# Proof of the kinetic conjecture in a weakly nonlinear Schrödinger equation with random initial data


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# Outline



- Dynamics and initial state
- Related kinetic theory
- Not to be neglected: 1st order effects
- Observable and main theorem
- Proof:
  - Selective iteration of the Duhamel formula
  - Control of fast oscillations via momentum graphs

# Dynamics and initial state



- We work with *dimension* $d \geq 4$
- Dynamics determined by a *discrete nonlinear Schrödinger equation*
- Hamiltonian system with conservation of *energy and norm*
  - We choose initial data randomly distributed according to a corresponding *Gibbs measure* ("thermal equilibrium initial state")

The initial measure is invariant under both the time-evolution and lattice translations.

# Main notations



- Finite lattice: $L \geq 2\,,\quad \Lambda = \{0, 1, \ldots, L-1\}^d$
  - Periodic BC: *All arithmetic* mod $L$
- Dual lattice: $\Lambda^* = \{0, \frac{1}{L}, \ldots, \frac{L-1}{L}\}^d$
  - All arithmetic mod 1
  - *"Integration"* = finite sum:

$$\int_{\Lambda^*} \mathrm{d}k\, f(k) := \frac{1}{|\Lambda|} \sum_{k\in\Lambda^*} f(k)$$

- Fourier transform: $(x \in \Lambda,\ k \in \Lambda^*)$

$$\hat{f}(k) = \sum_{y\in\Lambda} f(y) \mathrm{e}^{-\mathrm{i}2\pi k\cdot y} \Rightarrow f(x) = \int_{\Lambda^*} \mathrm{d}k'\, \hat{f}(k') \mathrm{e}^{\mathrm{i}2\pi k'\cdot x}$$

# Evolution equations



## Discrete nonlinear Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\psi_t(x) = \sum_{y\in\Lambda} \alpha_\Lambda(x-y)\psi_t(y) + \lambda|\psi_t(x)|^2\psi_t(x)$$

- $\psi_t : \Lambda \to \mathbb{C}, \quad t \in \mathbb{R}$
- $\lambda > 0$
- Harmonic coupling determined by $\alpha : \mathbb{Z}^d \to \mathbb{R}$,

$$\alpha_\Lambda(x) = \sum_{n\in\mathbb{Z}^d} \alpha(x+nL)$$

- $\alpha$ *exponentially decaying* (example: nearest neighbor)
- We assume also $\alpha(-x) = \alpha(x)$

# Conservation laws



## Hamiltonian function

$$H_\Lambda(\psi) = \sum_{x,y\in\Lambda} \alpha_\Lambda(x-y)\psi(x)^*\psi(y) + \frac{1}{2}\lambda \sum_{x\in\Lambda} |\psi(x)|^4$$

- Relate $q_x, p_x \in \mathbb{R}$ to $\psi$ by $\psi(x) = \frac{1}{\sqrt{2}}(q_x + i p_x)$
- NLS equivalent to the Hamiltonian equations

$$\dot{q}_x = \partial_{p_x} H_\Lambda \,, \quad \dot{p}_x = -\partial_{q_x} H_\Lambda$$

- Thus $H_\Lambda(\psi_t)$ is conserved
- By explicit differentiation, also $\sum_x |\psi_t(x)|^2$ is conserved

# Initial state



**Probability distribution of $\psi = \psi_0$**

$$\frac{1}{Z^{\lambda}_{\beta,\mu}} e^{-\beta\left(H_\Lambda(\psi)-\mu\|\psi\|_2^2\right)} \prod_{x\in\Lambda} [\mathrm{d}(\mathrm{Re}\,\psi(x))\,\mathrm{d}(\mathrm{Im}\,\psi(x))]$$

- Define $\omega : \mathbb{T}^d \to \mathbb{R}$ by $\omega = \mathcal{F}_{x\to k}\alpha$.
- We consider only $\beta > 0$ and $\mu < \min_k \omega(k)$
  $\Rightarrow$ Also the Gaussian measure at $\lambda = 0$ is well-defined
- $Z^{\lambda}_{\beta,\mu} > 0$ is the normalization constant
- Let $\mathbb{E}$ denote expectation over the initial data

