integrable flow), we have approximate or explicit expressions for $\psi_n(x)$ (separation of variables + WKB).

At the opposite: manifolds supporting a chaotic geodesic flow. We don't have any approximate expression for the ψ_n at our disposal.

(inbetween: manifolds with *mixed* phase space. The properties of the classical flow are even more complicated).

Several ways to describe eigenfunctions $\psi_n(x)$

One can study $\psi_n(x)$ on different levels/scales:

- microscopic: **statistical** fluctuations at the scale $\hbar_n$, nodal lines/domains, random wave models.

- macroscopic properties of ψ_n.
  For some fixed test function F on X, investigate the behaviour of $\int_X |\psi_n(x)|^2 F(x)\, dx$ when $n \to \infty$.

  $\to$ one can extract a subsequence $(n_j \to \infty)$ s.t. for any F,

$$\int F\, |\psi_{n_j}|^2\, dx \stackrel{j\to\infty}{\to} \tilde{\mu}(F)$$

  The probability measure $\tilde{\mu}$ on X is called a **quantum limit**. It describes the asymptotic localization on X of the states (ψ_{n_j}), measured at the scale unity.

Lift to the phase space: semiclassical measures (1)

A quantum limit $\tilde{\mu}$ can be connected with the geodesic flow by **lifting** it to a *phase space measure*.

This lift can be performed by using quantum observables ($\hbar$-pseudodifferential operators), which not only measure $|\psi_\hbar(x)|$, but also its *phase fluctuations* of at the scale $\hbar$ (phase fluctuations $\equiv$ *momentum* of the quantum particle).

Ex: $\psi_0(x) = \exp\left(\frac{-(x-x_0)^2 + i\xi_0 \cdot x}{\hbar}\right)$ localized at position x_0 AND at momentum ξ_0.

Observable $f(x,\xi)$ on phase space $T^*X \stackrel{quantization}{\longrightarrow}$ operator $\mathrm{Op}_\hbar(f)$ on $L^2(X)$.
Main property:

$$\mathrm{Op}_\hbar(f)\psi_0 = f(x_0,\xi_0)\,\psi_0 + \mathcal{O}(\hbar)$$

To measure the localization properties of the (ψ_n), consider the matrix elements

$$\langle \psi_n, \mathrm{Op}_{\hbar_n}(f)\,\psi_n\rangle \stackrel{\mathrm{def}}{=} \int f(x,\xi)\,\rho_n(x,\xi)\,dx d\xi$$

Depending on the choice of quantization $\mathrm{Op}_\hbar$, the function ρ_n is called the Wigner function, the Husimi function...

Lift to the phase space: semiclassical measures (2)

Def: from (ψ_n) one can extract a subsequence (ψ_{n_j}) such that, for any $f \in C_b^\infty(T^*X)$,

$$\langle \psi_n, \mathrm{Op}_{\hbar_n}(f)\, \psi_n \rangle \stackrel{j\to\infty}{\to} \mu_{sc}(f)$$

μ_{sc} is a probability measure on phase space. It is called the **semiclassical measure** associated with the subsequence (ψ_{n_j}).

μ_{sc} describes the asymptotic macroscopic distribution of the ψ_{n_j}, both in position and momentum.

Take $f(x,\xi) = F(x) \Longrightarrow \mu_{sc}$ is a **lift** of $\tilde{\mu}$.

Rmk: We have a priori no idea of the speed of convergence $\to$ difficult to identify μ_{sc} from numerics.

Quantum-classical correspondence

A semiclassical measure μ_{sc} satisfy simple properties:

• From the mode equation $\frac{-\hbar_n^2\Delta}{2}\psi_n = \mathrm{Op}_{\hbar_n}(p)\psi_n = \frac{1}{2}\psi_n$, one shows that μ_{sc} is supported on the energy shell $\{p(x,\xi) = \frac{1}{2}\} = S^*X$.

• Call $U_\hbar^t = e^{it\hbar\Delta/2}$ the Schrödinger propagator, and Φ^t the (geodesic) flow generated by p.

Egorov's theorem (quantum-classical correspondence): for any $f \in C_b^\infty(T^*X)$,

$$U_\hbar^{-t}\,\mathrm{Op}_\hbar(f)\,U_\hbar^t = \mathrm{Op}_\hbar(f\circ\Phi^t) + \mathcal{O}_t(\hbar)$$

$\rightsquigarrow$ μ_{sc} is **invariant through the geodesic flow**. This establishes a connection between quantum invariants (ψ_n) and classical ones (μ_{sc}).

