Kinetic models for shock collisions and Burgers turbulence

Govind Menon,
Division of Applied Mathematics,
Brown University.

Burgers turbulence is the study of the statistics of shocks in Burgers equation with random intial data or forcing. The main motivation is to provide insights and rigorous results on turbulence in fluids (Burgers, 1940).

It is a vastly simplified problem, but still of interest as a benchmark. There are fascinating links to problems in statistics, kinetic theory and other areas of mathematical physics.

We study the following general question. Let f be a convex flux function. What can we say about the statistics of the entropy solution to the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0, \quad x \in \mathbb{R}, t > 0,$$

when the initial data is random?

But first a quick refresher on Burgers equation. In this case, $f(u)=u^2/2$

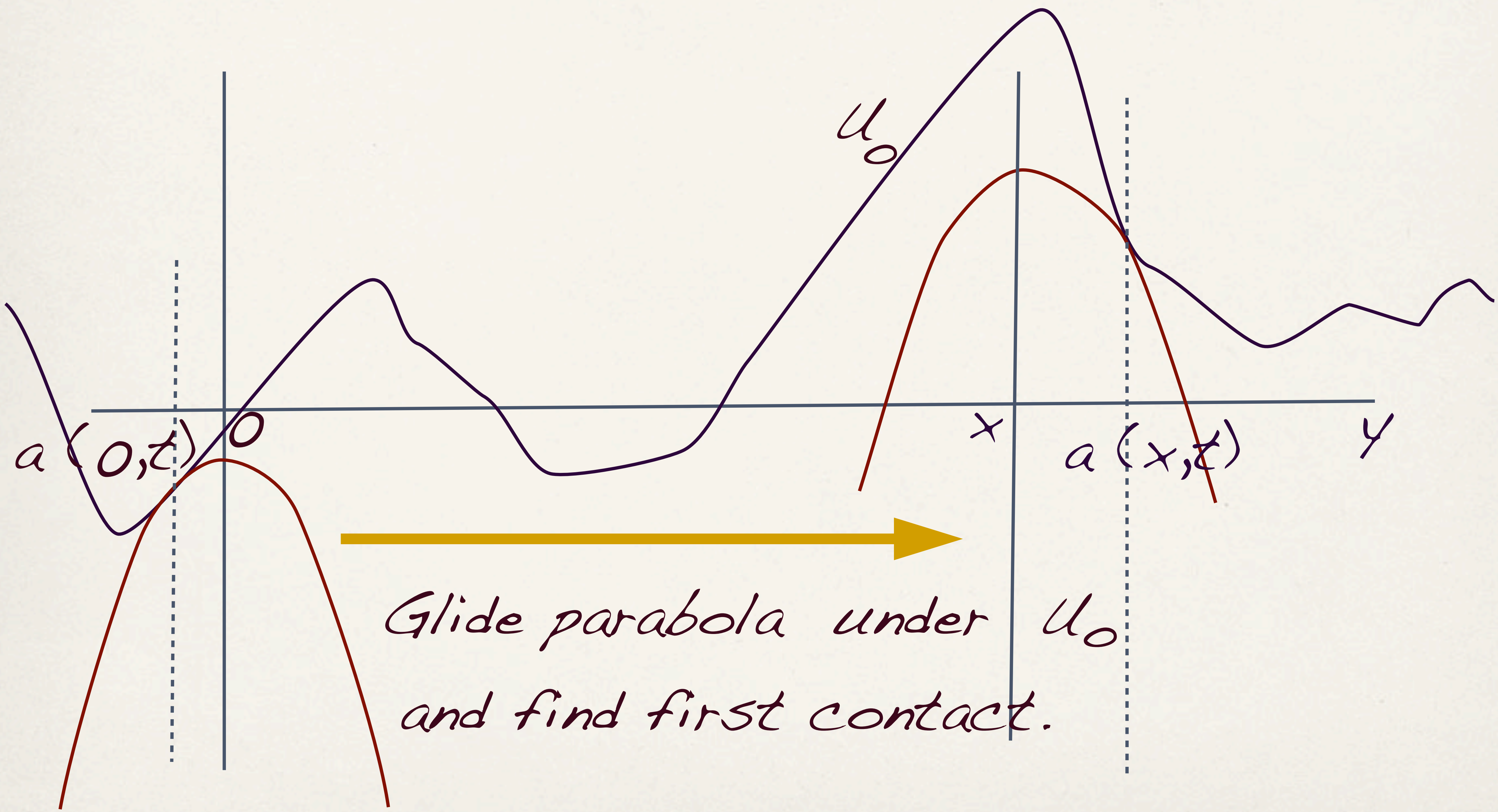
The unique entropy solution, or the Cole-Hopf solution, is given by a variational principle.

$$u(x,t) = \frac{x - a(x,t)}{t}$$

$$a(x,t) = \mathrm{argmin}_y^+ \left\{ U_0(y) + \frac{(x-y)^2}{2t} \right\}$$

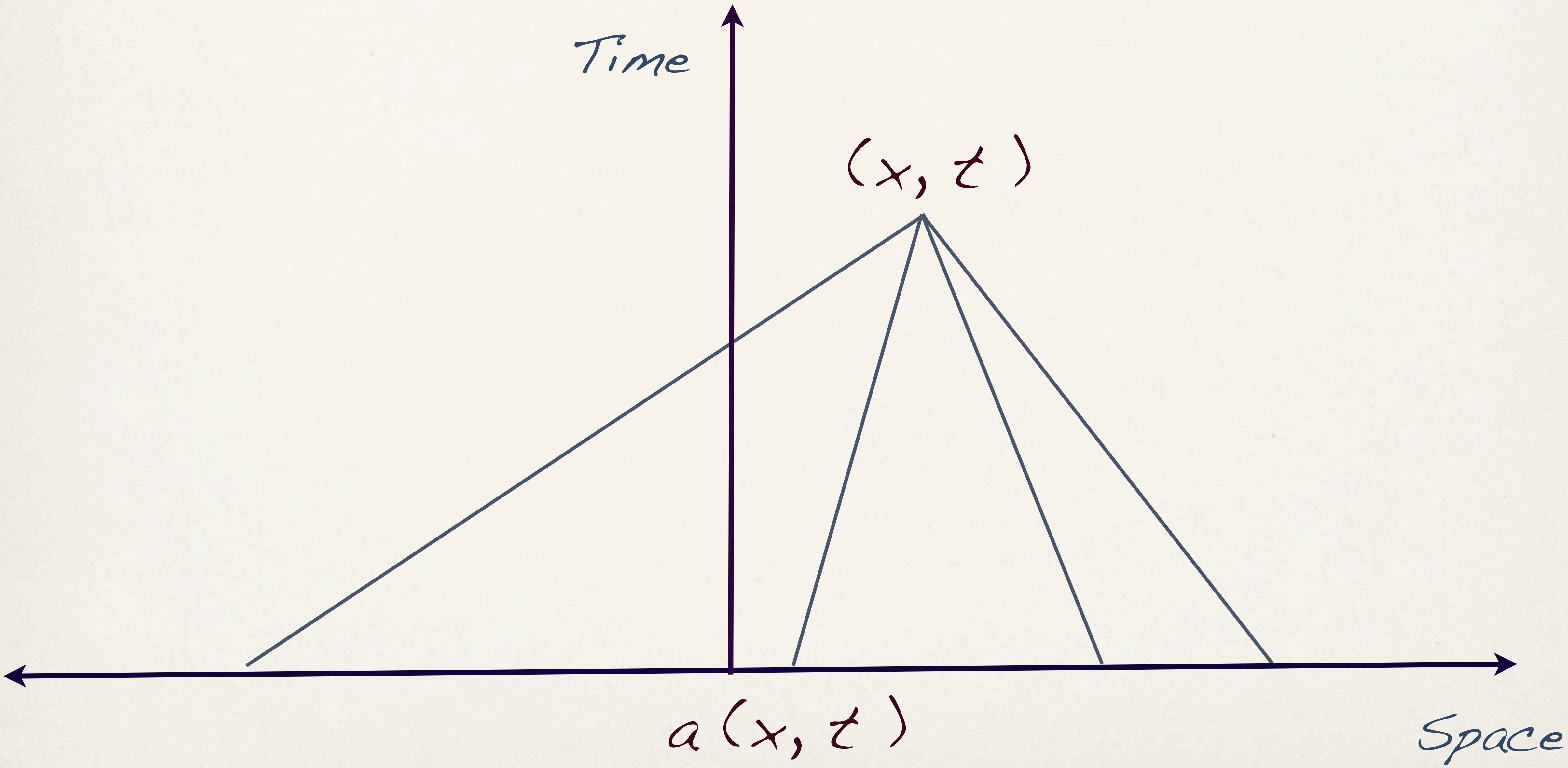
$$U_0(y) = \int_0^y u_0(s)\, ds$$

$u(x,t)$ is the <u>velocity field</u>. $\mathcal{U}$ is called the <u>potential</u> and $a(x,t)$ the <u>inverse Lagrangian function</u>. The variational principle is a geometric recipe that uses the potential.

