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Fragmentation equations and stochastic models

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Plan of the talk

1. Introduction
   - fragmentation equation
   - stochastic model 1 (direct simulation process)
   - probabilistic representation of the fragmentation kernel
   - stochastic model 2 (mass flow process)

2. Shattering phenomenon
   - non-conservative solutions
   - explosion in stochastic models
   - examples
   - references

Fragmentation equation

$$\frac{\partial}{\partial t} c(t,x) = \int_x^\infty b(x|y)\, a(y)\, c(t,y)\, dy - a(x)\, c(t,x) \qquad t > 0 \qquad x > 0$$

$c(t,x)$ $\sim$ concentration of particles of size x

$a(x)$ $\sim$ rate at which particles of size x break

$b(x|y)$ $\sim$ distribution of products from a particle of size y breaking

$$\int_0^y x\, b(x|y)\, dx = y \quad \sim \quad \text{mass conservation}$$

DS particle system $\big(X_1(t),\dots,X_{N(t)}(t)\big)$ $t\ge 0$ $N(0)=n$

Time evolution

1. exponentially distributed waiting time with parameter

$$a(x_i) \qquad i=1,\dots,N$$

2. replace x_i by fragments $z_1,\dots,z_k$ with distribution

$$F(x_i,dz)$$

- ▷ fragmentation rate $a(x)$ compactly bounded on $\mathcal{X}=(0,\infty)$
- ▷ fragmentation kernel $F(x,dz)$ from $\mathcal{X}$ to $\mathcal{Z}=\bigcup_{k=2}^{\infty}\mathcal{X}^k$
- ▷ mass conservation property

$$F\left(x,\{z\in\mathcal{Z}:\ z_1+\ldots+z_k=x\}\right)=F(x,\mathcal{Z})=1$$

state space $$\mathcal{Y}^{(n)} = \left\{ \frac{1}{n} \sum_{i=1}^{N} \delta_{x_i} \; : \quad N \geq 1\,, \quad x_i \in \mathcal{X}\,, \quad i = 1, \ldots, N \right\}$$

kernel $$q^{(n)}(\xi, \Gamma) = \sum_{i=1}^{N} \int_{\mathcal{Z}} 1_\Gamma(J_F(\xi, i, z))\, a(x_i)\, F(x_i, dz)$$

jump transformation $$J_F(\xi, i, z) \;=\; \xi - \frac{1}{n}\delta_{x_i} + \frac{1}{n}[\delta_{z_1} + \ldots + \delta_{z_k}]$$

DS process $$\xi^{(n)}(t, dx) = \tfrac{1}{n}\textstyle\sum_{i=1}^{N(t)} \delta_{X_i(t)}(dx) \quad \to \quad \mu(t, dx) \qquad (n \to \infty)$$

$$\frac{d}{dt}\int_{\mathcal{X}} \varphi(x)\,\mu(t, dx) = \int_{\mathcal{X}}\int_{\mathcal{Z}} [\varphi(z_1) + \ldots + \varphi(z_k) - \varphi(x)]\, a(x)\, F(x, dz)\, \mu(t, dx)$$

Define

$$F^{(k)}_{\mathrm{sym}}(x, dz_1, \ldots, dz_k) = \frac{1}{k\,!} \sum_{\pi} F^{(k)}(x, dz_{\pi(1)}, \ldots, dz_{\pi(k)}) \qquad k = 2, 3, \ldots$$

$$F^{(k|1)}_{\mathrm{sym}}(x, dy) = F^{(k)}_{\mathrm{sym}}(x, dy, \mathcal{X}, \ldots, \mathcal{X}) \qquad B(dy|x) = \sum_{k=2}^{\infty} k\, F^{(k|1)}_{\mathrm{sym}}(x, dy)$$