Examples of invariant measures: Liouville μ_L, δ_P on periodic trajectories.

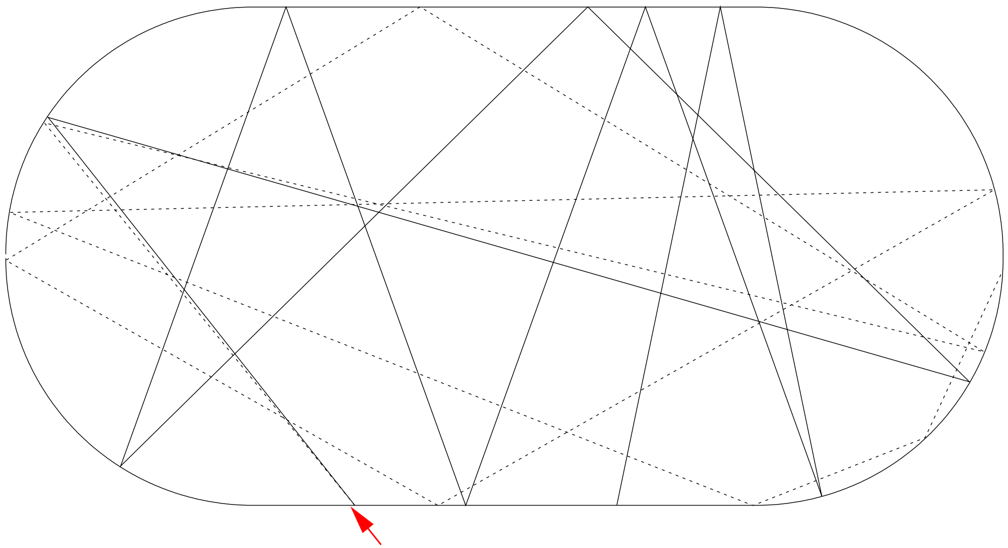
Can one obtain ANY invariant measure by extracting appropriate subsequences of (ψ_n)?

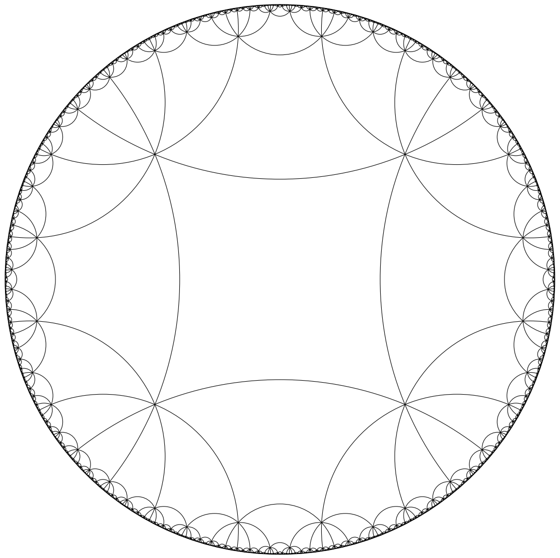
To address this question, we restrict ourselves to a certain type of manifolds.

Quantum ergodicity

From now on, assume that the geodesic flow on S^*X is ergodic w.r.to Liouville.

- X some Euclidean billiard (stadium, Sinai, cardioid,..).





- X boundaryless, of negative sectional curvature. Special case: $X = \Gamma/\mathbb{H}$, Γ subgroup of $SL_2(\mathbb{R})$. Even more special: Γ an *arithmetic* subgroup.

Quantum ergodicity theorem

[Shnirelman'74, Zelditch'87, Colin de Verdière'85] for negative curvature,
[Gérard-Leichtnam'93, Zelditch-Zworski'96] for Euclidean billiards.

There exists a subsequence (ψ_{n_j}) **of density 1** associated the Liouville measure μ_L.

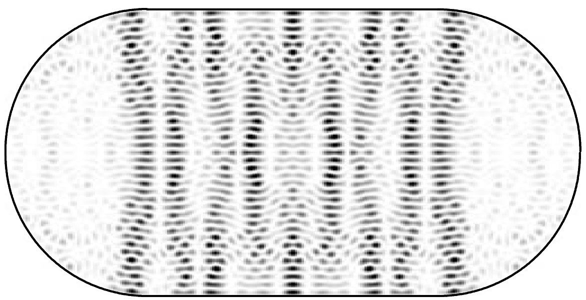
$\Longleftrightarrow$ **almost all** eigenstates ψ_n become equidistributed (in a weak sense) when $n \to \infty$:

Quantum (unique?) ergodicity

Assume the geodesic flow on X is ergodic. Is μ_L the only semiclassical measure for the whole sequence (ψ_n)?