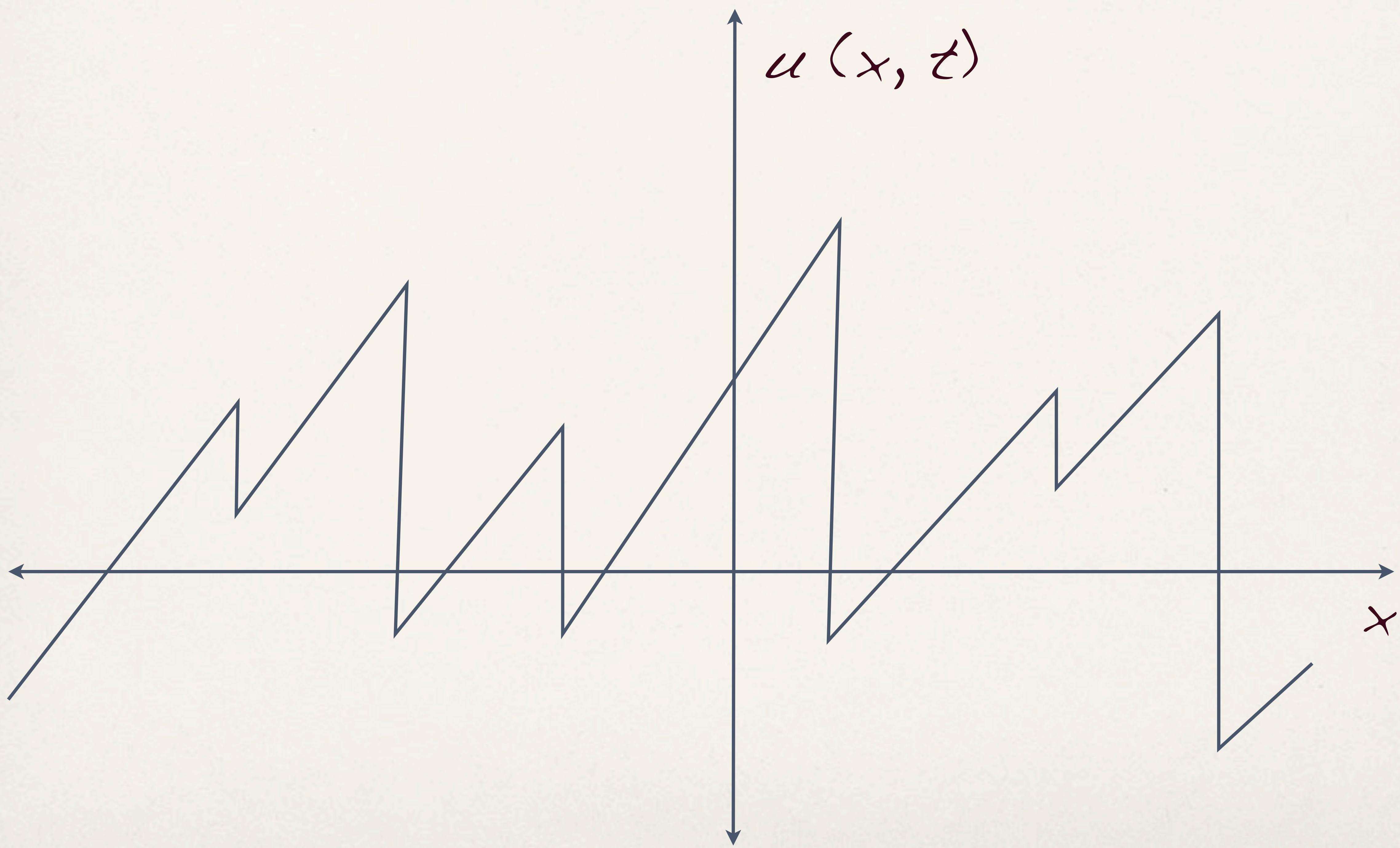


1) $a(x,t)$ is increasing in x.

2) $a(x,t)$ gives the 'correct' characteristic through the point (x,t) in space-time.



3) As a consequence, $u(x,t)$ is of bounded variation. Jumps in a give rise to shocks in u. These correspond to 'double-touches' in the geometric principle.



Numerical experiments with Brownian initial potential (white noise as initial velocity)

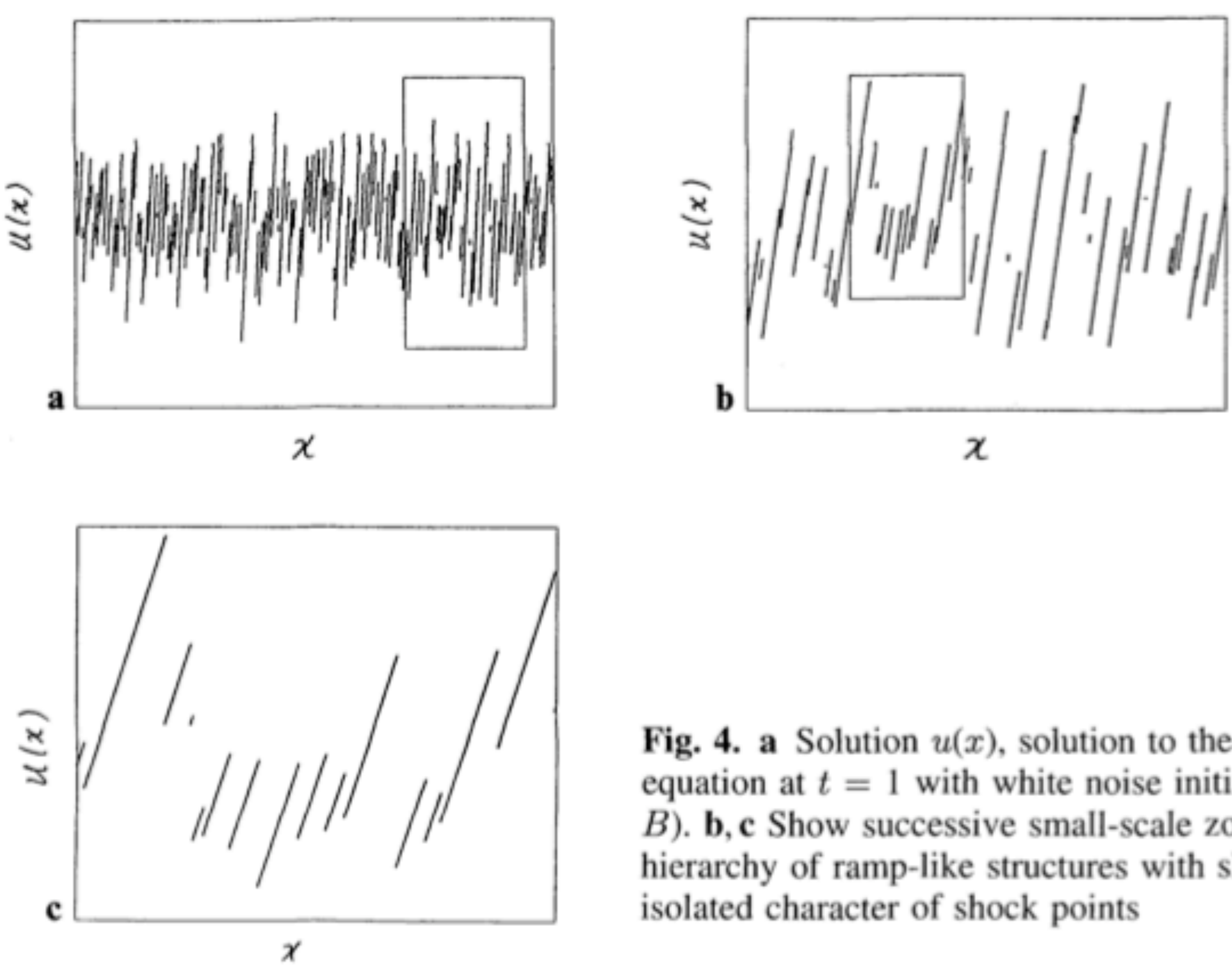


Fig. 4. a Solution $u(x)$, solution to the inviscid Burgers equation at $t = 1$ with white noise initial velocity (type B). **b, c** Show successive small-scale zooming. Note the hierarchy of ramp-like structures with slope 1 and the isolated character of shock points