Then

$$\int_{\mathcal{Z}} [\varphi(z_1) + \ldots + \varphi(z_k)]\, F(x, dz) =$$

$$\sum_{k=2}^{\infty} \int_{\mathcal{X}^k} [\varphi(z_1) + \ldots + \varphi(z_k)]\, F^{(k)}_{\mathrm{sym}}(x, dz_1, \ldots, dz_k)$$

$$= \sum_{k=2}^{\infty} k \int_{\mathcal{X}} \varphi(y)\, F^{(k|1)}_{\mathrm{sym}}(x, dy) = \int_{\mathcal{X}} \varphi(y)\, B(dy|x) \qquad (\star)$$

Fragmentation equation: transformation (continued)

According to

$$\int_{\mathcal{Z}} [\varphi(z_1) + \ldots + \varphi(z_k)]\, F(x, dz) = \int_{\mathcal{X}} \varphi(y)\, B(dy|x) \qquad (\star)$$

equation

$$\frac{d}{dt} \int_{\mathcal{X}} \varphi(x)\, \mu(t, dx) = \int_{\mathcal{X}} \int_{\mathcal{Z}} [\varphi(z_1) + \ldots + \varphi(z_k) - \varphi(x)]\, a(x)\, F(x, dz)\, \mu(t, dx)$$

transforms into

$$\frac{d}{dt} \int_{\mathcal{X}} \varphi(x)\, \mu(t, dx) = \int_{\mathcal{X}} \left[\int_{\mathcal{X}} \varphi(y)\, B(dy|x)\, a(x) - \varphi(x)\, a(x) \right] \mu(t, dx)$$

and

$$\boxed{\frac{\partial}{\partial t} \mu(t, dx) = \int_{\mathcal{X}} B(dx|y)\, a(y)\, \mu(t, dy) - a(x)\, \mu(t, dx)}$$

Fragmentation equation: transformation (continued-2)

If

$$\mu(t, dx) = c(t, x)\, dx$$

and

$$B(dy|x) = b(y|x)\, dy$$

then

$$\frac{\partial}{\partial t}\mu(t, dx) = \int_{\mathcal{X}} B(dx|y)\, a(y)\, \mu(t, dy) - a(x)\, \mu(t, dx)$$

implies

$$\boxed{\frac{\partial}{\partial t} c(t, x) = \int_x^\infty b(x|y)\, a(y)\, c(t, y)\, dy - a(x)\, c(t, x)}$$

Property

$$\mathbb{E} \sum_{i=1}^{k} \varphi(z_i) = \int_{\mathcal{Z}} [\varphi(z_1) + \ldots + \varphi(z_k)] \, F(x, dz) = \int_{\mathcal{X}} \varphi(y) \, B(dy|x) \qquad (\star)$$

implies

▷ average number of fragments with size in (u, v)

$$\mathbb{E} \sum_{i=1}^{k} 1_{(u,v)}(z_i) = \int_{\mathcal{X}} 1_{(u,v)}(y) \, B(dy|x) = \int_{u}^{v} b(y|x) \, dy$$

Property

$$\mathbb{E}\sum_{i=1}^{k}\varphi(z_i) = \int_{\mathcal{Z}}[\varphi(z_1)+\ldots+\varphi(z_k)]\,F(x,dz) = \int_{\mathcal{X}}\varphi(y)\,B(dy|x) \qquad (\star)$$

implies

▷ average number of fragments with size in (u,v)

$$\mathbb{E}\sum_{i=1}^{k}1_{(u,v)}(z_i) = \int_{\mathcal{X}}1_{(u,v)}(y)\,B(dy|x) = \int_u^v b(y|x)\,dy$$

▷ mass conservation on average

$$\sum_{i=1}^{k}z_i = x \qquad \Rightarrow \qquad \mathbb{E}\sum_{i=1}^{k}z_i = \int_{\mathcal{X}}y\,B(dy|x) = \int_0^x y\,b(y|x)\,dy = x$$

Special case: homogeneous binary fragmentation

Consider a probability density p on $[0,1]$ and

$$F(x,dz) = F^{(2)}(x,dz_1,dz_2) = \int_0^1 \delta_{(xu,x(1-u))}(dz_1,dz_2)\,p(u)\,du$$

Special case: homogeneous binary fragmentation

Consider a probability density p on $[0,1]$ and

$$F(x,dz) = F^{(2)}(x,dz_1,dz_2) = \int_0^1 \delta_{(xu,x(1-u))}(dz_1,dz_2)\,p(u)\,du$$

