ON THE ROBIN PROBLEM IN FRACTAL DOMAINS

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Collaborators

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Robin problem

\[ \Delta u(x) = 0, \quad x \in D \setminus B, \]
\[ \frac{\partial u}{\partial n}(x) = cu(x), \quad x \in \partial D, \]
\[ u(x) = 1, \quad x \in \partial B. \]
Applications of the Robin problem


A natural system: oxygen transport in human lungs.
Applications of the Robin problem


- A man made system: ion transport in battery electrolyte.
Is the whole surface active?

For what domains is it true that $\inf_{x \in D \setminus B} u(x) = 0$?
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\[ y = x^\alpha \]
Is the whole surface active?

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\[
y = x^\alpha
\]

**THEOREM (Bass, B, Chen)**

\[
\inf_{x \in D \setminus B} u(x) = 0 \text{ if and only if } \alpha \geq 2.
\]
Branching fractals
Example: von Koch snowflake.

The normal vector does not exist at almost all boundary points.
Approximate the snowflake domain $D$ with an increasing sequence of smooth domains $D_k$, such that $\bigcup_k D_k = D$. 

Let $u_k$ be the solution to the Robin boundary problem in $D_k$, with the same $c$ (adsorption rate) for all $k$, and let $u(x) = \lim_{k \to \infty} u_k(x)$. Then $u$ satisfies the Dirichlet boundary conditions $u(x) = 0$ on $\partial D$. 

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$$u(x) = \lim_{k \to \infty} u_k(x).$$

Then $u$ satisfies the Dirichlet boundary conditions $u(x) = 0$ on $\partial D$. 
Assuming that $D$ is smooth, the Green-Gauss formula implies that for $u, v \in C^2(D)$,

$$\int_D \nabla u(x) \cdot \nabla v(x) \, dx = - \int_D v(x) \Delta u(x) \, dx - \int_{\partial D} v(x) \frac{\partial u}{\partial n}(x) \sigma(dx),$$

where $\sigma$ is the surface measure on $\partial D$. 

A weak solution $u$ to the Robin problem is characterized by

$$\int_D \nabla u(x) \cdot \nabla v(x) \, dx = - \int_{\partial D} cu(x)v(x) \sigma(dx),$$

for every $v \in C^2(D)$ that vanishes on $B$. 

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Let $\mu$ be $d$-dimensional Hausdorff measure.
Solution to Robin problem in von Koch snowflake

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Let $\mu$ be $d$-dimensional Hausdorff measure.

**DEFINITION**

We will say that a function $u$ is a weak solution to the Robin problem in the snowflake domain if for all smooth $v$,

$$\int_D \nabla u(x) \cdot \nabla v(x) dx = - \int_{\partial D} cu(x)v(x) \mu(dx).$$
Alternative representation

\[ D \] – von Koch snowflake domain
\[ X \] – reflected Brownian motion in \( D \)
\[ \sigma_B \] – hitting time of \( B \)

\text{CONJECTURE (B, Chen)}

The continuous additive functional \( L \) with Revuz measure \( \mu \) exists.

The function \( u(x) = \mathbb{E}_x \left[ \exp \left( -c \int_0^{\sigma_B} ds \right) \right] \), \( x \in D \setminus B \), is the unique weak solution to the Robin problem.
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\(D\) – von Koch snowflake domain
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\(L\) – “local time” on \(\partial D\), i.e., a continuous additive functional of \(X\) with Revuz measure \(\mu\)

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**CONJECTURE (B, Chen)**

- The continuous additive functional \( L \) with Revuz measure \( \mu \) exists.
- The function

\[
u(x) = \mathbb{E}_x \left[ \exp \left( -\frac{c}{2} \int_0^{\sigma_B} dL_s \right) \right], \quad x \in \overline{D} \setminus B,
\]

is the unique weak solution to the Robin problem.
Generalization: $c(x)$

$$\int_D \nabla u(x) \cdot \nabla v(x) \, dx = - \int_{\partial D} c(x)u(x)v(x)\mu(dx).$$

$$u(x) = \mathbb{E}_x \left[ \exp \left( -\frac{1}{2} \int_0^{\sigma_B} c(X_s)\,dL_s \right) \right], \quad x \in \overline{D} \setminus B.$$
Smooth killing

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Robin problem
Increasing families of domains

$D \subset \mathbb{R}^d$ – open bounded connected set

$D_k \subset D_{k+1}, \bigcup_k D_k = D, \ D_k$ have smooth boundaries

THEOREM (B, Chen)

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Reflected Brownian motions \( X^k \) converge, as \( k \to \infty \), to reflected Brownian motion in \( D \).
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$$D_k = \{ x \in D : \text{dist}(x, \partial D) < 1/k \}.$$
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Suppose that $a_k$ are chosen so that $a_k 1_{D_k} m$ converge weakly to $\mu$ as $k \to \infty$. 

\*\*CONJECTURE (B, Chen)\*\*

Functions $u_k$ converge to $u$ (the Robin problem solution in $D$), as $k \to \infty$. 

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Let

$$u_k(x) = \mathbb{E}_x \left[ \exp \left( -\frac{a_k c}{2} \int_0^{\sigma_B} 1_{D_k}(X_s) ds \right) \right], \quad x \in \overline{D} \setminus B.$$
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Invariance principle for reflected random walks

\( D \) – von Koch snowflake
\( X^k \) – reflected random walk on \( D \cap (2^{-k}\mathbb{Z}^2) \)

THEOREM (B, Chen)
Reflected random walks \( X^k \), with sped-up clocks, converge weakly to reflected Brownian motion in \( D \), as \( k \to \infty \).

