The Landau damping: relaxation without dissipation for particle systems

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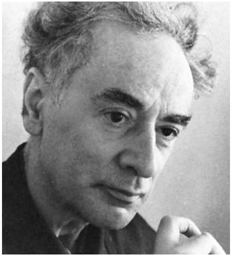
(short presentation note: arXiv:0905.2167)

What is the Landau damping? (I)



- Roughly speaking: **damping of spatial oscillations in a collisionless plasma** close to a spatially homogeneous steady state.
- Predicted by Landau in a famous paper in 1946 on the basis of a linearized study of the Vlasov-Poisson equation.
- (The other main contribution of Landau to plasma physics: Landau-Coulomb equation for collisional plasma 1936.)

What is the Landau damping? (II)

Vlasov-Poisson-Landau equation (1936)

$$\boxed{\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F[f] \cdot \nabla_v f = \frac{\log \Lambda}{2\pi\Lambda} Q_L(f,f)} \qquad \Lambda \sim 10^2 - 10^{30}$$

- For very long time ($O(\Lambda / \log \Lambda)$), dissipative phenomena, increase of entropy and (slow) cvg to Maxwellian
- But Landau 1946 made a much more subtle prediction: "stability" of many homogeneous equilibria (not necessarily gaussians) in shorter times ($O(1)$), by means of purely conservative mechanisms.
- Observed (with exponential rate)! Stunned the physics community: relaxation without dissipation (hamiltonian PDE)
- CM-Villani: first proof of the Landau damping for the non-linear Vlasov equation, global in time (or "quasi-global" for Coulomb-Newton interaction).

Many-particle system

- System of N interacting particles

$$\frac{d^2}{dt^2} x_i = \sum_{j \neq i} F(x_j - x_i), \quad F = -\nabla \phi, \quad 1 \leq i \leq N$$

- Interaction potential ϕ with long-range interaction: in dim. 3, Coulomb: $\phi(r) = r^{-1}$, Newton: $\phi(r) = -r^{-1}$, other smoother mathematical models...
- Limit $N \to +\infty$ on $f(t,x,v) \geq 0$ probability density:
    - Very low density: free transport eq. $\partial_t + v \cdot \nabla_x f = 0$;
    - Low density: Vlasov Eq.

$$\boxed{\partial_t f + v \cdot \nabla_x f - (\nabla \phi * \rho[f]) \nabla_v f = 0} \quad \rho[f](t,x) = \int_v f \, dv$$

- Long-time collision correction: cf. Landau-Coulomb eq...

The Vlasov-Poisson equation (I)

$$\boxed{\partial_t f + v \cdot \nabla_x f - (\nabla \phi * \rho[f]) \cdot \nabla_v f = 0}$$

Modelization:

- Repulsive $\phi \geq 0$ (cf. Coulomb): gaz of electrons in a plasma (heavy ions, no relativistic effects);
- Attractive $\phi \leq 0$ (cf. Newton): gaz of stars (!) in galactic dynamics (very large scale).

Cauchy theory:

- Lions-Perthame: weak solutions in $L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ with polynomial moments (key tool: averaging lemmas);
- Pfaffelmoser, Schaeffer, Bouchut-Golse-Pallard: Classical solutions by characteristics methods (needs compact support).

Still incomplete but important progress. But it does not tell much about the dynamics?

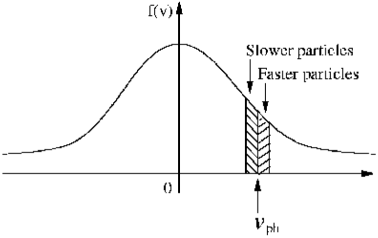
The Vlasov-Poisson equation (II)

$$\boxed{\partial_t f + v \cdot \nabla_x f - (\nabla \phi * \rho[f]) \cdot \nabla_v f = 0}$$

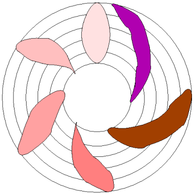
- Transport equation: time-reversible, preservation of entropy.
- Any spatially homogeneous $f(v)$ is stationary with vanishing force field. (More general stationary states: "BGK" waves...)
- Landau damping (LD): stability of homogeneous profiles $f(v)$ and "homogeneization" in their vicinity
- A tentative picture of time-scales from physics: Lagmuir waves / Landau damping / regularization by diffusion (?) / entropic convergence / appearance of correlations...
- Indeed study of Landau purely linearized and LD is controversial at the non-linear level since Landau: global in time? rate (exponential, polynomial)? for gravitation (Lynden-Bell'62)? explanation?

Physical interpretations

- Wave-particle interaction:



- Less known interpretation in terms of phase mixing (Van Kampen, Case, 1950')



Conceptual difficulty (I)

Dissipative behavior but equation is time-reversible

- Some answers: quasi-modes Van Kampen, add noise, coarse-graining...
- Our answer: Physically information still stored in large oscillations of the density distribution, not in observables.
- Lynden-Bell: "*A [galactic] system whose density has achieved a steady state will have information about its birth still stored in the particular velocities of its stars*"
- Mathematically weak convergence (dual of dispersion, the other key relaxation mechanism for reversible PDEs: here dispersion in frequency space!)
- Basic example: $u(t,x) = e^{itx}\, u_i(x)$

Conceptual difficulty (II)

Why would the large-time behavior of a nonlinear conservative equation be accurately predicted by the linearized system?