Quantum Unique Ergodicity conjecture: YES (for X of negative curvature) [Rudnick-Sarnak'93].

The contrary would be the possibility of *exceptional subsequences* (ψ_{n_j}) associated with $\mu_{sc} \neq \mu_L$. In particular, could there be **strong scars** $\mu_{sc} = \delta_{PO}$, or **bouncing-ball modes** $\mu_{sc} = \mu_{bb}$, or more complicated μ_{sc}?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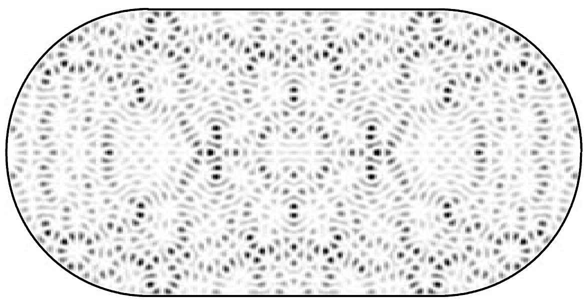
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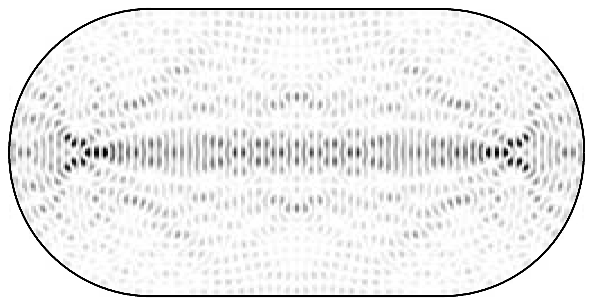
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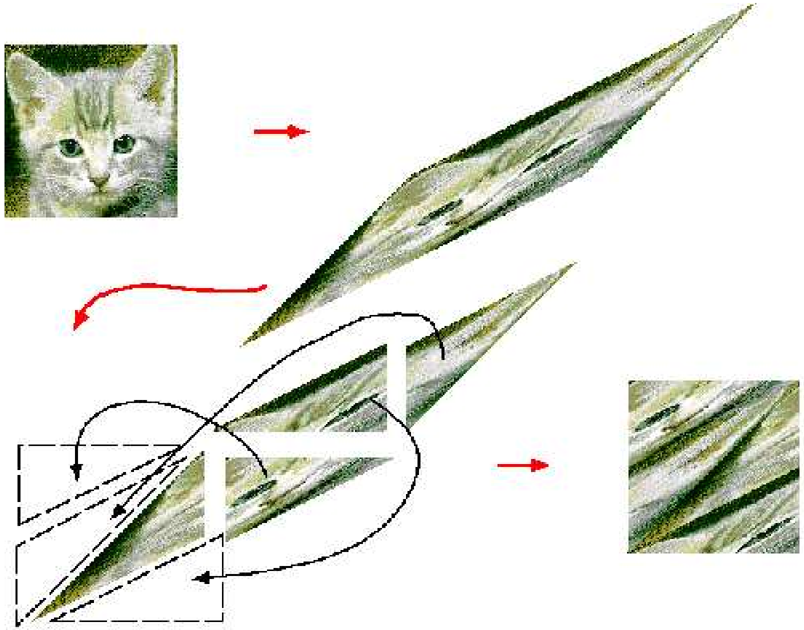
Few results on the QUE conjecture

Quantum Ergodicity can be proved for symplectic chaotic maps on compact phase spaces, like Arnold's cat map on the 2-torus phase space: $\binom{x}{\xi} \mapsto A\binom{x}{\xi}$, $A \in SL_2(\mathbb{Z})$.



This map can be $\hbar$-quantized into a family of (finite-dimensional) unitary propagators $(U_\hbar(A))_{\hbar=(2\pi N)^{-1}}$. The eigenstates of these propagators are models of chaotic eigenmodes.

Rmk: the eigenvalues of $U_\hbar(A)$ are often highly degenerate.

