ON THE ROBIN PROBLEM IN FRACTAL DOMAINS

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Collaborators

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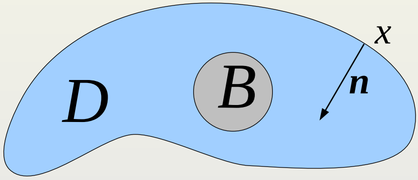
Robin problem

$$\Delta u(x) = 0, \qquad x \in D \setminus B,$$
$$\frac{\partial u}{\partial \mathbf{n}}(x) = cu(x), \qquad x \in \partial D,$$
$$u(x) = 1, \qquad x \in \partial B.$$



Applications of the Robin problem

- M. Felici, B. Sapoval, and M. Filoche (2003) Renormalized random walk study of oxygen absorption in the human lung, *Phys. Rev. Lett.* **92**, 068101-1—068101-4.

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- A natural system: oxygen transport in human lungs.
- A man made system: ion transport in battery electrolyte.

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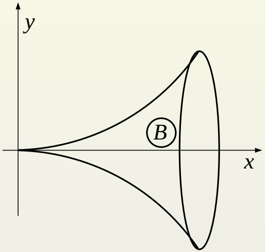

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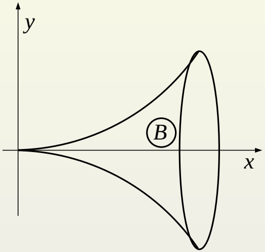

$$y = x^\alpha$$

THEOREM (Bass, B, Chen)

$\inf_{x \in D \setminus B} u(x) = 0$ if and only if $\alpha \geq 2$.

Branching fractals

Robin problem in truly fractal domains

Example: von Koch snowflake.

The normal vector does not exist at almost all boundary points.

Naive approach

Approximate the snowflake domain D with an increasing sequence of smooth domains D_k, such that $\bigcup_k D_k = D$.

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$$u(x) = \lim_{k \to \infty} u_k(x).$$

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Then u satisfies the Dirichlet boundary conditions $u(x) = 0$ on ∂D.

Reformulation of Robin problem

Assuming that D is smooth, the Green-Gauss formula implies that for $u, v \in C^2(\overline{D})$,

$$\int_D \nabla u(x) \cdot \nabla v(x) dx = - \int_D v(x) \Delta u(x) dx - \int_{\partial D} v(x) \frac{\partial u}{\partial \mathbf{n}}(x) \sigma(dx),$$

where σ is the surface measure on ∂D.

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where σ is the surface measure on ∂D.

A weak solution u to the Robin problem is characterized by

$$\int_D \nabla u(x) \cdot \nabla v(x) dx = -\int_{\partial D} cu(x) v(x) \sigma(dx),$$

for every $v \in C^2(\overline{D})$ that vanishes on B.

Solution to Robin problem in von Koch snowflake

$d = \log 4 / \log 3$
Let μ be d-dimensional Hausdorff measure.

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DEFINITION

We will say that a function u is a weak solution to the Robin problem in the snowflake domain if for all smooth v,

$$\int_D \nabla u(x) \cdot \nabla v(x) dx = - \int_{\partial D} c u(x) v(x) \mu(dx).$$

Alternative representation

D – von Koch snowflake domain
X – reflected Brownian motion in D
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CONJECTURE (B, Chen)

- The continuous additive functional L with Revuz measure μ exists.
- The function

$$u(x) = \mathbb{E}_x \left[\exp\left(-\frac{c}{2} \int_0^{\sigma_B} dL_s \right) \right], \qquad x \in \overline{D} \setminus B,$$

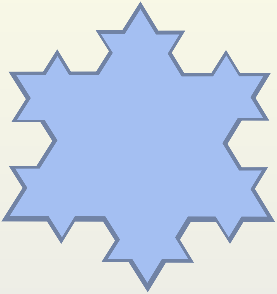
is the unique weak solution to the Robin problem.

Generalization: $c(x)$

$$\int_D \nabla u(x) \cdot \nabla v(x) dx = - \int_{\partial D} c(x) u(x) v(x) \mu(dx).$$

$$u(x) = \mathbb{E}_x \left[\exp \left(-\frac{1}{2} \int_0^{\sigma_B} c(X_s) dL_s \right) \right], \qquad x \in \overline{D} \setminus B.$$

Smooth killing



Increasing families of domains

$D \subset \mathbb{R}^d$ – open bounded connected set
$D_k \subset D_{k+1}$, $\bigcup_k D_k = D$, D_k have smooth boundaries

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THEOREM (B, Chen)

Reflected Brownian motions X^k converge, as $k \to \infty$, to reflected Brownian motion in D.