- Linearized case, $f_t \rightarrow \langle f_i \rangle(v)$ space-averaging of the initial datum, preserved (infinitely many conservation laws)
- Nonlinear case: no such conservation laws; no reason to cvg!
- The quasilinear theory of LD "establishes" convergence via ad hoc diffusion eq. in velocity space for statistical averages...
- Still the original work of Landau was in a linearized setting, and almost all works since on LD as well! Therefore validity of LD in the long-time is a debate among physicists...
- Our answer: homogeneization true for linearized Vlasov, and "KAM"-like feature of this ∞-dim. hamiltonian close to this "integrable case" (*cf.* Nash-Moser flavor in our proof)

Conceptual difficulty (III)

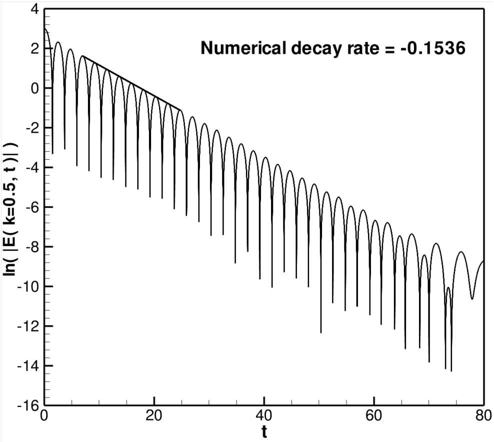
Analyticity of the data *versus* filamentation!

- What is filamentation? Velocity regularity of solution $f_t(x, v) = f_i(x - vt, v)$ of free transport grows in time whatever the initial regularity...
- Landau's computation is based on the analyticity of the "background" distribution $f^0(v)$ around which he linearizes
- But in the nonlinear pb, distribution evolves in time and depends on x: its analytic norm diverges, at best exp. fast
- Our answer: construct "gliding" analytic norms with a time-shift (as for free transport)
- Subdifficulty related: impossible to work only along characteristics of free transport to get rid of filamentation because the force depends on ρ! $\longrightarrow$ combine eulerian & lagrangian viewpoints with scattering-like estimates

Existing results

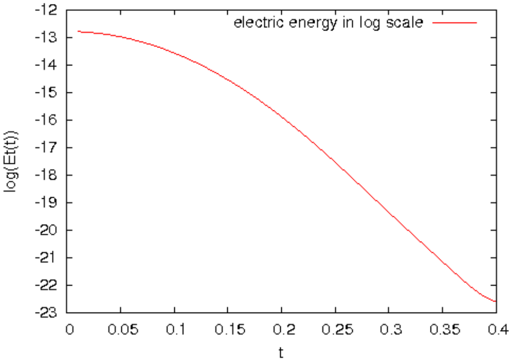
- Orbital stability: many works over past 15 years (Wolanski, Strauss, Guo, Rein...): Linear stability is easy, nonlinear stability much more tricky
- **Linear LD**: Saenz, Degond, Maslov-Fedoryuk...
  $\hookrightarrow$ We revisit linear LD without spectral theory, and also extend it from $f^0(v)$ to a damped $f^0(t, x, v)$
- **Nonlinear LD**: Caglioti-Maffei, Hwang-Velazquez: existence of some damped solutions (no information about which and how many initial data would yield a damped solution...)
- Although a short list, kind of exhaustive for math. study of LD!

Numerics (I) (performed by Francis Filbet)



Electric field (1-dim, repulsive, perturbation of Maxwellian)

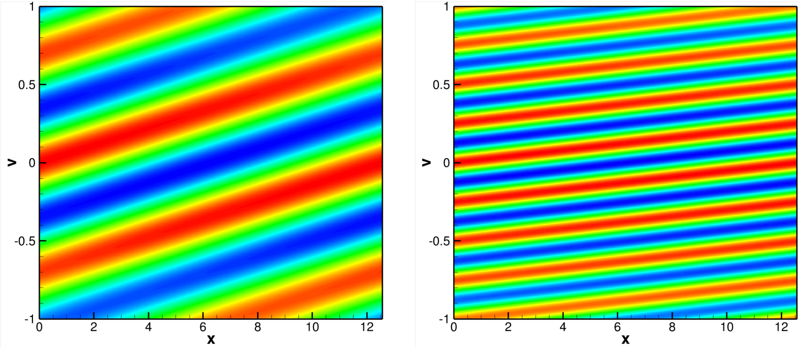
Numerics (II) (performed by Francis Filbet)



Gravitational field (1-dim, attractive, perturbation of Maxwellian)

Numerics (III) (performed by Francis Filbet)

Filamentation (repulsive case)...



Other simulations on the webpage of Francis Filbet...

The linear damping

$$(LVP) \qquad \boxed{\partial_t f + v \cdot \nabla_x f - (\nabla_x \phi * \rho_f) \cdot \nabla_v f^0(v) = 0}$$

with f^0 analytic, $x \in \mathbb{T}_L^d$ (torus with size L), $v \in \mathbb{R}^d$.