- QUE holds for *arithmetic* (desymmetrized) eigenstates of $U_\hbar(A)$ [Kurlberg-Rudnick'00]

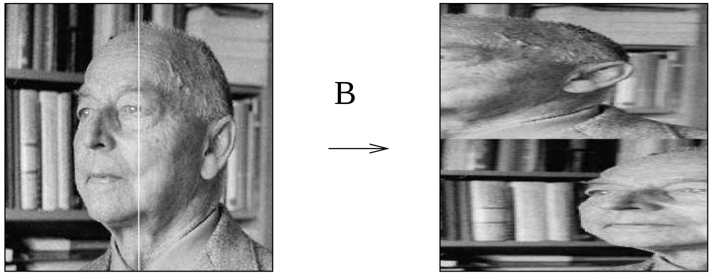
- QUE holds for *arithmetic* eigenstates on *arithmetic* surfaces $\Gamma/\mathbb{H}$ [Lindenstrauss'06].

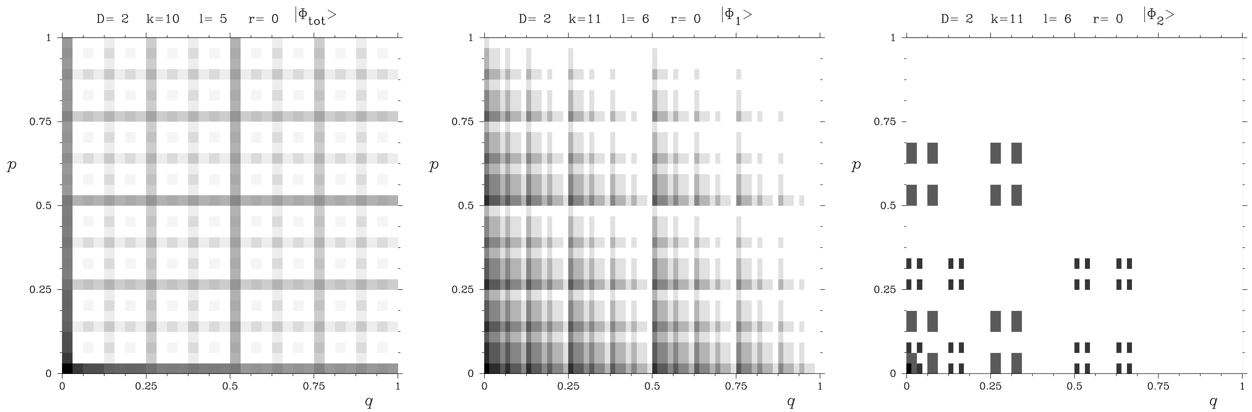
But..

Counterexamples to QUE for symplectic maps

$\exists$ sequences $(\psi_{\hbar_k})$ of eigenstates of $U_\hbar(A)$ associated with semiclassical measures $\mu_{sc} \neq \mu_L$ [FAURE-N-DEBIÈVRE'03].

Idem for the baker's map quantized *à la Walsh* [ANANTHARAMAN-N'06]





Examples of exceptional semiclassical measures:

• $\mu_{sc} = \frac{1}{2}(\nu + \mu_L)$, with ν arbitrary. In particular $\mu_{sc} = \frac{1}{2}(\delta_P + \mu_L)$

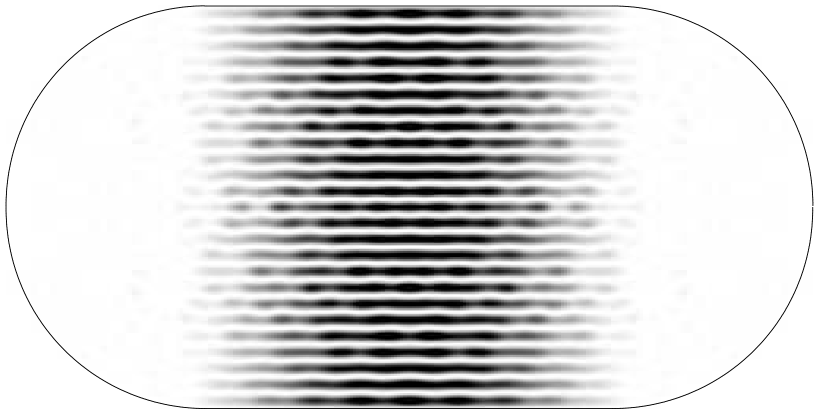
• μ_{sc} a "fractal" invariant measure, which may be supported on a strict (fractal) subset of the torus.

• higher-dimensional cat maps $A \in SL_4(\mathbb{Z})$ on $\mathbb{T}^4 \rightsquigarrow \mu_{sc} =$ Lebesgue measure on a co-isotropic subspace of $\mathbb{T}^4$ (if any) [KELMER'06],. . .

Almost all stadia are not QUE

[Hassel'08] shows that all stadium billiards (defined by the ratio $\frac{length}{height}$) admit at least one semiclassical measure different from Liouville.