# Properties of the system



- The solution $\psi_t$ exists and is unique for all $t \in \mathbb{R}$ with any initial data $\psi_0 \in \mathbb{C}^\Lambda$. (conservation laws)
- Initial state is *stationary*: $\mathbb{E}[F(\psi_t)] = \mathbb{E}[F(\psi_0)]$
- Also invariant under periodic translations:

  $$\mathbb{E}[F(\tau_x\psi)] = \mathbb{E}[F(\psi)], \quad (\tau_x\psi)(y) = \psi(y+x)$$

- Translations commute with the time-evolution:

  $$\tau_x\psi_t = \tilde{\psi}_t|_{\tilde{\psi}_0=\tau_x\psi_0}$$

- *"Gauge invariance"*: similar invariance properties hold for translations of total phase, $\psi_0(x) \mapsto e^{i\varphi}\psi_0(x)$, $\varphi \in \mathbb{R}$.
- Thus, for instance, $\mathbb{E}[\psi_t] = 0$, $\mathbb{E}[\psi_{t'}\psi_t] = 0$,

  $$\mathbb{E}[\psi_{t'}(x')^*\psi_t(x)] = \mathbb{E}[\psi_0(0)^*\psi_{t-t'}(x-x')]$$

# Related kinetic theory



- The "unperturbed" system ($\lambda = 0$) has *harmonic dynamics* with initial data distributed according to a *Gaussian measure*.
- This system can be explicitly solved by Fourier-transform and has *wave-like solutions*.
- *Kinetic theory postulates* that for small perturbations, and suitable initial data, the effect of the perturbation amounts to *Boltzmann-type collisions* of the waves.
- An explicit *kinetic conjecture* based on perturbation expansions states that the approximation should become exact in the following limit:
  - Consider only space-time scales $\mathcal{O}(\lambda^{-2})$ (here $t = \tau\lambda^{-2}$ with $\tau$ fixed)
  - Take first $\Lambda \to \infty$, and then $\lambda \to 0$.

# Conjecture: Inhomogeneous Gaussian initial data



More explicitly:

- Suppose the initial state is Gaussian with slowly varying covariance,

$$\mathbb{E}[\psi_0(x')^*\psi_0(x)] \approx \tilde{W}_0\Big(\lambda^2\frac{x+x'}{2}, x-x'\Big)$$

where $W_0(y,k) \in \mathcal{S}(\mathbb{R}^d \times \mathbb{T}^d)$ and $\tilde{W}_0 := \mathcal{F}^{-1}_{k\to z} W_0$.

- Then the approximate form is retained for times $\mathcal{O}(\lambda^{-2})$, and $\mathbb{E}[\psi_t(x')^*\psi_t(x)]$, $t = \tau\lambda^{-2}$, can be approximated similarly using some $W_\tau(x,k)$
- $W_\tau(x,k)$ solves an inhomogeneous non-linear Boltzmann-Peierls equation

$$\partial_\tau W_\tau(x,k) + \frac{\nabla\omega(k)}{2\pi}\cdot\nabla_x W_\tau(x,k) = \mathcal{C}_{\mathrm{NL}}[W_\tau(x,\cdot)]$$

# Conjecture: Homogeneous Gaussian initial data



- Assume that the initial state is Gaussian and translation invariant
- Then there always is $\tilde{w}_t(x)$ such that

$$\mathbb{E}[\psi_t(x')^*\psi_t(x)] = \tilde{w}_t(x'-x)$$

- Kinetic conjecture: $W_\tau = \lim_{\lambda\to 0}\lim_{\Lambda\to\infty}(\mathcal{F}\tilde{w}_{\tau\lambda^{-2}})$ solves a homogeneous non-linear Boltzmann-Peierls equation

$$\partial_\tau W_\tau(k) = \mathcal{C}_{\rm NL}[W_\tau(\cdot)],$$

$$\mathcal{C}_{\rm NL}[h](k_1) = 4\pi\int_{(\mathbb{T}^d)^3} {\rm d}k_2{\rm d}k_3{\rm d}k_4\,\delta(k_1+k_2-k_3-k_4)$$
$$\times\,\delta(\omega_1+\omega_2-\omega_3-\omega_4)\,(h_2h_3h_4+h_1h_3h_4-h_1h_2h_3-h_1h_2h_4)$$

# Conjecture: Equilibrium time-correlations (this talk)



- Consider initial states which are stationary and translation invariant (*equilibrium states*)
- Consider a sequence of such states with $\lambda \to 0$ such that the limit is Gaussian and

$$\lim_{\lambda \to 0} \lim_{\Lambda \to \infty} \mathbb{E}^{\lambda}[\psi_0(0)^* \psi_0(x)] = \tilde{W}(x)$$

- Suppose also that $\mathcal{C}_{\mathrm{NL}}[W] = 0$ (true automatically?)
- In the present setup, $W(k) = \frac{1}{\beta(\omega(k)-\mu)}$, a smooth function.