Proposition (CM-Villani, 2009)

Sufficient conditions for damping:
(1) Either

$$\left(\max_{k \neq 0} |\hat{\phi}(k)| \right) \left(\sup_{|\sigma|=1} \int_0^{+\infty} \left| \tilde{f}^0(r\sigma) \right| \, r \, dr \right) < \frac{1}{4\pi^2}$$

(good for Newton interaction below Jeans scale)
(2) Or $\hat{\phi} \geq 0$ and $z\, F'(z) < 0$ for any one-dimensional marginal F of f^0 (good for Coulomb interaction at all scales)

Non-linear Landau damping

$$(NLV) \qquad \boxed{\partial_t f + v \cdot \nabla_x f - (\nabla_x \phi * \rho_f) \cdot \nabla_v f = 0} \qquad x \in \mathbb{T}_L^d,\ v \in \mathbb{R}^d$$

Theorem (CM-Villani, 2009)

Assume (i) $f^0(v)$ and ϕ satisfy the linear damping condition;

(i) ϕ is not too singular: $\left|\hat{\phi}(k)\right| \leq \dfrac{C}{|k|^{1+\gamma}}$, $\gamma > 1$;

(ii) $\delta := |||f_i - f^0||| \leq \varepsilon << 1$: exponential localization in v, k, η.

Let f solve NLV with $f(t=0) = f_i$.

Then (a) $\rho[f](t,x) \xrightarrow{t \to \pm\infty} c$ strongly (exp. fast);

(b) $f(t,x,v) \rightharpoonup f_{\pm\infty}(v)$ as $t \to \pm\infty$ weakly;

(c) $\langle f \rangle(t,v) \xrightarrow{t \to \pm\infty} f_{\pm\infty}(v)$ strongly (space average).

Coulomb-Newton case $\gamma = 1$: same for $t = O(A^{1/\delta})$

Some new ingredients

- Mixed Lagrangian/Eulerian approach: estimate simultaneously the density and the trajectories
- Estimate regularity by comparison to the solution of free transport (gliding regularity: energy goes from large to small scales)
- Use new analytic norms that are well adapted to the geometry of the problem, and composition (!) as well
- Some new functional inequalities, quantifying how a background reacting to a plasma lend regularity to the plasma (flavor of averaging lemma but seems different)
- Understanding the destabilizing features (math. estimates lead us to study plasma echos)
- Newton scheme (as in KAM!) but no regularization

What do we learn from this result?

- f_t converges for all time without appealing to extra randomness;
- $f(t, x + vt, v)$ remains close to $f^0(v)$ in analytic norms ("super orbital stability");
- limit not determined by conservation laws or thermodynamical issues; keeps memory of initial datum and interaction;
- convergence "for no reason", just because near the "completely integrable" linear case: "KAM spirit"
- A whole neighborhood (in analytic topology) of f^0 is filled with homoclinic/heteroclinic (in weak topology) trajectories;
- From our proof: constructive scheme to approximate the whole dynamics and the limits $f_\pm(v)$ in terms of δ;
- Coulomb/Newton interaction is critical, as well as analyticity (echos. . .)!

Basic fundamental tool: Fourier transform

Consider $f = f(x, v)$ where $x \in \mathbb{T}^d_L := \mathbb{R}^d/(L\mathbb{Z}^d)$ and $v \in \mathbb{R}^d$

We denote

$$\hat{f}(k, v) := \int_{\mathbb{T}^d_L} e^{-2i\pi \frac{k}{L}\cdot x} f(x, v)\, dx, \qquad k \in \mathbb{Z}^d$$

$$\tilde{f}(k, \eta) := \int_{\mathbb{T}^d_L \times \mathbb{R}^d} e^{-2i\pi \frac{k}{L}\cdot x}\, e^{-2i\pi \eta\cdot v} f(x, v)\, dx\, dv, \qquad k \in \mathbb{Z}^d,\ \eta \in \mathbb{R}^d$$

For simplification, in most of the computations in the sequel, set $L = 1 \dots$

Phase mixing for the free transport (I)

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = 0 \text{ in } \mathbb{T}^d_L \times \mathbb{R}^d \text{ with initial data } f_i(x,v)$$

- total mass is of course preserved;
- also zero mode ($k = 0$) is preserved:

$$\forall\, v \in \mathbb{R}^d, \quad \int f(t,x,v)\, dx = \int f_i(x,v)\, dx$$

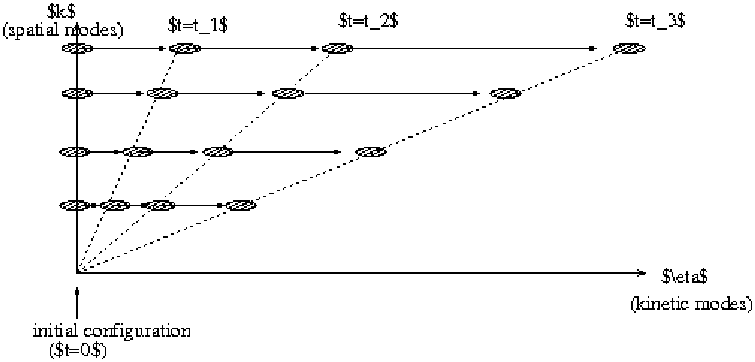
- Fourier transform:

$$\boxed{\tilde{f}(t,k,\eta) = f(t,k,\eta + kt)}$$

- Hence convergence to zero for any $k \neq 0$, with a rate given by the smoothness of f_i in v (analyticity $\to$ exponential rate!)