It is strongly believed that these measures correspond to bouncing-ball modes, which would then indeed survive in the high-frequency limit (cf. [Bäcker-Schubert-Stifter'98]).



k= 39.292

This is the first counter-example to QUE for an ergodic billiard (or manifold).

The entropy as a measure of localization

Idea: to characterize the localization of Φ^t-invariant measures on S^*X, use the **Kolmogorov-Sinai entropy** $H_{KS}(\mu)$, which quantifies the *information complexity* of μ w.r.to the flow.

- $H_{KS}(\mu) \in [0, H_{\max}]$.

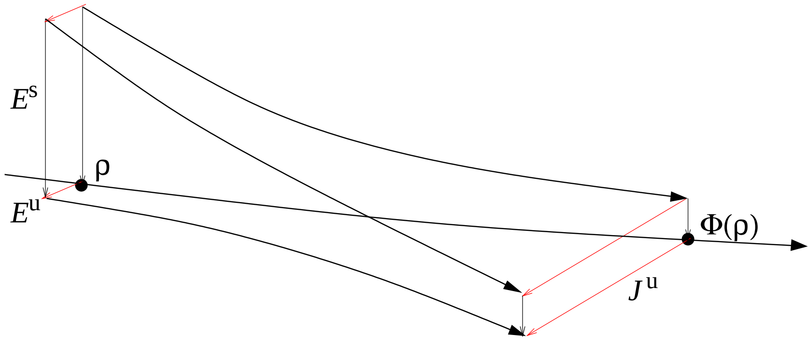
- Related to localization: $H_{KS}(\delta_P) = 0$, $H_{KS}(\mu_L) = \int \sum_{\lambda_i > 0} \lambda_i \, d\mu_L$ (positive Lyapunov exponents)

- Affine function of μ.

  What can be the entropy of a semiclassical measure?

Eigenmodes of Anosov manifolds are at least half-delocalized

We now restrict ourselves to X of negative curvature. The geodesic flow is then of Anosov type (uniformly hyperbolic). $J^u(\rho) = \det(d\Phi_{\restriction E^u(\rho)})$.



Theorem [Anantharaman'05]: for X of negative curvature, any semiclassical measure μ_{sc} satisfies

$$H_{KS}(\mu_{sc}) \geq \epsilon > 0.$$

In particular, "strong scars" $\mu = \delta_P$ are forbidden.

Theorem [Anantharaman-Koch-N'07]:

$$H_{KS}(\mu_{sc}) \geq \int \log J^u(\rho)\, d\mu_{sc}(\rho) - \frac{1}{2}\Lambda_{\max}(d-1).$$

$\Lambda_{\max}$ is the maximal expanding rate, so $\Lambda_{\max}(d-1) \geq \log J^u(\rho)$.Pb: if J^u varies too much, the RHS may become negative.

Eigenmodes of Anosov manifolds are at least half-delocalized

Theorem [Rivière'08]: for X a surface of **nonnegative curvature**,

$$H_{KS}(\mu_{sc}) \geq \frac{1}{2} \int \log J^u(\rho)\, d\mu_{sc}(\rho).$$

Rmk: Some of the exceptional measures mentioned above for chaotic maps *saturate* this lower bound.

[Gutkin'08] constructs such eigenstates for chaotic piecewise-linear maps.

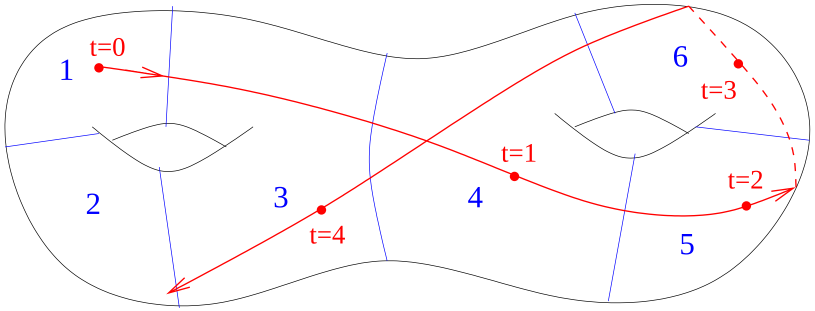
We expect the same bound to hold for d-dimensional mfold of negative (nonpositive?) curvature.

One has $H_{KS}(\mu) \leq \int \log J^u\, d\mu$ for any invariant measure, with equality iff $\mu = \mu_L$.