**Kinetic conjecture**

Equilibrium time-correlations are determined by an operator $\mathcal{L}$, a linearization of $\mathcal{C}_{\mathrm{NL}}$ around $W$

- Energy-type correlations ($\sum \mathrm{Var}(\psi^*_{\tau\lambda^{-2}}\psi_{\tau\lambda^{-2}}, \psi^*_0\psi_0)$) should have a limit which follows time-evolution determined by $\mathrm{e}^{-\mathcal{L}\tau}$.
- Typically, $(\mathcal{L}h)(k) = V(k)h(k) - (Ah)(k)$, $V(k)$ some nice function and $A$ a compact operator.
- Field correlations $\mathbb{E}[\psi^*_0\psi_{\tau\lambda^{-2}}]$, should be have a limit whose decay is determined by $\mathrm{e}^{-V\tau}$
(linearization of the loss term of $\mathcal{C}_{\mathrm{NL}}$).

# Not to be neglected: 1st order effects



**Evolution of the Fourier-transform**

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\psi}_t(k) = -\mathrm{i}\omega(k)\widehat{\psi}_t(k) - \mathrm{i}\lambda \int_{(\Lambda^*)^3} \mathrm{d}k_1'\mathrm{d}k_2'\mathrm{d}k_3'\, \delta_\Lambda(k + k_1' - k_2' - k_3')\widehat{\psi}_t(k_1')^*\widehat{\psi}_t(k_2')\widehat{\psi}_t(k_3')$$

where

$$\delta_\Lambda(k) = |\Lambda|\mathbb{1}(k \bmod 1 = 0)\,.$$

Therefore, with $R_0 = 2\mathbb{E}[|\psi_0(0)|^2] \neq 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[\widehat{\psi}_0(k')^*\mathrm{e}^{\mathrm{i}t\omega(k)}\widehat{\psi}_t(k)\right]_{t=0} = -\mathrm{i}\lambda\delta_\Lambda(k' - k)\,[R_0 + \mathcal{O}(\lambda)]$$

Thus there is are corrections to the wave-evolution starting from the natural time-scale of the perturbation, $t \propto \lambda^{-1}$.

- In general, cannot be removed by a simple shift of $\omega$ (unlike for the earlier treated random linear perturbations)
- However, can be treated here by *truncation of "internal pairings":*

$$\widehat{\mathcal{P}}\,[a_1 a_2 a_3] := a_1 a_2 a_3 - \mathbb{E}[a_1 a_2]a_3 - \mathbb{E}[a_1 a_3]a_2 - \mathbb{E}[a_2 a_3]a_1$$

$$\begin{aligned} \Rightarrow \quad \widehat{\psi}_t(k_1')^*\widehat{\psi}_t(k_2')\widehat{\psi}_t(k_3') &= \widehat{\mathcal{P}}\big[\widehat{\psi}_t(k_1')^*\widehat{\psi}_t(k_2')\widehat{\psi}_t(k_3')\big] \\ &\quad + \delta_\Lambda(k_1' - k_2')\mathbb{E}[\psi_0(0)^*\widehat{\psi}_0(k_2')]\,\widehat{\psi}_t(k_3') \\ &\quad + \delta_\Lambda(k_1' - k_3')\mathbb{E}[\psi_0(0)^*\widehat{\psi}_0(k_3')]\,\widehat{\psi}_t(k_2') \end{aligned}$$

# Final complication: slow decay from coherent collisions 

- This would remove all $\mathcal{O}(\lambda)$ -effects.
- However, there is a bad set $S \subset (\mathbb{T}^d)^3$ of wave numbers with "degenerate" collisions.
- Let $\Phi_0^\lambda$ have support within $\lambda^{\frac{3}{4}}$-radius around $S$, with $\Phi_0^\lambda|_S = 1$. Define $\Phi_1^\lambda = 1 - \Phi_0^\lambda$.
- Insert $1 = \Phi_1^\lambda + \Phi_0^\lambda$ inside the $k'$-integral, and use the pairing truncation *only* for the second term (for the bad set).