Property

$$\int_{\mathcal{Z}} [\varphi(z_1)+\ldots+\varphi(z_k)]\,F(x,dz) = \int_{\mathcal{X}} \varphi(y)\,B(dy|x) \qquad (\star)$$

implies

$$\begin{aligned}\int_{\mathcal{X}} \varphi(y)\,B(dy|x) &= \int_0^1 [\varphi(x\,u)+\varphi(x\,(1-u))]\,p(u)\,du \\ &= \int_0^x \varphi(y)\,p\left(\frac{y}{x}\right)\frac{1}{x}\,dy + \int_0^x \varphi(y)\,p\left(1-\frac{y}{x}\right)\frac{1}{x}\,dy\end{aligned}$$

so that

$$b(y|x) = \frac{1}{x}\left[p\left(\frac{y}{x}\right)+p\left(1-\frac{y}{x}\right)\right]$$

Special case: homogeneous binary fragmentation (continued)

Note

$$F(x, dz) = F^{(2)}(x, dz_1, dz_2) = \int_0^1 \delta_{(xu, x(1-u))}(dz_1, dz_2)\, p(u)\, du$$

and

$$b(y|x) = \frac{1}{x}\left[p\left(\frac{y}{x}\right) + p\left(1 - \frac{y}{x}\right)\right]$$

Example 1: $p(u) = 1 \quad \Rightarrow \quad b(y|x) = \frac{2}{x}$

Example 2: $p(u) = 2u \quad \Rightarrow \quad b(y|x) = \frac{2}{x}$

$\Rightarrow$ Examples 1, 2 correspond to the equation

$$\frac{\partial}{\partial t} c(t, x) = \int_x^\infty \frac{2}{y}\, a(y)\, c(t, y)\, dy - a(x)\, c(t, x)$$

If

$$\frac{\partial}{\partial t} c(t,x) = \int_x^\infty b(x|y)\, a(y)\, c(t,y)\, dy - a(x)\, c(t,x)$$

then

$$\tilde{c}(t,x) = x\, c(t,x)$$

satisfies

$$\boxed{\frac{\partial}{\partial t} \tilde{c}(t,x) = \int_x^\infty \tilde{b}(x|y)\, a(y)\, \tilde{c}(t,y)\, dy - a(x)\, \tilde{c}(t,x)}$$

where

$$\tilde{b}(x|y) = \frac{x}{y}\, b(x|y)$$

$$\frac{\partial}{\partial t}\,\tilde{c}(t,x) = \int_x^\infty \tilde{b}(x|y)\,a(y)\,\tilde{c}(t,y)\,dy - a(x)\,\tilde{c}(t,x) \qquad \tilde{b}(x|y) = \frac{x}{y}\,b(x|y)$$

Note

$$\int_0^y x\,b(x|y)\,dx = y \qquad \Rightarrow \qquad \int_0^y \tilde{b}(x|y)\,dx = 1$$

and

$$\int_0^\infty \int_x^\infty \tilde{b}(x|y)\,a(y)\,\tilde{c}(t,y)\,dy\,dx = \int_0^\infty \int_0^y \tilde{b}(x|y)\,a(y)\,\tilde{c}(t,y)\,dx\,dy = \int_0^\infty a(y)\,\tilde{c}(t,y)\,dy$$

so that

$$\frac{d}{dt}\int_0^\infty \tilde{c}(t,x)\,dx = 0 \qquad \text{and} \qquad \int_0^\infty \tilde{c}(t,x)\,dx = \text{const}$$

$\Rightarrow$ random process with jump rate $a(x)$ and jump distribution $\tilde{b}(y|x)\,dy$

$$\frac{\partial}{\partial t}\tilde{c}(t,x) = \int_x^\infty \tilde{b}(x|y)\, a(y)\, \tilde{c}(t,y)\, dy - a(x)\, \tilde{c}(t,x) \qquad \tilde{b}(x|y) = \frac{x}{y}\, b(x|y)$$