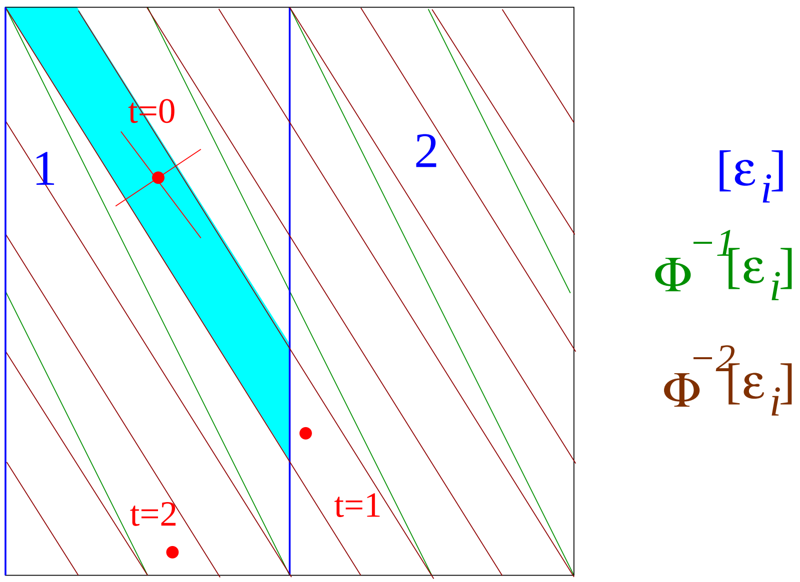
$\rightsquigarrow$ In some sense, μ_{sc} is at least half-delocalized.

Definition of the KS entropy (1)

Take a finite partition $\mathcal{P}$ of the phase space (S^*X or $\mathbb{T}^2$). Each trajectory will be represented by a symbolic sequence $\cdots\epsilon_{-1}\epsilon_0\epsilon_1\cdots\cdots$ according to its *history*.



At each time n, the *rectangle* $[\epsilon_0\cdots\epsilon_n]\subset S^*X$ consists of all points sharing the same "symbolic history" between times 0 and n (ex: [121]).



Definition of the KS entropy (2)

Let μ be an invariant proba. measure. The time-n entropy

$$H_n(\mu, \mathcal{P}) = - \sum_{\epsilon_0, \ldots, \epsilon_n} \mu([\epsilon_0 \cdots \epsilon_n]) \, \log \mu([\epsilon_0 \cdots \epsilon_n])$$

measures the distribution of the probability weights $\mu([\epsilon_0 \cdots \epsilon_n])$.

The limit (uses subadditivity)

$$H_{KS}(\mu, \mathcal{P}) = \lim_{n \to \infty} \frac{H_n(\mu, \mathcal{P})}{n}$$

measures the average rate of exponential decay of these weights.

If the diameter of $\mathcal{P}$ is small enough, $H_{KS}(\mu, \mathcal{P}) = H_{KS}(\mu)$ is called the Kolmogorov-Sinai entropy.

The entropy is positive iff $typical$ weights $\mu([\epsilon_0 \cdots \epsilon_n])$ decay exponentially when $n \to \infty$.

Quantum partition of unity

Need to adapt the notions to the quantum framework. Assume $(\psi_\hbar)_{\hbar\to 0} \rightsquigarrow \mu_{sc}$.

Use *quasi-projectors* $P_j = \mathrm{Op}_\hbar(\chi_j)$ on the components of the partition to construct a **quantum partition of unity**

$$Id = \sum_{j=1}^{J} P_j^2$$

Improved Egorov thm: $U_\hbar^{-t}\,\mathrm{Op}_\hbar(f)\,U_\hbar^t = \mathrm{Op}_\hbar(f\circ\Phi^t) + \mathcal{O}(\hbar\, e^{\Lambda_{\max} t})$

$\Rightarrow$ for n smaller than the Ehrenfest time $T_E = \frac{|\log\hbar|}{\Lambda_{\max}}$, the operator

$$P_{\epsilon_0\cdots\epsilon_n} \stackrel{\text{def}}{=} (U_\hbar^{-n} P_{\epsilon_n} U_\hbar^n)\cdots(U_\hbar^{-1} P_{\epsilon_1} U_\hbar) P_{\epsilon_0}$$

is a quasi-projector on the rectangle $[\epsilon_0\cdots\epsilon_n]$.