CONJECTURE (B, Chen)
Feynman-Kac transforms \( u^k(x) = \mathbb{E}_x \left[ \exp \left( -b \sum_{0 \leq n \leq \sigma^B} \partial_r \left( D \cap (2^{-k}\mathbb{Z}^2) \right)(X^k_n) \right) \right] \), converge to the solution of the Robin problem in \( D \), as \( k \to \infty \).
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Feynman-Kac transforms

$$u_k(x) = \mathbb{E}_x \left[ \exp \left( -b_k c \sum_{0 \leq n \leq \sigma_B} \mathbbm{1}_{\partial(D\cap(2^{-k}\mathbb{Z}^2))}(X^k_n) \right) \right],$$

converge to the solution of the Robin problem in $D$, as $k \to \infty$. 
Invariance principle – open problem

\( D \) – bounded domain above the graph of a Hölder function

OPEN PROBLEM
Is it true that reflected random walks \( X_k \), with sped-up clocks, converge weakly to reflected Brownian motion in \( D \), when \( k \to \infty \)?

Remark added after the talk: The answer is yes. (B, Chen).
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**THEOREM (B, Chen)**

There exists a bounded domain $D \subset \mathbb{R}^2$ such that reflected random walks $X^k$, with sped-up clocks, do not converge weakly to reflected Brownian motion in $D$, when $k \to \infty$. 
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Example: Remove suitable dust from a square.
Consider $N$ non-overlapping discs with radius $r > 0$ in the square $[0,1]$. Their configuration can be represented as a point in $D \subset [0,1]^N$.

Metropolis algorithm

(i) Pick a disc at random (uniformly).
(ii) Pick a vector $v$ at random (uniformly) from $B(0,\epsilon)$ and move the disc in direction $v$, provided the dislocated disc does not intersect any other disc.
(iii) The stationary distribution of the discs is uniform in $D$.

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**Packing of hard discs and Metropolis algorithm**

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Consider $N$ non-overlapping discs with radius $r > 0$ in the square $[0, 1]$. Their configuration can be represented as a point in $D \subset [0, 1]^N$.

**THEOREM (Diaconis, Lebeau and Michel)**

i) $D$ is Lipschitz if $Nr < \alpha$.

ii) $D$ is not Lipschitz if $N \cdot 2^r = 1$. 
Consider $N$ non-overlapping discs with radius $r > 0$ in the square $[0, 1]$. Their configuration can be represented as a point in $D \subset [0, 1]^N$. 

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Myopic conditioning

\[ D \subset \mathbb{R}^d \] – open bounded connected set
\[ \varepsilon > 0 \]
\[ X^\varepsilon_t \] – a continuous process in \( D \)
Myopic conditioning

$D \subset \mathbb{R}^d$ – open bounded connected set

$\varepsilon > 0$

$X_\varepsilon^t$ – a continuous process in $D$

**DEFINITION**

Given $\{X_\varepsilon^t, 0 \leq t \leq k\varepsilon\}$, the process $\{X_\varepsilon^t, k\varepsilon \leq t \leq (k + 1)\varepsilon\}$ is Brownian motion conditioned not to hit $D^c$ (during the time interval $[k\varepsilon, (k + 1)\varepsilon]$).
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**THEOREM (B, Chen)**

Processes $X_\varepsilon^t$ converge weakly, as $\varepsilon \to 0$, to reflected Brownian motion in $D$. 
Myopic conditioning – a technical observation

\[ D \subset \mathbb{R}^d \] – open bounded connected set, \( \varepsilon > 0 \)
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Given $\{X_t^\varepsilon, 0 \leq t \leq k\varepsilon\}$, the process $\{X_t^\varepsilon, k\varepsilon \leq t \leq (k + 1)\varepsilon\}$ is Brownian motion conditioned not to hit $D^c$ during the time interval $[k\varepsilon, (k + 1)\varepsilon]$. 
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$B$ – Brownian motion in $\mathbb{R}^d$, $\tau_D = \inf\{t \geq 0 : B_t \notin D\}$

$Y^\varepsilon_k = X^\varepsilon_{k\varepsilon}, \quad k \geq 1$

$m^\varepsilon(dx) = P^x(\tau_D > \varepsilon)dx$
Myopic conditioning – a technical observation

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OBSERVATIONS (B, Chen)

(i) $m^\varepsilon \to$ Lebesgue measure on $D$ as $\varepsilon \to 0$.  

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OBSERVATIONS (B, Chen)

(i) $m^\varepsilon \to$ Lebesgue measure on $D$ as $\varepsilon \to 0$.
(ii) $m^\varepsilon(dx)$ is a reversible measure for $Y^\varepsilon_k$. 
Increasing families of domains – a technical observation

$D \subset \mathbb{R}^d$ – open bounded connected set

$D_k \subset D_{k+1}$, $\bigcup_k D_k = D$, $D_k$ have smooth boundaries

$X^k$ – reflected Brownian motion in $D_k$

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Reflected Brownian motions $X^k$ converge, as $k \to \infty$, to reflected Brownian motion in $D$. 

$W_k$ – $d$-dimensional Brownian motion

$(1)$ is a special case of Lyons–Zheng's forward-backward martingale decomposition.
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$r_T$ – time reversal operator for $X^k$ at time $T$

$X^k_t(r_T(\omega)) = X^k_{T-t}(\omega)$
Increasing families of domains – a technical observation

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$$X^k_t - X^k_0 = \frac{1}{2} W^k_t - \frac{1}{2} \left( W^k_T \circ r_T - W^k_{T-t} \circ r_T \right) \tag{1}$$

$W^k$ – $d$-dimensional Brownian motion
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\( W^k \) – \( d \)-dimensional Brownian motion

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