MF particle system $\big(\tilde{X}_1(t), \ldots, \tilde{X}_n(t)\big) \qquad t \geq 0$

Time evolution

1. exponentially distributed waiting time with parameter

$$a(x_i) \qquad i = 1, \ldots, n$$

2. replace x_i by “fragment” y with distribution

$$\tilde{B}(dy|x_i) \;=\; \tilde{b}(y|x_i)\, dy$$

MF process $\tilde{\xi}^{(n)}(t,dx) = \frac{1}{n}\sum_{i=1}^n \delta_{\tilde{X}_i(t)}(dx) \quad \longrightarrow \quad \tilde{c}(t,x)\, dx \qquad (n \to \infty)$

According to

$$\int_{\mathcal{Z}} [\varphi(z_1) + \ldots + \varphi(z_k)]\, F(x, dz) = \int_{\mathcal{X}} \varphi(y)\, B(dy|x) \qquad (\star)$$

one obtains

$$B(dy|x) = \int_{\mathcal{Z}} [\delta_{z_1}(dy) + \ldots + \delta_{z_k}(dy)]\, F(x, dz)$$

and the MF “fragmentation” kernel

$$\tilde{B}(dy|x) = \frac{y}{x}\, B(dy|x) = \int_{\mathcal{Z}} \left[\frac{z_1}{x}\, \delta_{z_1}(dy) + \ldots + \frac{z_k}{x}\, \delta_{z_k}(dy)\right] F(x, dz)$$

Example: uniform binary fragmentation

Consider

$$F(x,dz) = F^{(2)}(x,dz_1,dz_2) = \int_0^1 \delta_{(xu,x(1-u))}(dz_1,dz_2)\,p(u)\,du \qquad \text{with} \qquad p(u) \equiv 1$$

and

$$b(y|x) = \frac{2}{x} \qquad\qquad \tilde{b}(y|x) = \frac{y}{x}\,b(y|x) = \frac{2\,y}{x^2}$$

DS process

$$x \quad \longrightarrow \quad u\,x \quad \text{and} \quad (1-u)\,x \qquad \text{with} \quad u \quad \text{uniform on} \quad [0,1]$$

MF process

$$x \quad \longrightarrow \quad y \qquad \text{with} \quad y \sim \frac{2\,y}{x^2} \quad \text{on} \quad [0,x]$$

$$\frac{\partial}{\partial t} c(t,x) = \int_x^\infty b(x|y)\, a(y)\, c(t,y)\, dy - a(x)\, c(t,x) \qquad \int_0^y x\, b(x|y)\, dx = y$$

Note

$$\int_0^\infty x \int_x^\infty b(x|y)\, a(y)\, c(t,y)\, dy\, dx = \int_0^\infty \int_0^y x\, b(x|y)\, a(y)\, c(t,y)\, dx\, dy$$
$$= \int_0^\infty y\, a(y)\, c(t,y)\, dy \qquad \Rightarrow \qquad \frac{d}{dt} \int_0^\infty x\, c(t,x)\, dx = 0$$

$$\frac{\partial}{\partial t} c(t,x) = \int_x^\infty b(x|y)\, a(y)\, c(t,y)\, dy - a(x)\, c(t,x) \qquad \int_0^y x\, b(x|y)\, dx = y$$

Note

$$\int_0^\infty x \int_x^\infty b(x|y)\, a(y)\, c(t,y)\, dy\, dx = \int_0^\infty \int_0^y x\, b(x|y)\, a(y)\, c(t,y)\, dx\, dy$$

$$= \int_0^\infty y\, a(y)\, c(t,y)\, dy \qquad \Rightarrow \qquad \frac{d}{dt} \int_0^\infty x\, c(t,x)\, dx = 0$$

However, for appropriate functions a and b

$$\int_0^\infty x\, c(t,x)\, dx \quad \searrow \quad \text{for} \quad t > 0$$

[Filippov-1961] formation of dust

[McGrady/Ziff-1987] shattering phenomenon

Non-conservative solutions (continued)

$$\frac{\partial}{\partial t} c(t,x) = \int_x^\infty b(x|y)\, a(y)\, c(t,y)\, dy - a(x)\, c(t,x)$$