For $n \geq 0$ fixed, we have

$$\|P_{\epsilon_0\cdots\epsilon_n}\psi_\hbar\|^2 \stackrel{\hbar\to 0}{\longrightarrow} \mu_{sc}([\epsilon_0\cdots\epsilon_n]).$$

A hyperbolic dispersion estimate

Aim: obtain a lower bound on the **quantum entropy**

$$H_n(\psi_\hbar, \mathcal{P}) = -\sum \|P_{\epsilon_0\cdots\epsilon_n}\psi_\hbar\|^2 \log \|P_{\epsilon_0\cdots\epsilon_n}\psi_\hbar\|^2$$

valid for $n \gg 0$ (fixed).

Can we show that the weights $\|P_{\epsilon_0\cdots\epsilon_n}\psi_\hbar\|^2$ decay expon. with n?

Proposition [ANANTHARAMAN'05] Consider a cutoff $\chi(\rho)$ localized in an energy interval $\{|p(\rho) - 1/2| \leq \varepsilon\}$, and $M > 0$ *arbitrary*. Then, for $\hbar$ small enough and $n \leq M|\log \hbar|$, one has

$$\|P_{\epsilon_0\cdots\epsilon_n} \mathrm{Op}_\hbar(\chi)\| \leq C\,\varepsilon\, h^{-d/2}\, J_u^n(\epsilon_0\cdots\epsilon_n)^{-1/2}$$

In constant curvature -1, and taking the optimal cutoff $\varepsilon \gtrsim \hbar$, this reads

$$\|P_{\epsilon_0\cdots\epsilon_n} \mathrm{Op}_\hbar(\chi)\| \leq C\,\varepsilon\, h^{-(d-1)/2}\, e^{-n(d-1)/2}\,.$$

Pb: this hyperbolic dispersion estimate is **trivial for times** $t \leq T_E$.

To finish: use an entropic uncertainty principle

[ANANTHARAMAN'06] used this estimate for $n \gg T_E$ and a (clever) subadditivity argument to show that $H_{KS}(\mu) > 0$.

[ANAN.-N,ANAN.-KOCH-N.'07] (assume constant curvature -1): split $P_{\epsilon_0\cdots\epsilon_{2n}}$ as

$$U_\hbar^n P_{\epsilon_0\cdots\epsilon_{2n}} = P_{\epsilon_{n+1}\cdots\epsilon_{2n}} U_\hbar^n P_{\epsilon_0\cdots\epsilon_n}$$

Interpret each such operator as a "block matrix element" $\pi_j U_\hbar^n \pi_k$ of the unitary propagator $U_\hbar^n$, expressed in the block-basis $(P_{\epsilon_0\cdots\epsilon_n}) = (\pi_k)$.

For $n = T_E = |\log \hbar|$, the hyperbolic estimate for $P_{\epsilon_0\cdots\epsilon_{2n}}$ with $\varepsilon \gtrsim \hbar$ shows that all these block matrix elements satisfy $\|\pi_j U_\hbar^{T_E} \pi_k \,\mathrm{Op}_\hbar(\chi)\| \leq C\,\hbar^{\frac{d-1}{2}}$.

An **entropic uncertainty principle** [MAASSEN-UFFINK'88] shows that the quantum entropy built from $\psi_\hbar \propto U_\hbar^{T_E}\,\psi_\hbar$ satisfies

$$H_{T_E}(\psi_\hbar) \geq |\log \hbar^{\frac{d-1}{2}}| = \frac{T_E\,(d-1)}{2}\,.$$

Finally, one uses subadditivity and improved Egorov to get a similar bound at fixed time $n = n_0$, and then the bound $H_{KS}(\mu_{sc}) \geq \frac{d-1}{2}$. □

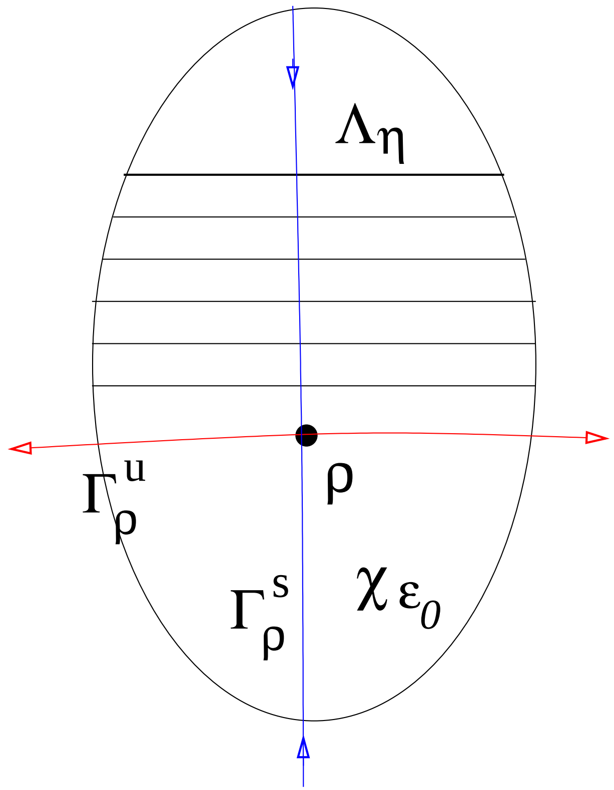
Proof of the hyperbolic dispersion estimate (1)