She, Aurell, Frisch, Commun. Math. Phys. 148, (1992)

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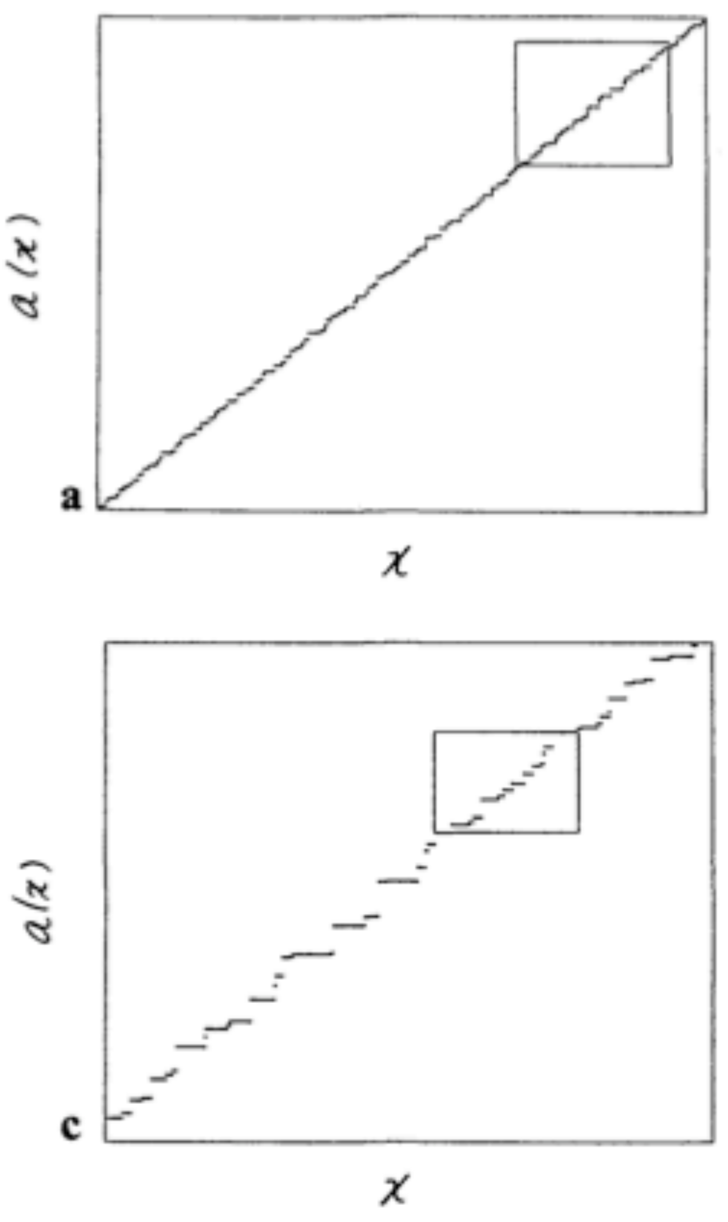


Fig. 3. a Inverse Lagrangian function $a(x)$ corresponding to the solution to the inviscid Burgers equation at $t = 1$ with white noise initial velocity (type B). **b, c** Show successive zooming, displaying smaller and smaller structures. Note the sparseness of small scale structures, compared to Fig. 1a–c

She, Aurell, Frisch, Commun. Math. Phys. 148, (1992)

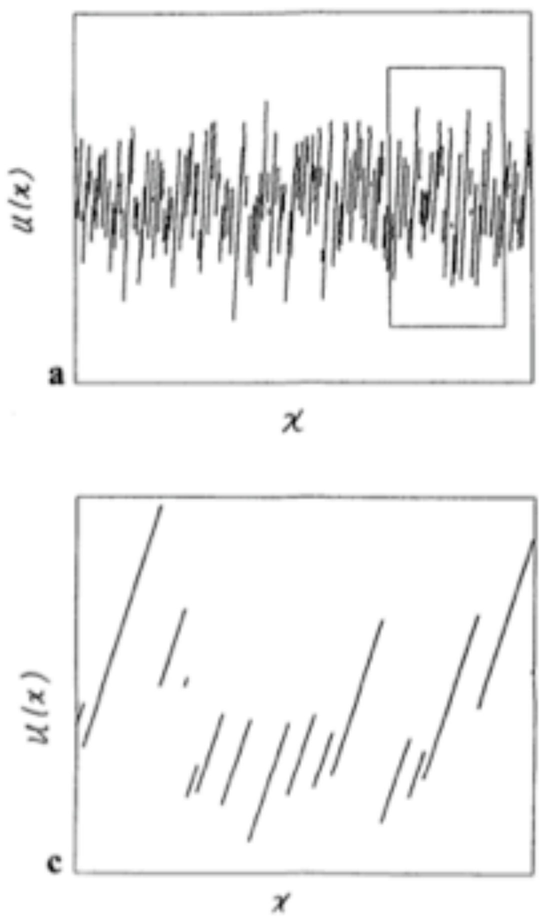


Fig. 4. **a** Solution $u(x)$, solution to the inviscid Burgers equation at $t = 1$ with white noise initial velocity (type B). **b, c** Show successive small-scale zooming. Note the hierarchy of ramp-like structures with slope 1 and the isolated character of shock points