Under some regularity conditions on b

$$\int_0^\infty x\, c(t,x)\, dx \searrow \quad \Longleftrightarrow \quad \int_0^\varepsilon \frac{1}{x\, a(x)}\, dx < \infty$$

Sufficient condition for shattering

$$a(x) \geq \frac{C}{x^\alpha} \qquad \forall\, x > 0 \qquad \text{for some} \quad \alpha > 0 \quad \text{and} \quad C > 0$$

Minimal jump process

$\mathcal{Y}$ - locally compact separable metric space

$\lambda(y)$ - jump rate, compactly bounded on $\mathcal{Y}$

$p(y, d\tilde{y})$ - jump kernel on $\mathcal{Y}$

$Y_0, Y_1, \ldots$ - Markov chain in $\mathcal{Y}$ $\sim$ p with initial distribution p_0

$T_0, T_1, \ldots$ - i.i.d. $\mathrm{Exp}(1)$ independent of (Y_k)

jump times

$$\tau_0 = 0 \qquad \tau_l = \sum_{k=0}^{l-1} \frac{T_k}{\lambda(Y_k)} \qquad l = 1, 2, \ldots$$

explosion time $\tau_\infty = \lim_{l \to \infty} \tau_l$

process on $\mathcal{Y} \cup \{\Delta\}$ (one-point compactification)

$$Y^\Delta(t) = \begin{cases} Y_l & : \quad \tau_l \le t < \tau_{l+1} \\ \Delta & : \quad t \ge \tau_\infty \end{cases}$$

Regularity and explosion

regularity: $\mathbb{P}(\tau_\infty = \infty) = 1$ explosion: $\mathbb{P}(\tau_\infty < \infty) > 0$

regularity $\iff$ $\displaystyle\sum_{k=0}^{\infty} \frac{1}{\lambda(Y_k)} = \infty$ a.s.

λ bounded $\Rightarrow$ regularity

explosion $\sim$ λ unbounded

$\&$ $Y_k \to \Delta$ with appropriate speed

Theorem

Assume

$$a(x) \geq \frac{C}{x^\alpha} \qquad \forall\, x > 0 \qquad \text{for some} \quad \alpha > 0 \quad \text{and} \quad C > 0\,.$$

Then the direct simulation process explodes almost surely, for any initial distribution.

recall:

DS process $\sim$ jump rate $a(x)$ and jump distribution $F(x,dz)$

Theorem
Assume

$$a(x) \geq \frac{C}{x^\alpha} \qquad \forall\, x > 0 \qquad \text{for some} \quad \alpha > 0 \quad \text{and} \quad C > 0$$

and

$$\int_0^x \left(\frac{y}{x}\right)^\alpha \tilde{B}(dy|x) \leq \beta < 1 \qquad \forall\, x > 0\,.$$

Then the mass flow process explodes almost surely, for any initial distribution.

recall: MF process $\sim$ jump rate $a(x)$ and jump distribution

$$\tilde{B}(dy|x) = \int_{\mathcal{Z}} \left[\frac{z_1}{x}\,\delta_{z_1}(dy) + \ldots + \frac{z_k}{x}\,\delta_{z_k}(dy)\right] F(x, dz)$$

Special case: deterministic binary fragmentation

Consider a function κ such that $\kappa(x) \in (0, x) \quad \forall\, x > 0$ and

$$F(x, dz) = F^{(2)}(x, dz_1, dz_2) = \delta_{(\kappa(x), x-\kappa(x))}(dz_1, dz_2)$$

Homogeneous case $\sim \quad \kappa(x) = \gamma\, x \qquad \gamma \in (0, 1)$

Consider a function κ such that $\kappa(x) \in (0, x) \quad \forall\, x > 0$ and

$$F(x, dz) = F^{(2)}(x, dz_1, dz_2) \;=\; \delta_{(\kappa(x), x-\kappa(x))}(dz_1, dz_2)$$