Phase mixing for the free transport (II)

Free transport "cascade"



Regularity improves in x, deteriorates in v

Recalls on the linearized Vlasov equation (I)

Consider the VP equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F[f] \cdot \nabla_v f = 0, \quad x \in \mathbb{T}^d, \; v \in \mathbb{R}^d$$

Linearize around a spatially homogeneous profile $f^0 = f^0(v)$: $F[f^0] = 0$ and therefore:

$$\boxed{\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F[f] \cdot \nabla_v f^0 = 0}$$

Recall that

$$F[f] = -(\nabla_x \phi) \star \rho[f]$$

where

$$\rho[f](t,x) = \int_{\mathbb{R}^d} f(t,x,v)\, dv$$

Recalls on the linearized Vlasov equation (II)

Duhamel formulation along free transport:

$$f(t,x,v) = f_i(x - vt, v) + \int_0^t (\nabla\phi \star \rho)(s, x - v(t-s)) \cdot \nabla_v f^0(v)\, ds$$

Integrate in v and Fourier transform in x:

$$\hat{\rho}(t,k) = \tilde{f}_i(k, kt) - 4\pi^2 \hat{\phi}(k) \int_0^t \hat{\rho}(s,k)\, \tilde{f}^0(k(t-s))\, |k|^2\, (t-s)\, ds$$

$\rightarrow$ **Series of Volterra-like equations decoupled for each mode k for the density**

$$\varphi(t) = a_k(t) + \int_0^t K_k(t-s)\, \varphi(s)\, ds$$

What do we want to prove?

$$\partial_t f + v \cdot \nabla_x f - (\nabla_x \phi * \rho_f) \cdot \nabla_v f^0(v) = 0$$

with f^0 analytic, $x \in \mathbb{T}^d_L$ (torus with size L), $v \in \mathbb{R}^d$.

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(2) Or $\hat{\phi} \geq 0$ and $z\, F'(z) < 0$ for any one-dimensional marginal F of f^0

An elementary computation (assumption (1))

Assumption (1) means for the kernel K_k of the Volterra equation of any k mode: For some $\lambda > 0$:

$$\int_0^{+\infty} e^{2\pi\lambda s}\, K_k(s)\, ds < 1 - \kappa, \qquad \kappa \in (0,1)$$

Then by simple computations one gets

$$\int_0^{+\infty} e^{2\pi\lambda t}\, \varphi(t)\, dt \leq \frac{\int_0^{+\infty} e^{2\pi\lambda t}\, a_k(t)\, dt}{1 - \int_0^{+\infty} e^{2\pi\lambda s}\, K_k(s)\, ds} \leq \frac{1}{\kappa} \int_0^{+\infty} e^{2\pi\lambda t}\, a_k(t)\, dt$$

Therefore exponential decay of the modes of the density in terms of the initial regularity in v

Analysis of a Volterra equation (assumption (2))

Let us first discuss informally:

$$\varphi(t) = a_k(t) + \int_0^t K_k(t-s)\,\varphi(s)\,ds$$

Laplace transform: $\boxed{\varphi^L = a_k^L + K_k^L\,\varphi^L}$
Hence formally φ should be given by

$$\varphi = (\mathrm{Laplace})^{-1}\left(\frac{a^L}{1-K^L}\right)$$

We expect the correct condition is: K^L does not approach 1 in a strip including some $\{0 < \Re\xi < \lambda\}$ in the complex plane

Analysis of a Volterra equation (assumption (2))

More precise statement:

$$\varphi(t) = a(t) + \int_0^t K(t-s)\,\varphi(s)\,ds$$

(i) Assume $K(t) = O(e^{-2\pi\lambda t})$
(ii) Define $\mathcal{L}(\xi) := \int_0^{+\infty} e^{2\pi\xi^* t}\,K(t)\,dt$ and assume (for some $\epsilon > 0$)

$$(L) \qquad \forall\,\xi \in \{-\epsilon < \Re\xi < \lambda\}, \quad |\mathcal{L}(\xi) - 1| \geq \kappa > 0$$

Then for all $\lambda' < \lambda$:

$$\sup_{t\geq 0} |\varphi(t)|e^{2\pi\lambda' t} \leq C(\lambda,\lambda',\kappa) \sup_{t\geq 0}\left(|a(t)|e^{2\pi\lambda t}\right)$$

Application to the case of assumption (2)

It remains to prove the following proposition:
The condition

$$(L) \qquad \forall\, \xi \in \{-\epsilon < \Re\xi < \lambda\}, \quad |\mathcal{L}(\xi) - 1| \geq \kappa > 0$$

is satisfied as soon as one of the following assumptions is true:
(1) Either

$$\left(\max_{k \neq 0} |\hat{\phi}(k)| \right) \left(\sup_{|\sigma|=1} \int_0^{+\infty} \left| \tilde{f}^0(r\sigma) \right| \, r\, dr \right) < \frac{1}{4\pi^2}$$

(2) Or $\hat{\phi} \geq 0$ and $z\, F'(z) < 0$ for any one-dimensional marginal F of f^0 (good for Coulomb interaction at all scales)

Requirements

- Analyticity is compulsory in order to get exponential decay (remark that however at the linear level one could relax the regularity at the price of a slower decay)
- Analytic norms based on Fourier seem well-adapted in the x variable in view of the linearized equation
- Analyticity of f^0 (and therefore f in the non-linear case) needed, but we shall need also analytic estimate (in v) of the characteristics, which are unbounded functions of v, therefore the analytic norm in v should be of L^∞ type rather
- The "gliding regularity" means that we should include a time-shift in the definition of the norm, accounting for the "cascade" of free transport

Analytic norms in one variable

Two natural families of analytic norms:

$\mathcal{C}^{\lambda;p}$ norm: $\displaystyle\sum_{n\in\mathbb{N}^d} \frac{\lambda^n}{n!}\|\nabla^n f\|_p$

$\mathcal{F}^{\lambda}$ norm: $\displaystyle\sum_{k\in\mathbb{Z}^d} e^{2\pi|k|}|\hat{f}(k)|$

$\mathcal{F}^{\lambda}$ and $\mathcal{C}^{\lambda,\infty}$ are algebra and as a consequence will satisfy nice composition properties.