White noise

Brownian motion

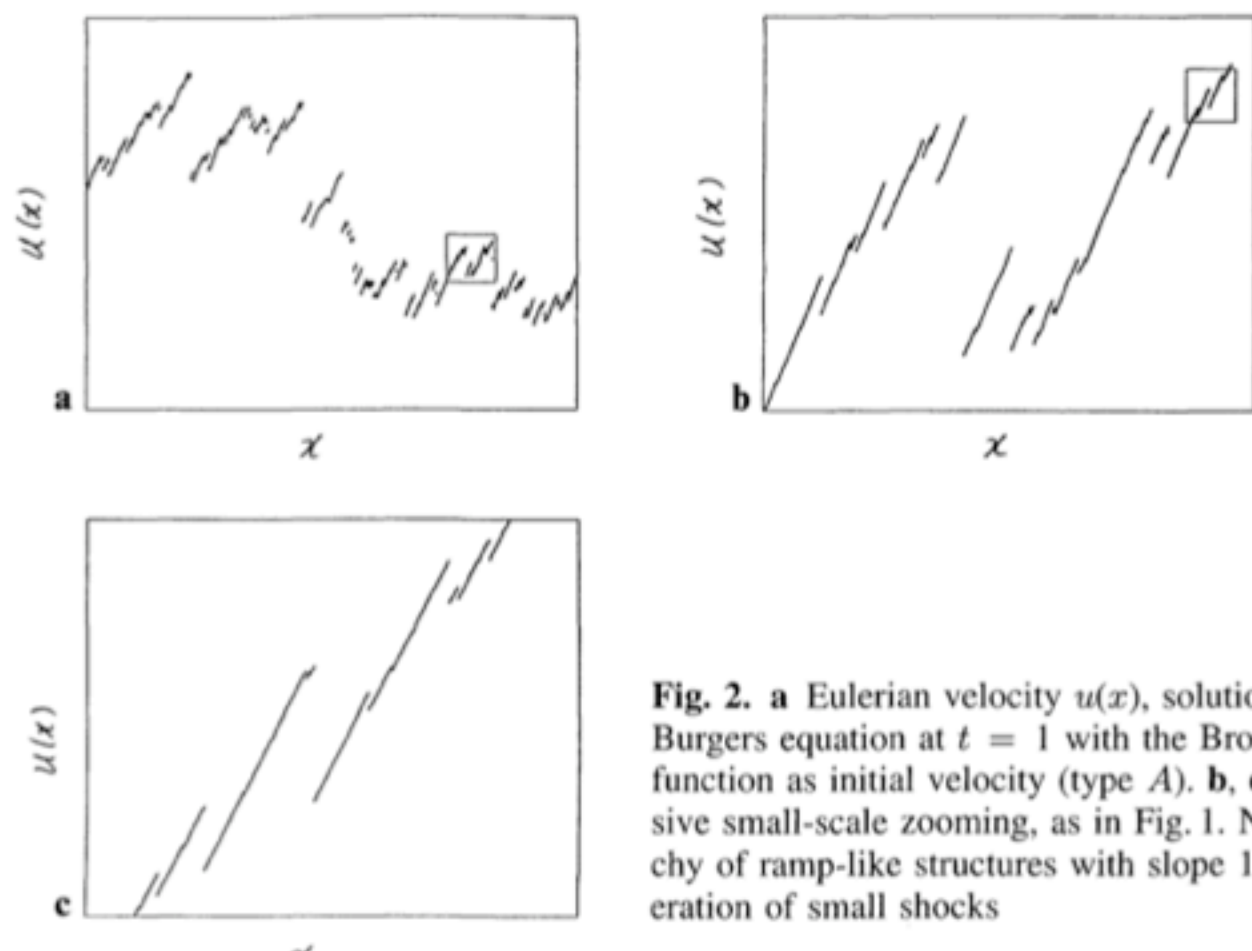


Fig. 2. **a** Eulerian velocity $u(x)$, solution to the inviscid Burgers equation at $t = 1$ with the Brownian motion function as initial velocity (type A). **b, c** Show successive small-scale zooming, as in Fig. 1. Note the hierarchy of ramp-like structures with slope 1 and the proliferation of small shocks

Some fundamental questions

1) How do we describe the n point statistics for $u(x,t)$? More precisely, how can we understand the shock structure and the coalescence of shocks?

2) Can we understand the fine structure of $u(x,t)$ -- eg. Hausdorff dimension of Lagrangian regular points?

3) How special are these choices of initial data. In what sense are white noise and Brownian motion 'typical'?

Both problems have remarkable exact solutions!

White noise initial velocity:

P. Groeneboom, Brownian motion with a parabolic drift and Airy Functions, Prob.Th. Rel. Fields, 81, (1989).

L. Frachebourg, P. Martin, Exact statistical properties of the Burgers equation, J. Fluid Mech, 417, (2000).

Brownian motion initial velocity :

L. Carraro, J. Duchon, C.R. Acad. Sc. Paris. Math 319, (1994) , Ann. IHP Anal. Nonlineaire 15, (1998).

J. Bertoin, Commun. Math. Phys. 193, (1998).

A first glimpse at Groeneboom's solution

The one-point distribution of u at time 1 has density

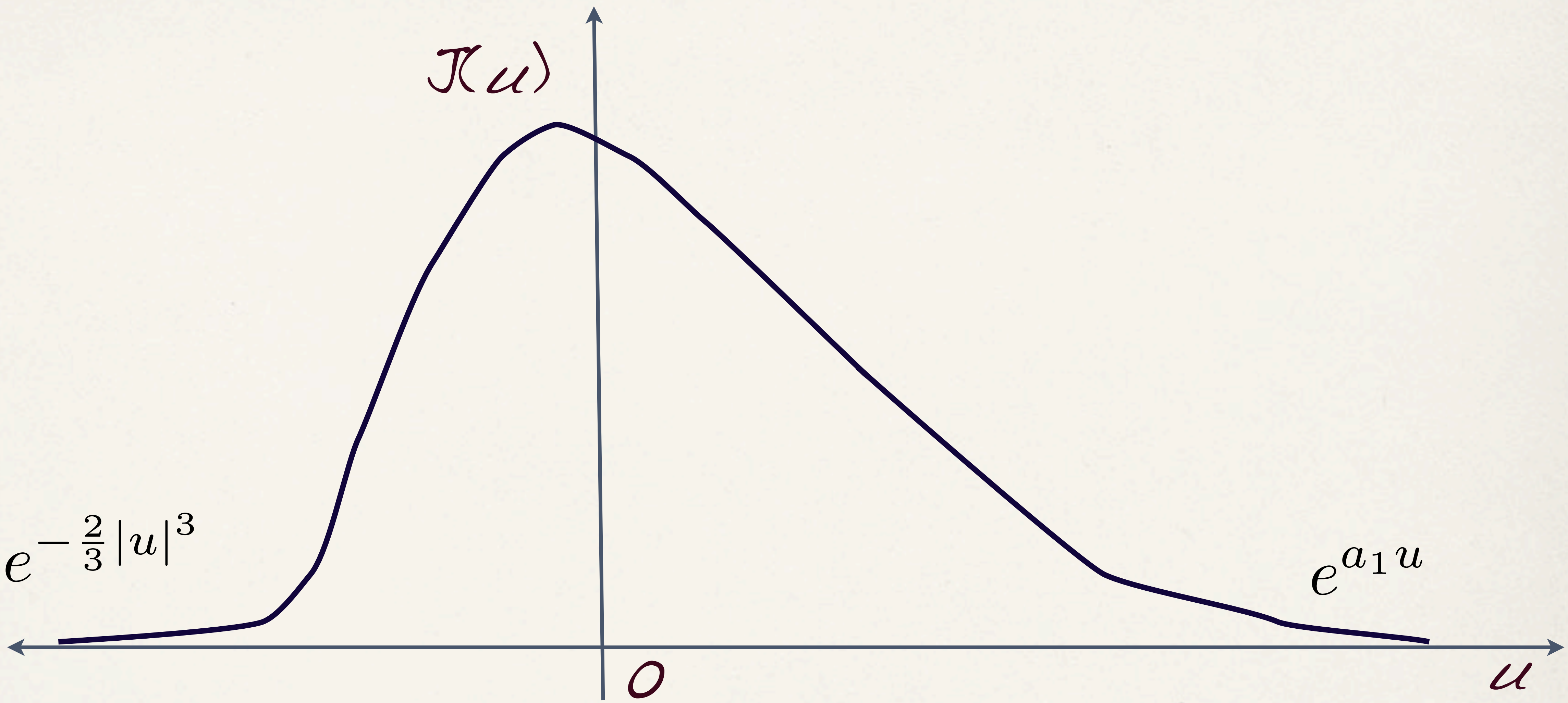
$$p(u) = J(u)J(-u), \quad u \in \mathbb{R}.$$

The function J has an explicit Laplace transform

$$\int_{-\infty}^{\infty} e^{-qu} J(u)\, du = \frac{1}{Ai(q)},$$

where $Ai(q)$ is the (first) Airy function.