Homogeneous case $\sim$ $\kappa(x) = \gamma\, x \qquad \gamma \in (0, 1)$

MF "fragmentation" kernel

$$\begin{aligned} \tilde{B}(dy|x) &= \int_{\mathcal{Z}} \left[\frac{z_1}{x}\, \delta_{z_1}(dy) + \ldots + \frac{z_k}{x}\, \delta_{z_k}(dy) \right] F(x, dz) \\ &= \frac{\kappa(x)}{x}\, \delta_{\kappa(x)}(dy) + \left(1 - \frac{\kappa(x)}{x} \right) \delta_{x-\kappa(x)}(dy) \end{aligned}$$

Markov chain $Y_k \qquad k = 0, 1, \ldots$

$$Y_{k+1} = \begin{cases} \kappa(Y_k) & \text{with probability } \frac{\kappa(Y_k)}{Y_k} \\ Y_k - \kappa(Y_k) & \text{with probability } 1 - \frac{\kappa(Y_k)}{Y_k} \end{cases}$$

Consider a function κ such that $\kappa(x) \in (0, x) \quad \forall\, x > 0$ and

$$F(x, dz) = F^{(2)}(x, dz_1, dz_2) \;=\; \delta_{(\kappa(x), x-\kappa(x))}(dz_1, dz_2)$$

Homogeneous case $\sim \quad \kappa(x) = \gamma\, x \qquad \gamma \in (0, 1)$

MF "fragmentation" kernel

$$\tilde{B}(dy|x) \;=\; \int_{\mathcal{Z}} \left[\frac{z_1}{x}\,\delta_{z_1}(dy) + \ldots + \frac{z_k}{x}\,\delta_{z_k}(dy)\right] F(x, dz)$$

$$= \frac{\kappa(x)}{x}\,\delta_{\kappa(x)}(dy) + \left(1 - \frac{\kappa(x)}{x}\right)\delta_{x-\kappa(x)}(dy)$$

Markov chain $Y_k \qquad k = 0, 1, \ldots$

$$Y_{k+1} = \begin{cases} \kappa(Y_k) & \text{with probability } \frac{\kappa(Y_k)}{Y_k} \\ Y_k - \kappa(Y_k) & \text{with probability } 1 - \frac{\kappa(Y_k)}{Y_k} \end{cases}$$

Note $\kappa(x) = \frac{x}{2} \qquad Y_0 = \frac{1}{2} \qquad \Rightarrow \qquad Y_k = \frac{1}{2^{k+1}}$ (deterministic)

$$F(x,dz) = F^{(2)}(x,dz_1,dz_2) = \delta_{(\kappa(x),x-\kappa(x))}(dz_1,dz_2) \qquad \text{with} \qquad \kappa(x) = \frac{x}{2}$$

$$\text{explosion} \quad \Leftrightarrow \quad \sum_{k=0}^{\infty} \frac{1}{a(Y_k)} < \infty \qquad\qquad Y_k = \frac{1}{2^{k+1}}$$

If

$$a(x) \geq C\,|\log x|^{\alpha} \qquad \forall\, x>0 \qquad \text{for some} \quad \alpha > 1 \quad \text{and} \quad C>0$$

then

$$\sum_{k=0}^{\infty} \frac{1}{a(Y_k)} \leq \frac{1}{C} \sum_{k=0}^{\infty} \frac{1}{(k+1)^{\alpha}} < \infty$$

$$F(x,dz) = F^{(2)}(x,dz_1,dz_2) = \delta_{(\kappa(x),x-\kappa(x))}(dz_1,dz_2)$$ with $$\boxed{\kappa(x) \; = \; \frac{x}{2}}$$

explosion $\Leftrightarrow$ $\displaystyle\sum_{k=0}^{\infty} \frac{1}{a(Y_k)} < \infty$ $\qquad$ $\displaystyle Y_k = \frac{1}{2^{k+1}}$

If

$a(x) \;\geq\; C\,|\log x|^{\alpha} \qquad \forall\, x > 0$ for some $\alpha > 1$ and $C > 0$

then

$$\sum_{k=0}^{\infty} \frac{1}{a(Y_k)} \;\leq\; \frac{1}{C} \sum_{k=0}^{\infty} \frac{1}{(k+1)^{\alpha}} \;<\; \infty$$