Example:

$$\|f\circ(a\mathrm{Id}+G)\|_{\mathcal{Y}^\lambda} \leq C\,\|f\|_{\mathcal{Y}^{a(\lambda+\nu)}}$$

where $\nu = \|G\|_{\mathcal{Y}^\lambda}$

Analytic norms in two variables

Of course one could define $\mathcal{C}^{\lambda;p}$ and $\mathcal{F}^{\lambda}$ with two variables. From the previous requirements discussion we shall hydridize these two spaces:

$$\|f\|_{\mathcal{Z}^{\lambda,\mu;p}} := \sum_{k\in\mathbb{Z}^d} \sum_{n\in\mathbb{N}^d} \frac{\lambda^n}{n!} e^{2\pi\mu|k|} \|\nabla_v^n \hat{f}(k,v)\|_{L^p(dv)}$$

Remark: In the proof we need in fact even more flexibility, with an additionnal index for a Sobolev correction in the x variable:

$$\|f\|_{\mathcal{Z}^{\lambda,(\mu,\gamma);p}} := \sum_{k\in\mathbb{Z}^d} \sum_{n\in\mathbb{N}^d} (1+|k|)^{\gamma} \frac{\lambda^n}{n!} e^{2\pi\mu|k|} \|\nabla_v^n \hat{f}(k,v)\|_{L^p(dv)}$$

For $p=\infty$, this is still an algebra!

Gliding norms (I)

Now we introduce the time-shift, the guiding principle is (remember the discussion on the free transport):

$$\|f\|_{\mathcal{Y}^{**}_\tau} = \|f \circ S^0_\tau\|_{\mathcal{Y}^{**}_0} \qquad \text{(and thus also } \|f\|_{\mathcal{Y}^{**}_{t+\tau}} = \|f \circ S^0_\tau\|_{\mathcal{Y}^{**}_t}\text{)}$$

Therefore we obtain

$$\|f\|_{\mathcal{Z}^{\lambda,(\mu,\gamma);p}_\tau} := \sum_{k \in \mathbb{Z}^d} \sum_{n \in \mathbb{N}^d} (1+|k|)^\gamma \frac{\lambda^n}{n!} e^{2\pi\mu|k|} \|(\nabla_v + 2i\pi\tau k)^n \hat{f}(k,v)\|_{L^p(dv)}$$

For $p = \infty$, still an algebra!

Gliding norms (II)

We have subtle composition properties like:

$$\|f(x+bv+X, av+V)\|_{\mathcal{Z}_\tau^{\lambda,(\mu,\gamma);p}} \leq |a|^{-d/p} \|f\|_{\mathcal{Z}_\sigma^{\alpha,(\beta,\gamma);p}}$$

with (!)

$$\alpha = \lambda|a| + \|V\|_{\mathcal{Z}_\tau^{\lambda,\mu}}, \qquad \beta = \mu + \lambda|b + \tau - a\sigma| + \|X - \sigma V\|_{\mathcal{Z}_\tau^{\lambda,\mu}}$$

(Composition properties crucial for treating characteristics)

Finally it can be proved (fastidious!) some comparison results with more "usual norms" such as for instance:

$$\|f\|_{\lambda,\mu,\beta} = \sup_{k,\eta}\left(|\tilde{f}(k,\eta)|\, e^{2\pi\lambda|\eta|}\, e^{2\pi\mu|k|}\right) + \iint_{\mathbb{T}^d\times\mathbb{R}^d} |f(x,v)|\, e^{2\pi\beta|v|}\, dv\, dx.$$

The linearized theorem reframed

Assume $\|\nabla \phi\|_{L^1} \leq C_W$, and $f_i(x, v)$ such that

(i) (L) holds for some constants $\lambda, \kappa > 0$

(ii) $\|f^0\|_{\mathcal{C}^{\lambda;1}} \leq C_0$

(iii) $\|f_i\|_{\mathcal{Z}^{\lambda,\mu;1}} \leq \delta$ for some $\mu > 0$

Then for any $\lambda' < \lambda$, $\mu' < \mu$,

$$\sup_{t \in \mathbb{R}} \left\| f(t, \cdot) \right\|_{\mathcal{Z}_t^{\lambda',\mu';1}} \leq C\, \delta$$

for some constant $C = C(d, C_W, C_0, \lambda, \lambda', \mu, \mu', \kappa)$. In particular, $\rho = \int f\, dv$ satisfies

$$\sup_{t \in \mathbb{R}} \left\| \rho(t, \cdot) \right\|_{\mathcal{F}^{\lambda'|t|+\mu'}} \leq C\, \delta$$