Classical Tauberian theorems yield asymptotics of J and p

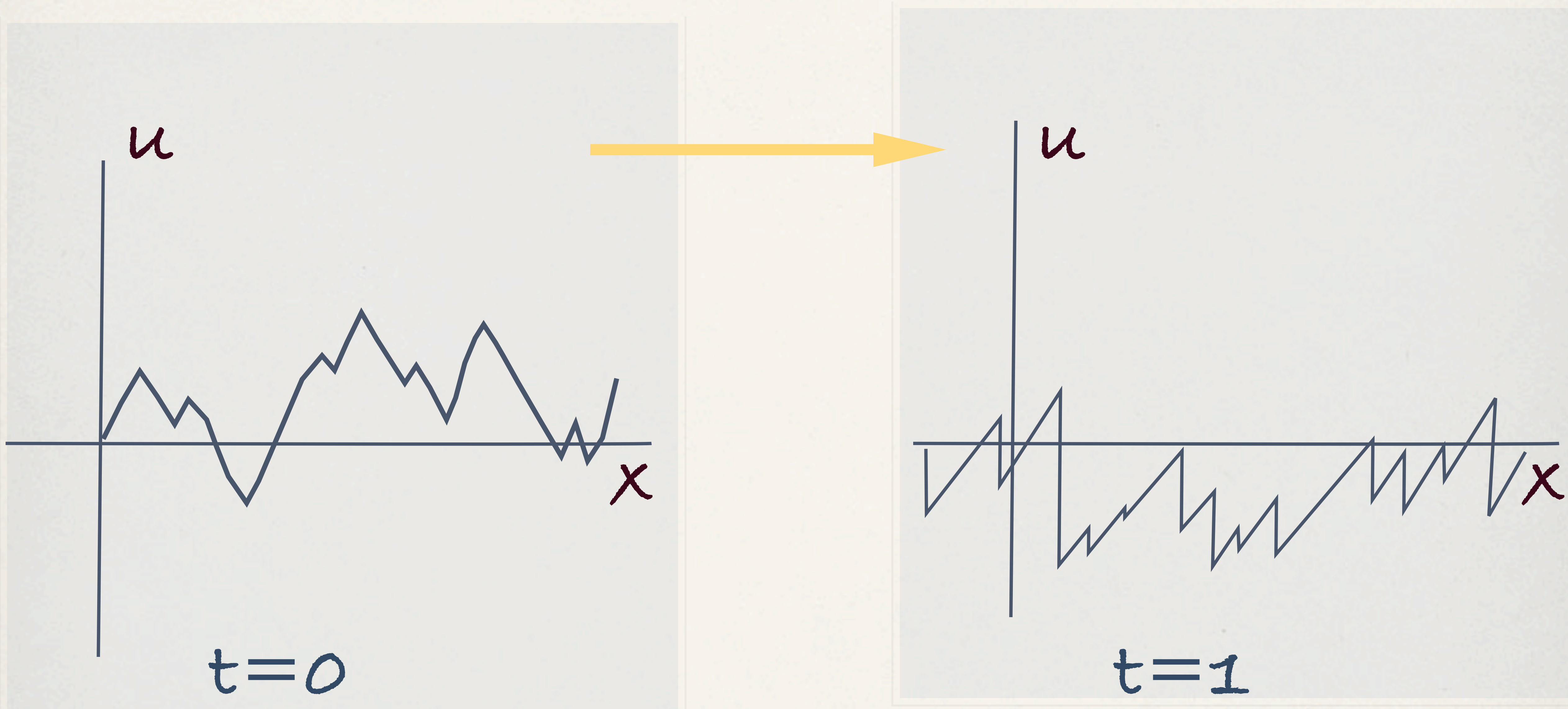


a_1 is the first zero of the Airy function.

$$p(u) = J(u)J(-u) \sim e^{-\frac{2}{3}|u|^3}, \quad |u| \to \infty.$$

Brownian motion as initial velocity, and Lévy processes.

Brownian motion as initial velocity



Initial data is a one sided Brownian motion (for simplicity).

We then study $u(x,t) - u(0,t)$.

Generators of Markov processes

A Markov process is characterized by its transition semigroup Q and generator. For suitable test functions

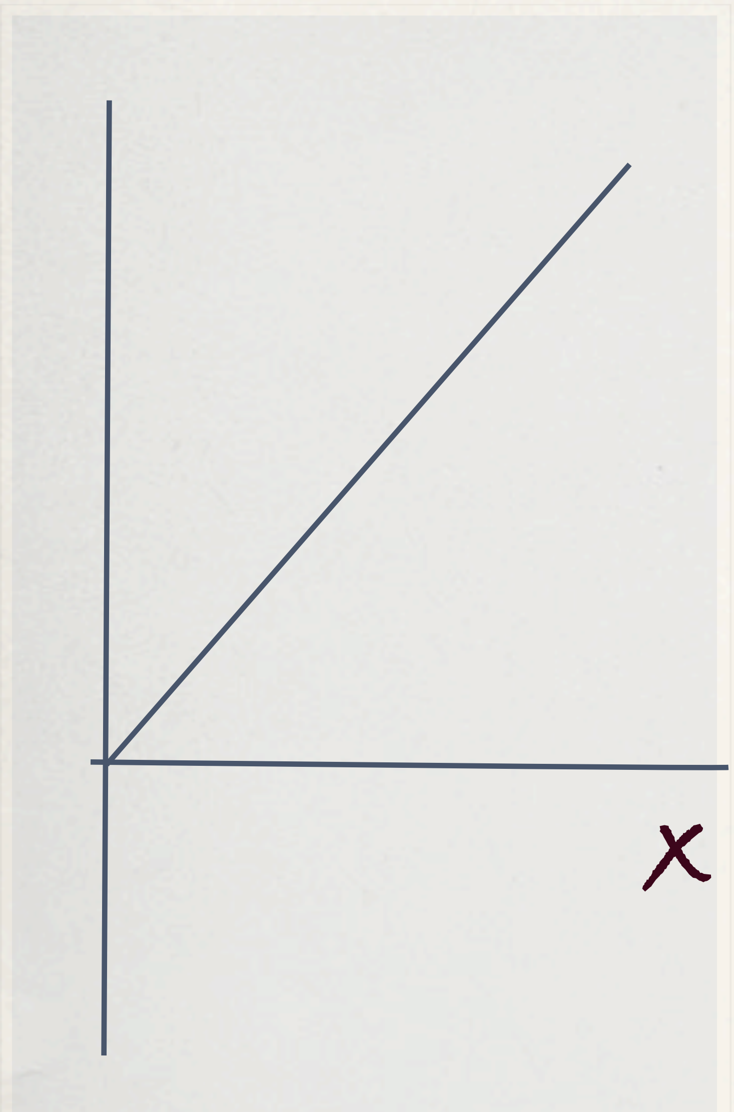
$$\mathcal{A}\varphi = \lim_{h \downarrow 0} \frac{Q_h \varphi - \varphi}{h}.$$

For Levy processes we may use Fourier analysis. That is, we consider exponentials as test functions and find

$$\mathcal{A}e^{iks} = \psi(ik)e^{iks}, \quad k \in \mathbb{R}.$$

For example, if $u(x)$ is a Brownian motion, $\psi(ik) = -\dfrac{k^2}{2}$.

The building blocks of our Levy processes



Pure drift

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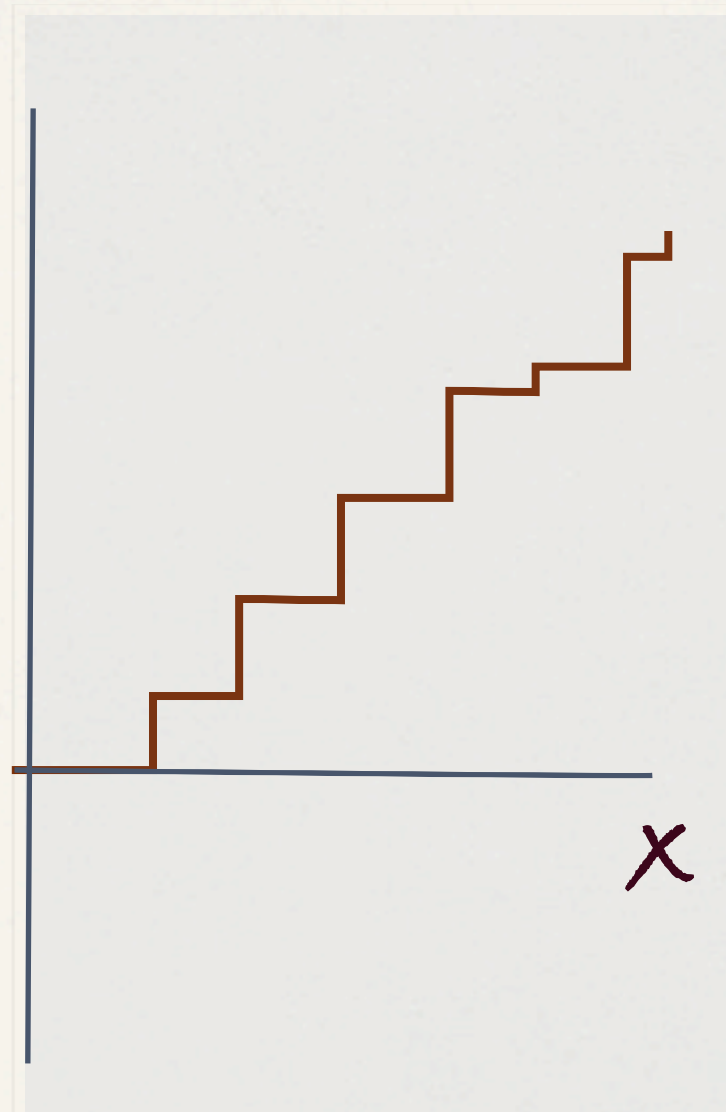

Increasing compound Poisson (inverse Lagrangian $a(x,t)$)