Consistent with

$$\int_0^{\varepsilon} \frac{1}{x\,a(x)} \;<\; \infty$$

Special case: deterministic binary fragmentation - Example 2

$$F(x,dz) = \delta_{(\kappa(x),x-\kappa(x))}(dz_1,dz_2) \qquad \text{with}$$

$$\kappa(x) = \begin{cases} \frac{x}{2} + \frac{1}{4} & \text{if } x > \frac{1}{2} \\ \frac{x}{2} & \text{otherwise} \end{cases}$$

If $x_0 > \frac{1}{2}$ then

$$x_k = \frac{1}{2} + \frac{2x_0 - 1}{2^{k+1}} \qquad k = 0, 1, \ldots$$

satisfies

$$\kappa(x_k) = \frac{1}{4} + \frac{2x_0 - 1}{2^{k+2}} + \frac{1}{4} = x_{k+1}$$

Transition $x_k \to \kappa(x_k) = x_{k+1}$ with probability $\frac{\kappa(x_k)}{x_k}$

▷ trajectory of "slowest decay" (x_k) has non-zero probability

$$\lim_{k\to\infty} \frac{\kappa(x_0)}{x_0} \frac{\kappa(x_1)}{x_1} \ldots \frac{\kappa(x_k)}{x_k} = \frac{1}{x_0} \lim_{k\to\infty} \kappa(x_k) = \frac{1}{2\,x_0}$$

▷ no explosion on this trajectory, since $\lim_{k\to\infty} x_k = \frac{1}{2}$

$$F(x,dz) = \delta_{(\kappa(x),x-\kappa(x))}(dz_1,dz_2) \qquad \text{with} \qquad \kappa(x) = \begin{cases} \frac{x}{2}+\frac{1}{4}, & \text{if } x > \frac{1}{2} \\ \frac{x}{2}, & \text{otherwise} \end{cases}$$

If

$$a(x) \geq \frac{C}{x^\alpha} \qquad \forall\, x > 0 \qquad \text{for some} \quad \alpha > 0 \quad \text{and} \quad C > 0$$

then

- DS explodes with probability one, for any x_0

$$x \quad \longrightarrow \quad \kappa(x) \quad \text{and} \quad x - \kappa(x)$$

- MF does not explode with probability one, for $x_0 > \frac{1}{2}$

$$x \quad \longrightarrow \quad \begin{cases} \kappa(x) & \text{with probability } \frac{\kappa(x)}{x} \\ x-\kappa(x) & \text{with probability } 1-\frac{\kappa(x)}{x} \end{cases}$$

$$F(x,dz) = \delta_{(\kappa(x),x-\kappa(x))}(dz_1,dz_2) \qquad \text{with} \qquad \kappa(x) = \begin{cases} \frac{x}{2}+\frac{1}{4}, & \text{if } x > \frac{1}{2} \\ \frac{x}{2}, & \text{otherwise} \end{cases}$$

MF condition for a.s. explosion

$$\int_0^x \left(\frac{y}{x}\right)^\alpha \tilde{B}(dy|x) \;\le\; \beta < 1 \qquad \forall\, x > 0$$

with

$$\tilde{B}(dy|x) \;=\; \frac{\kappa(x)}{x}\,\delta_{\kappa(x)}(dy) + \left(1 - \frac{\kappa(x)}{x}\right)\delta_{x-\kappa(x)}(dy)$$

takes the form

$$\left(\frac{\kappa(x)}{x}\right)^{\alpha+1} + \left(1 - \frac{\kappa(x)}{x}\right)^{\alpha+1} \;\le\; \beta < 1 \qquad \forall\, x > 0$$

and is not fulfilled for $x \searrow \frac{1}{2}$

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▷ phase transition (loss of mass phenomenon)

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“raspylenie” (“pyl”∼“dust”), translation: “decomposition”, “disintegration”; MF explosion (irregular), homogeneous case

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very precise conditions, homogeneous case

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