Ideas of the proof

First estimate

$$\sup_{t\geq 0}\left\|\rho(t,\,\cdot\,)\right\|_{\mathcal{F}^{\lambda' t+\mu'}} = \sup_{t\geq 0}\sum_{k\in\mathbb{Z}^d} e^{2\pi(\lambda' t+\mu')|k|}|\hat{\rho}(t,k)|$$

by summing the estimate we have for each mode and using the margin on μ to get k-summability

Then estimate

$$\sup_{t\in\mathbb{R}}\left\|f(t,\,\cdot\,)\right\|_{\mathcal{Z}_t^{\lambda',\mu';1}}$$

using the Hölder like inequality

$$\left\|fg\right\|_{\mathcal{Z}_t^{\lambda',\mu';1}} \leq \left\|f\right\|_{\mathcal{Z}_t^{\lambda',\mu';\infty}} \times \left\|g\right\|_{\mathcal{Z}_t^{\lambda',\mu';1}}$$

and the fact that $\mathcal{Z}_t^{\lambda',\mu';1} = \mathcal{F}^{\lambda' t+\mu'}$ for functions only of x.

Abstract Newton scheme (I)

$\partial_t f = Q(f)$ around stat. solution f^0 ($Q(f^0) = 0$)
Write Cauchy problem with initial datum $f_i \simeq f^0$ as:

$$\Phi(f) := \Big(\partial_t f - Q(f),\ f(0,\cdot)\Big) = (0, f_i).$$

Newton iteration: start from f^0 and solve inductively
$\Phi(f^{n-1}) + \Phi'(f^{n-1}) \cdot (f^n - f^{n-1}) = 0$ for $n \geq 1$:

$$\partial_t h^1 = Q'(f^0) \cdot h^1 \quad h^1(0,\cdot) = f_i - f^0$$

$$\forall n \geq 1, \quad \partial_t h^{n+1} = Q'(f^n) \cdot h^{n+1} + \Big[Q(f^n) - \partial_t f^n\Big], \quad h^{n+1}(0,\cdot) = 0.$$

By substraction, for $n \geq 1$ this is the same as

$$\partial_t h^{n+1} = Q'(f^n) \cdot h^{n+1} + \Big[Q(f^{n-1} + h^n) - Q(f^{n-1}) - Q'(f^{n-1}) \cdot h^n\Big]$$

Abstract Newton scheme (II)

$$\partial_t h^{n+1} = Q'(f^n) \cdot h^{n+1} + \left[Q(f^{n-1} + h^n) - Q(f^{n-1}) - Q'(f^{n-1}) \cdot h^n \right]$$

$$h^{n+1}(0, \cdot) = 0$$

- Green: linearized semi-group around solution of the previous step
- Red: Source term quadratic in terms of the previous step if "Q twice differentiable"
- Blue: Apart from the first step, zero initial data
- So it replaces a nonlinear problem by an infinite system of linearized problems with source term
- BUT linearization around a non-stationnary solution! Yields new difficulties.
- Formally if the linearized semi-group can be solved correctly, we expect convergence like δ^{2^n}, typical of a Newton scheme

Concrete Newton scheme for our problem

$$f^n = f^0 + h^1 + \ldots + h^n,$$

First step:

$$\partial_t h^1 + v \cdot \nabla_x h^1 + F[h^1] \cdot \nabla_v f^0 = 0, \quad h^1(0, \cdot) = f_i - f^0$$

$n \geq 1$:

$$\partial_t h^{n+1} + v \cdot \nabla_x h^{n+1} + F[f^n] \cdot \nabla_v h^{n+1} + F[h^{n+1}] \cdot \nabla_v f^n = -F[h^n] \cdot \nabla_v h^n$$

$$h^{n+1}(0, \cdot) = 0$$

In Nash-Moser approach: regularization operators at each step.
Here no (we need the full power of the convergence), but we shall allow summable losses of regularity.

New issues to be solved

- Initial data enters through the first step only in the scheme, and the smallness δ should propagate
- First step is exactly the linearized study of the second course! Therefore we need the linearized stability condition (L).
- For $n \geq 1$, 3 new difficulties:
  1. Green: Perturbation of the free transport characteristics by $F[f^n]$ (force field small but not going to 0 as $n \to \infty$)
  2. Yellow: The background in the reaction term is now depending on t, x, v $\to$ growth of its ∇_v!
  3. Red: An additionnal quadratic source term, which should yield a quadratic error in some norm if we want the scheme to converge very fast

General strategy (I)

We propagate a number of estimates along the scheme; the most important are (slightly simplifying)

$$\sup_{\tau \geq 0} \left\| \int_{\mathbb{R}^d} h^n(\tau, \cdot, v)\, dv \right\|_{\mathcal{F}^{\lambda_n \tau + \mu_n}} \leq \delta_n,$$

$$\sup_{t \geq \tau \geq 0} \left\| h^n\left(\tau, \Omega^n_{t,\tau}\right) \right\|_{\mathcal{Z}^{\lambda_n(1+b),\mu_n;1}_{\tau - \frac{bt}{1+b}}} \leq \delta_n, \qquad b = b(t) = \frac{B}{1+t},$$

$$\left\| \Omega^n_{t,\tau} - \mathrm{Id} \right\|_{\mathcal{Z}^{\lambda_n(1+b),(\mu_n,\gamma);\infty}_{\tau - \frac{bt}{1+b}}} \leq C \left(\sum_{k=1}^{n} \frac{\delta_k\, e^{-2\pi(\lambda_k - \lambda_{n+1})\tau}}{2\pi(\lambda_k - \lambda_{n+1})^2} \right) \min\{t - \tau\, ;\, 1\}.$$

$\Omega^n_{t,\tau}$ denote the (finite-time) scattering operators, see below
Remark: Stratification of all the estimates. . .