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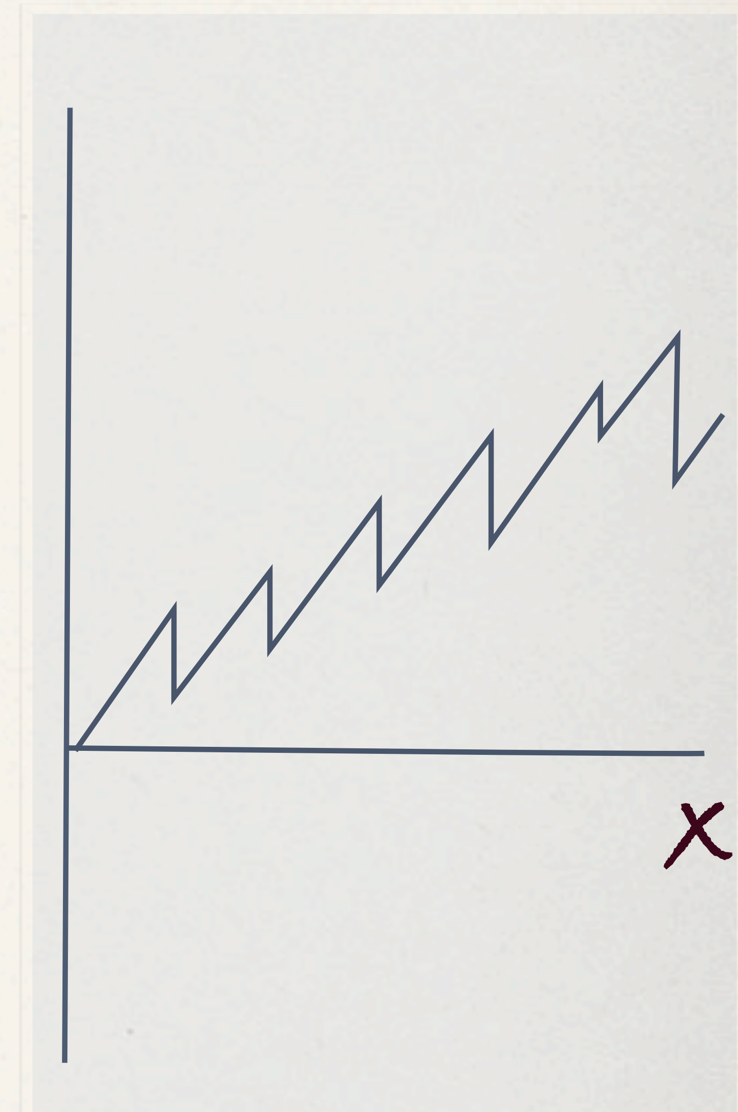


Velocity field

The Levy-Khintchine formula

The processes we consider have one-sided jumps because of the entropy condition. The multiplier ψ is called the Laplace exponent and has the representation:

$$\psi(q) = \int_0^\infty \left(e^{-qs} - 1 + qs\right) \Lambda(ds), \quad q \in \mathbb{C}_+.$$

The jump measure Λ describes the jumps in the compound Poisson process (in fact, by taking limits we do not need to assume that this measure is finite).

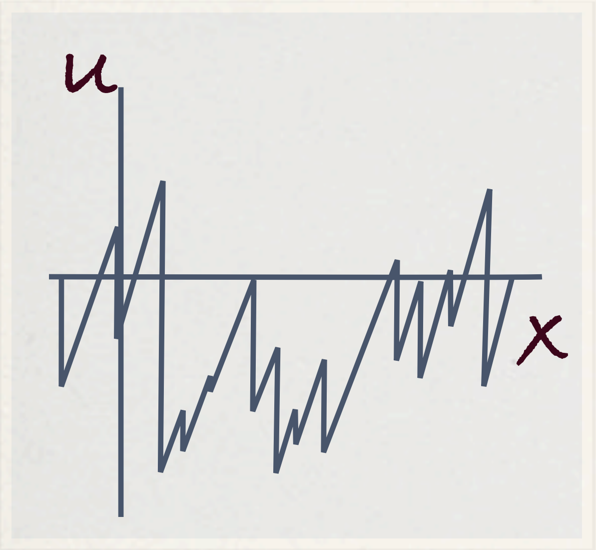
Closure for spectrally negative processes

The entropy solution to Burgers equation admits a remarkable closure property: Assume that the initial data is a Levy process with only downward jumps (spectrally negative). Then so is the solution for every $t > 0$.

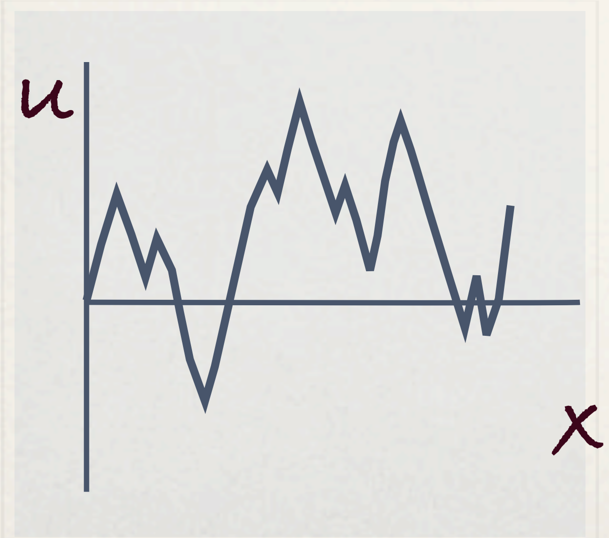
This was observed formally by Carraro and Duchon (1994), and proved rigorously by Bertoin (1998). It yields the exact solution for Brownian motion initial data, and a lot more.

Here by the term closure we mean that this class of random processes is preserved by the entropy solution.

The solution for spectrally negative Levy processes



Entropy solution to Burgers equation.



$$\begin{tikzcd} u(x,t) \arrow[rr, leftrightarrow, "\text{Closure theorem}"'] & & \psi(q,t) \\ u(x,0) \arrow[u] \arrow[rr, leftrightarrow, "\text{Closure theorem}"] & & \psi(q,0) \arrow[u, "??"'] \end{tikzcd}$$

The kinetic equation for clustering of shocks

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\psi(q,t) \arrow[rr, leftrightarrow, "\text{Levy-Khintchine formula}"'] & & \Lambda_t \\
\psi(q,0) \arrow[u, "\partial_t\psi + \psi\partial_q\psi = 0"] \arrow[rr, leftrightarrow, "\text{Levy-Khintchine formula}"] & & \Lambda_0 \arrow[u, "\text{Smoluchowski's coagulation equation with additive kernel}"']
\end{tikzcd}
$$

Assume the jump measure Λ has a density n. Then the Smoluchowski coagulation equation with additive kernel is