Killing in boundary layer

Let m denote the Lebesgue measure on $\mathbb{R}^2$ and

$$D_k = \{x \in D : \quad \text{dist}(x, \partial D) < 1/k\}.$$

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Let

$$u_k(x) = \mathbb{E}_x \left[\exp \left(-\frac{a_k c}{2} \int_0^{\sigma_B} \mathbf{1}_{D_k}(X_s) ds \right) \right], \qquad x \in \overline{D} \setminus B.$$

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CONJECTURE (B, Chen)

Functions u_k converge to u (the Robin problem solution in D), as $k \to \infty$.

Invariance principle for reflected random walks

D – von Koch snowflake
X^k – reflected random walk on $D \cap (2^{-k}\mathbb{Z}^2)$

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CONJECTURE (B, Chen)

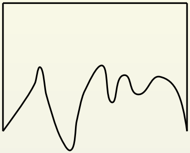
Feynman-Kac transforms

$$u_k(x) = \mathbb{E}_x \left[\exp \left(-b_k c \sum_{0 \le n \le \sigma_B} \mathbf{1}_{\partial(D \cap (2^{-k}\mathbb{Z}^2))}(X_n^k) \right) \right],$$

converge to the solution of the Robin problem in D, as $k \to \infty$.

Invariance principle – open problem

D – bounded domain above the graph of a Hölder function



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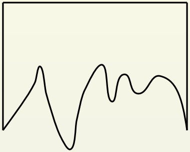

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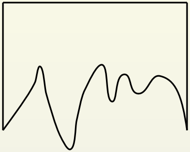

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OPEN PROBLEM

Is it true that reflected random walks X^k, with sped-up clocks, converge weakly to reflected Brownian motion in D, when $k \to \infty$?

Remark added after the talk: The answer is yes. (B, Chen).

Invariance principle – counterexample

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THEOREM (B, Chen)

There exists a bounded domain $D \subset \mathbb{R}^2$ such that reflected random walks X^k, with sped-up clocks, do not converge weakly to reflected Brownian motion in D, when $k \to \infty$.

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Example: Remove suitable dust from a square.

Packing of hard discs and Metropolis algorithm

N. Metropolis, A. Rosenbluth, M. Rosenbluth, A. Teller, and E. Teller. Equations of state calculations by fast computing machines. *J. Chem. Phys.*, 21:10871092, 1953.

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Consider N non-overlapping discs with radius $r > 0$ in the square $[0, 1]$. Their configuration can be represented as a point in $D \subset [0, 1]^N$.

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Metropolis algorithm

(i) Pick a disc at random (uniformly).

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(i) Pick a disc at random (uniformly).
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(iii) The stationary distribution of the discs is uniform in D.

Packing of hard discs – geometry of state space

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THEOREM (Diaconis, Lebeau and Michel)

(i) D is Lipschitz if $Nr < \alpha$.

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THEOREM (Diaconis, Lebeau and Michel)

(i) D is Lipschitz if $Nr < \alpha$.
(ii) D is not Lipschitz if $N \cdot 2r = 1$.

Myopic conditioning

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$\varepsilon > 0$
X_t^ε – a continuous process in D

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Given $\{X_t^\varepsilon, 0 \le t \le k\varepsilon\}$, the process $\{X_t^\varepsilon, k\varepsilon \le t \le (k+1)\varepsilon\}$ is Brownian motion conditioned not to hit D^c (during the time interval $[k\varepsilon, (k+1)\varepsilon]$).

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THEOREM (B, Chen)

Processes X^ε converge weakly, as $\varepsilon \to 0$, to reflected Brownian motion in D.

Myopic conditioning – a technical observation

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B – Brownian motion in $\mathbb{R}^d$, $\tau_D = \inf\{t \ge 0 : B_t \notin D\}$
$Y_k^\varepsilon = X_{k\varepsilon}^\varepsilon, \quad k \ge 1$
$m_\varepsilon(dx) = P^x(\tau_D > \varepsilon)dx$

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OBSERVATIONS (B, Chen)

(i) $m_\varepsilon \to$ Lebesgue measure on D as $\varepsilon \to 0$.

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OBSERVATIONS (B, Chen)

(i) $m_\varepsilon \to$ Lebesgue measure on D as $\varepsilon \to 0$.
(ii) $m_\varepsilon(dx)$ is a reversible measure for Y^ε_k.

Increasing families of domains – a technical observation

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$$X^k_t - X^k_0 = \frac{1}{2} W^k_t - \frac{1}{2}\Big(W^k_T \circ r_T - W^k_{T-t} \circ r_T \Big) \tag{1}$$

W^k – d-dimensional Brownian motion

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(1) is a special case of Lyons–Zheng's forward-backward martingale decomposition