General strategy (II)

- **Step 1.** estimate $\Omega^n - \mathrm{Id}$ (the bound should be uniform in n);
- **Step 2.** estimate $\Omega^n - \Omega^k$ ($k \leq n-1$; the error should be small when $k \to \infty$);
- **Step 3.** estimate $\nabla\Omega^n - \mathrm{Id}$;
- **Step 4.** estimate $(\Omega^k)^{-1} \circ \Omega^n$;
- **Step 5.** estimate h^k and its derivatives along the composition by Ω^n;
- **Step 6.** estimate $\rho[h^{n+1}]$;
- **Step 7.** estimate $F[h^{n+1}]$ from $\rho[h^{n+1}]$;
- **Step 8.** estimate $h^{n+1} \circ \Omega^n$;
- **Step 9.** estimate derivatives of h^{n+1} composed with Ω^n;
- **Step 10.** show that for h^{n+1}, ∇ and composition by Ω^n asymptotically commute.

Implication of the convergence of the scheme

The goal is to prove sthg like

$$\forall\, t \geq 0, \quad \|h_n(t,\cdot)\|_{\mathcal{Z}_t^{\lambda_n,\mu_n}} \leq \delta_n$$

where δ_n converges very fast to 0 (summable) and λ_n decaying to $\lambda_\infty > 0$, μ_n decaying to $\mu_\infty > 0$
Then we deduce by summation that $f^n = f^0 + h^1 + \ldots + h^n \to f^\infty$ with

$$\forall\, t \geq 0, \quad \|f^\infty(t,\cdot)\|_{\mathcal{Z}_t^{\lambda_\infty,\mu_\infty}} \leq C\,\delta$$

This conclues the proof of the theorem as in the (reframed) linearized case.
Also it allows for rigorous expansion of the solution, and therefore its limiting profiles, by computing the solution to a finite number of linearized problems.

Remark on the short-time estimate

As a preliminary, in our proof, we shall need small times estimates

Cauchy-Kowalevskaya style problem (loss of one derivative)

To compensate for the loss of one derivative, allow for a loss of regularity:

$$\lambda(t) = \lambda - Kt, \quad \mu(t) = \mu - Kt$$

and use

$$\frac{d}{dt}^{+}\bigg|_{t=\tau} \|f\|_{\mathcal{Z}_\tau^{\lambda(t),\mu(t);p}} \leq -\frac{K}{1+\tau} \|\nabla f\|_{\mathcal{Z}_\tau^{\lambda(\tau),\mu(\tau);p}}$$

The characteristics method

Linearize around $\bar{f}$:

$$\partial_t f + v \cdot \nabla_x f + F[f] \cdot \nabla_v \bar{f} + F[\bar{f}] \cdot \nabla_v f = 0$$

We want to get rid of the last term by characteristics method:
$(X, V)_{s,t}(x, v)$ position/velocity at time t, starting at time s from (x, v), driven by $F[\bar{f}]$ (non autonomous):

$$\frac{dX}{dt} = V, \qquad \frac{dV}{dt} = F[X]$$

Hence: $\partial_t f(t, X_{0,t}, V_{0,t}) = \partial_t f_| + v \cdot \nabla_x f_| + F \cdot \nabla_v f_|$ and so

$$f(t, x, v) = f_i(X_{t,0}(x, v), V_{t,0}(x, v))$$
$$+ \int_0^t F[f](X_{t,s}(x, v)) \cdot \nabla_v \bar{f}(s, X_{t,s}(x, v), V_{t,s}(x, v))\, ds$$

Finite-time scattering

- $F = -\nabla \phi \star \rho$, with $\|\rho_t\|_{\mathcal{F}^{\lambda t + \mu}} \leq C$
- $S_{s,t} = (X_{s,t}, V_{s,t})$: characteristics induced by F
- $S^0_{s,t} = (x + v(t-s), v)$: characteristics of free transport
- $\Omega_{t,s} = S_{t,s} \circ S^0_{s,t}$ "scattering operators" (be careful: not semigroup)
- the kind of estimates we establish:

$$\|\Omega_{t,s} - \mathrm{Id}\|_{\mathcal{Z}^{\lambda',\mu'}_{s'}} \leq C' \min\{(t-s), 1\}\, e^{-\alpha s}$$

- Uniform in $t >> s$, small for $s \sim t$ and $s \to +\infty$
- Proof: Fixed point theorems combined with properties of $\mathcal{Z}$ spaces...