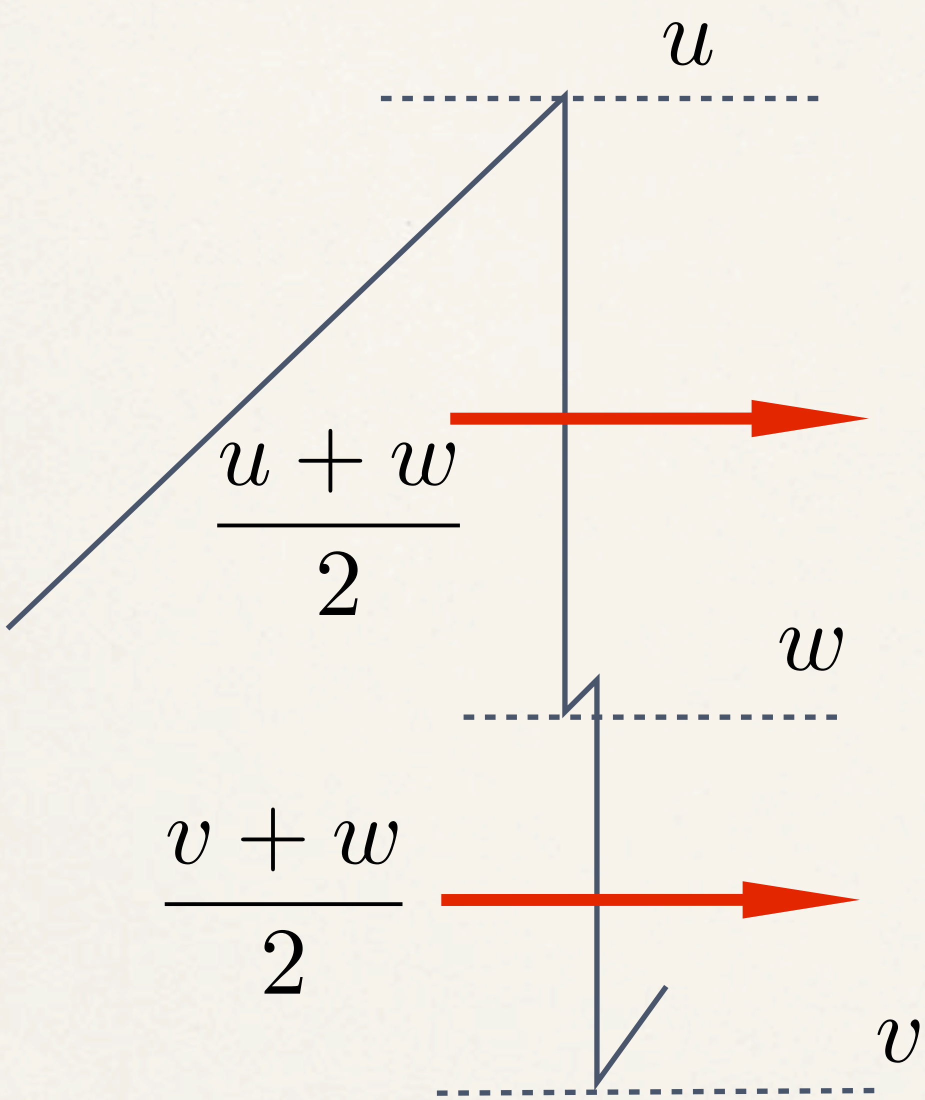
$$\partial_t n(t,s) = Q_1 - Q_2$$

$$Q_1 = \frac{s}{2}\int_0^s n(t,s')n(t,s-s')\,ds' \longleftarrow \text{Birth}$$

$$Q_2 = \int_0^\infty \frac{s+s'}{2} n(t,s)n(t,s')\,ds' \longleftarrow \text{Death}$$

If we take the Laplace exponent of this equation, we obtain Burgers equation. This has long been known (Golovin, 1963) but the link with Burgers turbulence is very recent.

An elementary argument for the additive kernel



1) Since u is a Levy process, shock sizes are independent of spatial location and velocity.

2) The shock speed is given by the Rankine-Hugoniot condition. Relative velocity in a collision is the sum of shock sizes.

It has taken a while for this simple description to arise. We can use it as a basis for a complete description of self-similarity in shock statistics, and various 'turbulence heuristics'. Our perspective, based on the closure theorem may be found in:

G. Menon and R. Pego, Commun. Math. Phys., 273 (2007).

Several precise links between stochastic coalescence and Burgers turbulence may be found in:

J. Bertoin, Some aspects of additive coalescents, Proc. ICM 2002.

White noise as initial velocity and general convex flux functions.

Despite the elegance of the previous solution, Levy processes are too rigid. The analysis does not apply to:

1) Noise initial data, in particular, for white noise, which was the initial motivation for Burgers study.

2) Any scalar conservation law aside from Burgers equation, even for Levy process initial data.

The Hopf-Lax formula

The entropy solution to the scalar conservation law

$$\partial_t u + \partial_x f(u) = 0, \quad x \in \mathbb{R}, t > 0,$$

with convex flux f is given by the Hopf-Lax formula

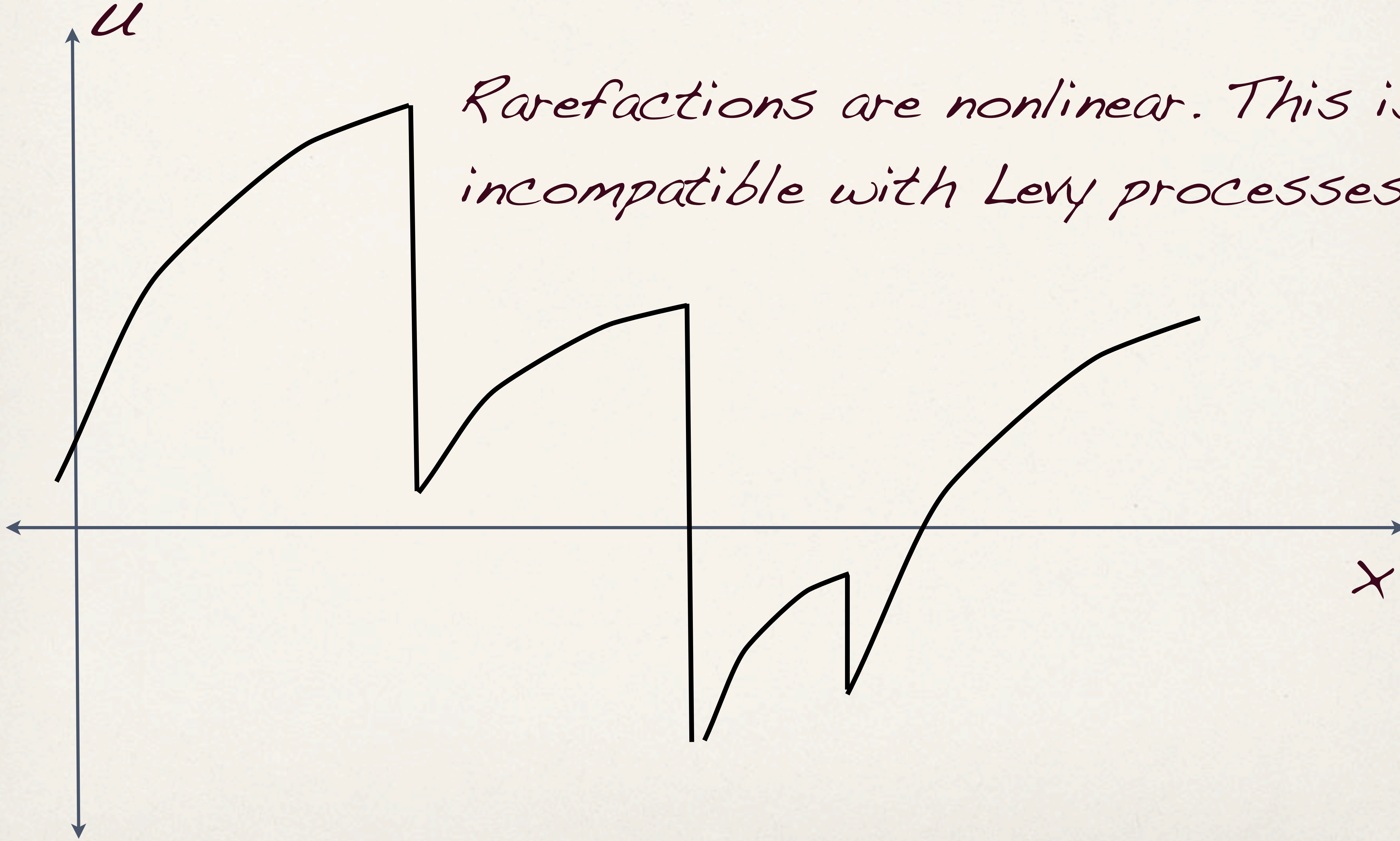
$$f'(u(x,t)) = \frac{x - a(x,t)}{t}$$

$$a(x,t) = \operatorname{argmin}_y^+ \left\{ U_0(y) + t f^* \left(\frac{x-y}{t} \right) \right\}.$$

Here f^* is the convex dual of f.

Typical profile of solutions

Rarefactions are nonlinear. This is incompatible with Levy processes.



The motivation for our work

1) Groeneboom's exact solution is a spectrally negative Markov process (but not Levy). In fact, he computed the generator of this process explictly (1988).

2) In a very interesting formal analysis, Chabanol and Duchon, (J. Stat. Phys. 114 (2004) found evolution equations (in t) for the generator of a velocity field that is Markov (in x). Our work relies strongly on this.

3) The precise links to coalescence for Levy processes, described earlier. This suggests a unified viewpoint.

Generators of spectrally negative Markov processes

A spectrally negative Feller process with BV sample paths has an infinitesimal generator of the form

$$\mathcal{A}\varphi(u) = \underbrace{b(u)}_{\text{Drift at level } u.}\varphi'(u) + \int_{-\infty}^{u} \underbrace{n(u, dv)}_{\text{Jumps from } u \text{ to } v.}\left(\varphi(v) - \varphi(u)\right).$$

Sample paths look locally like a Lévy process, but the infinitesimal 'Lévy' characteristics depend on the level u. This allows curved paths for example.