A state $P_{\epsilon_0}\psi$ is first decomposed into an appropriate family of *elementary* states (ψ_η):

$$P_{\epsilon_0}\psi = \hbar^{-d/2} \int d\eta \, \psi_\eta \, f(\eta) \, .$$

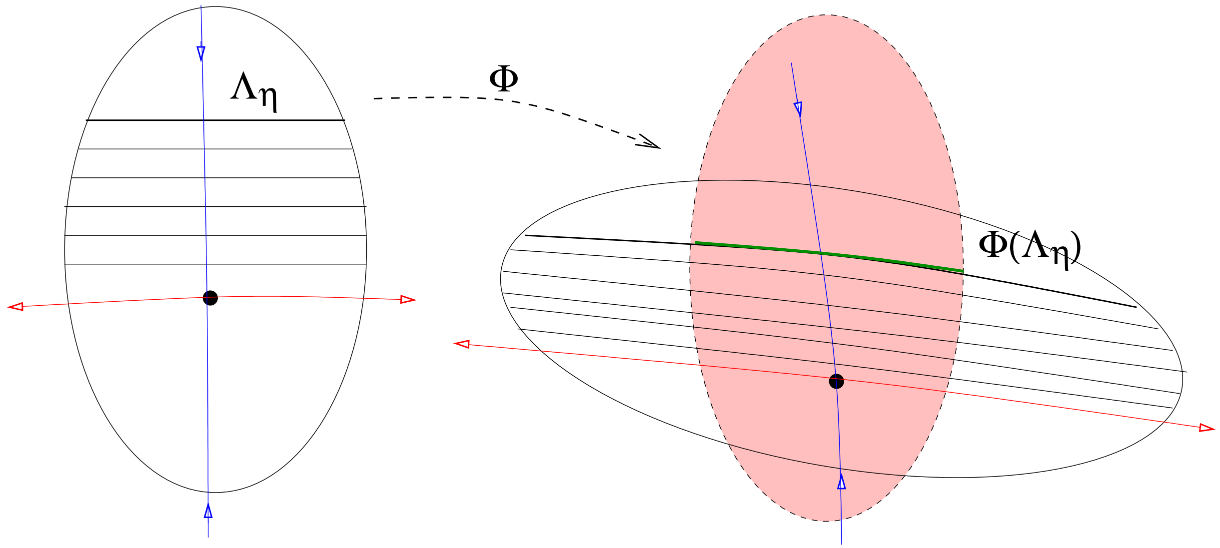
Here (ψ_η) is a family of Lagrangian states associated with Lagrangian manifolds (Λ_η) *close to* the unstable foliation:

$$\psi_\eta(x) = a(x)\, e^{iS_\eta(x)/\hbar} \quad \text{is localized on } \Lambda_\eta = \{(x, \xi = \nabla S_\eta(x))\}$$



Proof of the dispersion estimate (2)

We will compute each evolution $P_{\epsilon_n} \cdots U_\hbar P_{\epsilon_1} U_\hbar \psi_\eta$ separately.



Through the sequence of stretching (U) and cutting (P_{ϵ_i}), the transformed state remains Lagrangian, supported on the transported Lagrangian mfold (which gets exponentially close to the unstable mfold).

The amplitude of $P_{\epsilon_n} \cdots U_\hbar P_{\epsilon_1} U_\hbar \psi_\eta$ is transformed as a half-density. Its decay is governed by the unstable Jacobian along the path $\epsilon_0 \cdots \epsilon_n$:

$$\|P_{\epsilon_n} \cdots U_\hbar P_{\epsilon_1} U_\hbar \psi_\eta\| \sim J_u^n(\epsilon_0 \cdots \epsilon_n)^{-1/2}$$

Summing up the decomposition to recover $P_{\epsilon_0}\psi$, one gets the hyperbolic estimate.