The bilinear term to be estimated

$$\sigma(t,x) = \int_0^t \int_{\mathbb{R}^d} \left(F[f] \cdot \nabla_v \bar{f}\right)\left(\tau, x - v(t-\tau), v\right) dv\, d\tau.$$

This quantity can be interpreted as follows:

If particles distributed according to f exert a force on particles distributed according to $\bar{f}$, then σ is the variation of density $\int f\, dv$ caused by the *reaction* of $\bar{f}$ on f

Regularity extorsion in short times

- Straight trajectories must be replaced by characteristics (this reflects the fact that $\bar{f}$ also exerts a force on f)
- $\longrightarrow$ source of considerable technical difficulties
- Key tool (together with finite-time scattering estimates) used to overcome them: regularity extorsion in the short time (reminding somehow velocity-averaging lemmas)
- Here is a simplified version:

$$\left\|\sigma(t,\,\cdot\,)\right\|_{\dot{\mathcal{F}}^{\lambda t+\mu}} \leq \int_0^t \left\|F\left[f(\tau,\,\cdot\,)\right]\right\|_{\mathcal{F}^{\lambda[\tau-b(t-\tau)]+\mu,\gamma}} \times \left\|\nabla f(\tau,\,\cdot\,)\right\|_{\mathcal{Z}^{\lambda(1+b),(\mu,0);1}_{\tau-bt/(1+b)}}\,d\tau.$$

- Observe that the regularity of σ is better than that of $F[f]$, with a gain that degenerates as $t \to \infty$ or $\tau \to t$.

Regularity extorsion in long times

We show that if $\bar{f}$ has a high gliding regularity, then the decay of σ in large time is better than what would be expected:

$$\|\sigma(t,\,\cdot\,)\|_{\dot{\mathcal{F}}^{\lambda t+\mu}} \leq \int_0^t K(t,\tau)\,\big\|F\big[f(\tau,\,\cdot\,)\big]\big\|_{\mathcal{F}^{\lambda\tau+\mu,\gamma}}\,d\tau,$$

where

$$K(t,\tau) = \left[\sup_{0\leq s\leq t}\left(\frac{\big\|\nabla_v \bar{f}(s,\,\cdot\,)\big\|_{\mathcal{Z}_s^{\bar{\lambda},\bar{\mu};1}}}{1+s}\right)\right] K_0$$

$$K_0 = (1+\tau)\sup_{k\neq 0,\ \ell\neq 0}\frac{e^{-2\pi(\bar{\lambda}-\lambda)|k(t-\tau)+\ell\tau|}\,e^{-2\pi(\bar{\mu}-\mu)|\ell|}}{1+|k-\ell|^{\gamma}}.$$

The time-response kernel (I)

Recall

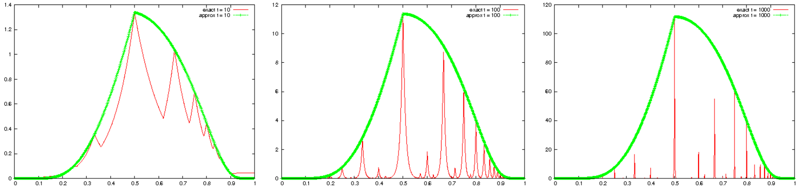
$$\left\|\sigma(t,\,\cdot\,)\right\|_{\mathcal{F}^{\lambda t+\mu}} \leq \int_0^t K(t,\tau)\,\Big\|F\big[f(\tau,\,\cdot\,)\big]\Big\|_{\mathcal{F}^{\lambda\tau+\mu,\gamma}}\,d\tau,$$

where

$$K(t,\tau) = \left[\sup_{0\leq s\leq t}\left(\frac{\left\|\nabla_v \bar{f}(s,\,\cdot\,)\right\|_{\mathcal{Z}_s^{\bar{\lambda},\bar{\mu};1}}}{1+s}\right)\right] K_0$$

$$K_0 = (1+\tau)\sup_{k\neq 0,\ \ell\neq 0}\frac{e^{-2\pi(\bar{\lambda}-\lambda)|k(t-\tau)+\ell\tau|}\,e^{-2\pi(\bar{\mu}-\mu)|\ell|}}{1+|k-\ell|^{\gamma}}.$$

The time-response kernel (II)



The kernel $K_0(t,\tau)$, together with an approximate upper bound for $\alpha = 0.5$ and $t = 10$, $t = 100$, $t = 1000$.

Stabilization by echoes (I)

- Kernel $K(t,\tau)$ has integral $O(t)$ as $t \to \infty \to$ risk of violent unstability
- But it is also more and more concentrated on discrete times $\tau = kt/(k-\ell)$
- This is the effect of plasma echoes, discovered and experimentally observed in the sixties.
- The stabilizing role of the echo phenomenon, related to the Landau damping, is uncovered in our study.

Stabilization by echoes (II)

- Case $\gamma < 1 \rightarrow$ subexponential response, so that it can be controlled by an arbitrarily small loss of gliding regularity, at the price of a gigantic constant, which later will be absorbed by the ultrafast convergence of the Newton scheme.
- In the end, part of the gliding regularity of $\bar{f}$ has been converted into a large-time decay.
- Case $\gamma = 1$: exponential growth due to echoes!
- Hence analyticity seems the critical regularity for the Landau damping in the Coulomb/Newton case (whereas L^∞ is the critical regularity in the Cauchy theory of Lions-Perthame...)

Perspectives

- Coulomb-Newton case under more investigation;
- Work in progress to try to prove the inviscid damping for two-dimensional incompressible fluids (asymptotic stability and "attraction" of very large scale structures like radially symmetric vortex and shear flows);
- Stability of BGK waves in a plasma...
- Wigner-Poisson / Schrödinger-Poisson? Link with dispersion results?...
- Link with weak KAM theory?...