Closure theorems (Srinivasan, Ph.D thesis 2009)

$\partial_t u + \partial_x f(u) = 0.$ f strictly convex.

Thm. 1. Assume the initial velocity $u(x,0)$ is a spectrally negative Markov process. Then so is the entropy solution $u(x,t)$ for every $t>0$.

Thm. 2. Assume the initial potential $U(x,0)$ is a spectrally negative Lévy process. Then the entropy solution is a spectrally negative Markov process for every $t>0$.

Since the process is Markov, it has an infinitesimal generator that depends on (x,t). Conceptually, we have the following picture.

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
u(x,t) \arrow[r, leftrightarrow, "\text{Closure theorem}"'] & \mathcal{A}_{(x,t)} \\
u(x,0) \arrow[u, "\partial_t u + \partial_x f(u) = 0."] \arrow[r, leftrightarrow, "\text{Closure theorem}"] & \mathcal{A}_{(x,0)} \arrow[u, "??"']
\end{tikzcd}
$$

The `generator` in time

First recall the definition of the generator:

$$\mathcal{A}\varphi(u) = b(u)\varphi'(u) + \int_{-\infty}^{u} n(u, dv)\left(\varphi(v) - \varphi(u)\right).$$

Now define an associated operator (here f is the flux function in the scalar conservation law):

$$\mathcal{B}\varphi(u) = -f'(u)b(u)\varphi'(u) - \int_{-\infty}^{u} n(u, dv)\frac{f(v) - f(u)}{v - u}\left(\varphi(v) - \varphi(u)\right).$$

The operator $\mathcal{B}$ corresponds to the stochastic process $u(x,t)$ now viewed as a process in t.

u is not Markov in t, but this works! Simply consider how the paths evolve without collisions.

$f'(u)$ is the speed at level u, so we have

$\partial_t u = -f'(u)b(u)$ for evolution by the drift.

Similarly, $\dfrac{f(u)-f(v)}{u-v}$ is the speed of a shock connecting levels u and v.

The equation of evolution

Consider compatibility of the semigroups in x, t generated by the forward Kolmogorov equations

$$\partial_x \varphi = \mathcal{A}\varphi \qquad \text{and} \qquad \partial_t \varphi = \mathcal{B}\varphi \ .$$

That is, we impose $\partial_t \partial_x \varphi = \partial_x \partial_t \varphi$ to obtain the Lax equation

$$\partial_t \mathcal{A} - \partial_x B = [\mathcal{B}, \mathcal{A}].$$

The general kinetic equation for shock coalescence

The Lax equation expands into a kinetic equation. For simplicity, assume no x dependence (stationarity).

$$\partial_t b(u,t) = -b^2(u,t) f''(u)$$

$$\partial_t n(u,dv,t) = D_f(b,n) + Q_f(n,n)$$

$$D_f(b,n) = \partial_u \left\{ n(u,dv) b(u) \left(f'(u) - [f']_{u,v} \right) \right\} + \partial_v \left\{ n(u,dv) b(v) \left(f'(v) - [f']_{v,u} \right) \right\}$$

$$- n(u,dv) \left\{ b(u) f''(u) + b'(u) \left(f'(u) - [f']_{u,v} \right) \right\}$$

$$\begin{aligned} Q_f(n,n) = &\int_{(v,u)} n(u,dw) n(w,dv) \left([f']_{u,w} - [f']_{w,v} \right) \\ &- n(u,dv) \int_{(-\infty,v)} n(v,dw) \left([f']_{u,v} - [f']_{v,w} \right) \\ &- n(u,dv) \int_{(-\infty,u)} n(u,dw) \left([f']_{u,w} - [f']_{u,v} \right) \end{aligned}$$

$$[f']_{a,b} = \frac{f(a) - f(b)}{a - b}$$

The kinetic equation for clustering (Burgers)

$$\dot{b} = -b^2, \qquad \partial_t n(u, dv, t) = \underbrace{D(b,n)}_{\text{Drift}} + \underbrace{Q(n,n)}_{\text{Collisions}}$$

$$D(b,n) = \left(\frac{u-v}{2}\right)\left(b(u)\partial_u n - \partial_v(b(v)n)\right)$$

$$Q(n,n) = \frac{u-v}{2}\int_v^u n(u,dw)n(w,dv)$$

Birth

$$-n(u,dv)\int_{-\infty}^{v} n(v,dw)\left(\frac{u-w}{2}\right)$$

$$-n(u,dv)\int_{-\infty}^{u} n(u,dw)\left(\frac{w-v}{2}\right)$$

Death

Self-similar solution for white noise (Groeneboom '88)

$$b(u,t) = \frac{1}{t}, \quad n(u,v,t) = \frac{1}{t^{1/3}} n_* \left(\frac{u}{t^{2/3}}, \frac{v}{t^{2/3}} \right).$$

The profile decomposes into:

$$n_*(u,v) = \frac{J(v)}{J(u)} K(u-v),$$

where J and K have Laplace transforms:

$$j(q) = \frac{1}{Ai(q)}, \quad k(q) = 2\log(j)'' = -2\left(\frac{Ai'(q)}{Ai(q)}\right)'.$$

More on Groeneboom's solution

In order to verify that this is a solution we need to use some interesting identities. These are best written in terms of the variable $e = j'/j$. Then

$$e' = -q + e^2,$$

Riccati eqn.

$$k' = -2(1 - ek),$$

$$k''' = 6kk' + 4qk' + 2k.$$

These yield three moment identities, such as

$K * J(x) = x^2 J - J'$ and some amazing cancellations.

Complete integrability and the Painleve property

In fact, e is the first Airy solution to Painleve 2!

$$w'' = 2w^3 + 2wq + \frac{1}{2}.$$

Self-similar solutions to several completely integrable systems (Kdv, NLS, Sine-Gordon) can be expressed in terms of Painleve transcendents. They also appear in famous 'solvable' problems in mathematical physics. In particular, the celebrated Tracy-Widom distributions in random matrix theory are described by Painleve 2 with parameter 0.

Burgers turbulence and random matrices

The Wigner semicircle law is the law of large numbers for the distribution of eigenvalues of random matrices from the GOE, GUE and GSE ensembles. The Tracy-Widom distributions describe universal fluctuations around the largest eigenvalues.

I cannot connect Groeneboom's solution to Tracy-Widom, but I can connect the first exact solution (Cararro-Duchon, Bertoin) very nicely to the Wigner semicircle law through Dyson's Brownian motion!

Much remains to be done...

1) Rigorous analysis of state-dependent coagulation:

   a) Persistence of Feller property.

   b) Well-posedness of Lax equation.

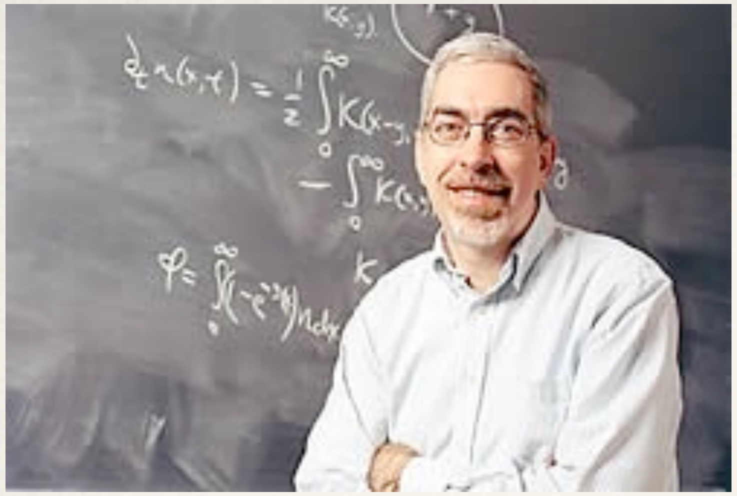
2) A deeper conceptual understanding of Lax equation:

   a) Inverse scattering/ Riemann-Hilbert problems.

   b) Connections to Tracy-Widom laws?

   c) Cube-root asymptotics in statistics.

Muchas Gracias!



Bob Pego

(Carnegie Mellon)

Ravi Srinivasan

(Brown)

John Hartigan

Mokshay Madiman (Yale)

David Pollard

Jonathan Mattingly

Scott McKinley

(Duke)

DMS 0605006, and

0748482 (career).