# Summary: Fields suitable for a perturbation expansion



## (Re)definitions

$$\omega^\lambda(k) := \omega(k) + \lambda R_0$$

$$\widehat{a}_t(k,1) := \mathrm{e}^{\mathrm{i}t\omega^\lambda(k)}\widehat{\psi}_t(k)$$

$$\widehat{a}_t(k,-1) := \widehat{a}_t(-k,1)^*$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{a}_t(k,\sigma) = -\mathrm{i}\lambda\sigma\int_{(\Lambda^*)^3}\mathrm{d}k_1'\mathrm{d}k_2'\mathrm{d}k_3'\,\delta_\Lambda(k-k_1'-k_2'-k_3')\mathrm{e}^{-\mathrm{i}t\Omega(k',\sigma)}$$
$$\times\Big\{\Phi_1^\lambda(k')\widehat{a}_t(k_1',-1)\widehat{a}_t(k_2',\sigma)\widehat{a}_t(k_3',1)$$
$$+\Phi_0^\lambda(k')\widehat{\mathcal{P}}\big[\widehat{a}_t(k_1',-1)\widehat{a}_t(k_2',\sigma)\widehat{a}_t(k_3',1)\big]\Big\},\quad k\in\Lambda^*,\ \sigma\in\{\pm1\}$$

$$\Omega((k_1,k_2,k_3),\sigma) := \omega(k_3)-\omega(k_1)+\sigma\omega(k_2)-\sigma\omega(k_1+k_2+k_3)$$

# Observable and main theorem



Fix *test-functions* $f, g \in \ell_2$, and assume they have finite support.

## Observable

$$Q_\Lambda^\lambda(\tau) := \mathbb{E}[\langle \widehat{f}, \widehat{\psi}_0\rangle^* \langle \mathrm{e}^{-\mathrm{i}\omega^\lambda \tau\lambda^{-2}} \widehat{g}, \widehat{\psi}_{\tau\lambda^{-2}}\rangle] = \mathbb{E}[\langle \widehat{f}, \widehat{a}_0\rangle^* \langle \widehat{g}, \widehat{a}_{\tau\lambda^{-2}}\rangle]$$

Under additional assumptions on the *decay of equilibrium correlations* and on the *dispersion relation*:

## Theorem

There is $\tau_0 > 0$ such that for all $|\tau| < \tau_0$

$$\lim_{\lambda\to 0} \lim_{\Lambda\to\infty} Q_\Lambda^\lambda(\tau) = \int_{\mathbb{T}^d} \mathrm{d}k\, \widehat{g}(k)^* \widehat{f}(k) W(k) \mathrm{e}^{-\Gamma_1(k)|\tau| - \mathrm{i}\tau\Gamma_2(k)}$$

- $W(k) = (\beta(\omega(k) - \mu))^{-1}$ is the earlier $\lambda = 0$ covariance function.
- $\Gamma_j(k)$ are real, and $\Gamma(k) = \Gamma_1(k) + \mathrm{i}\Gamma_2(k)$ is given by

$$\Gamma(k_1) = -2\int_0^\infty \mathrm{d}t \int_{(\mathbb{T}^d)^3} \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4 \delta(k_1 + k_2 - k_3 - k_4) \times \mathrm{e}^{\mathrm{i}t(\omega_1+\omega_2-\omega_3-\omega_4)} (W_3 W_4 - W_2 W_4 - W_2 W_3)$$

with $\omega_i = \omega(k_i)$, $W_i = W(k_i)$.

$$\Rightarrow \quad \Gamma_1(k_1) = 2\pi \frac{1}{W(k_1)^2} \int_{(\mathbb{T}^d)^3} \mathrm{d}k_2 \mathrm{d}k_3 \mathrm{d}k_4 \delta(k_1 + k_2 - k_3 - k_4) \times \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \prod_{i=1}^{4} W(k_i)$$

# Summary of the main result



- *Loosely*: for all not too large $t = \mathcal{O}(\lambda^{-2})$,

$$\mathbb{E}[\widehat{\psi}_0(k')^* \widehat{\psi}_t(k)] \approx \delta_\Lambda(k'-k) W(k) \mathrm{e}^{-\mathrm{i}\omega^\lambda_{\text{ren}}(k)t} \mathrm{e}^{-\left|\lambda^2 t\right| \Gamma_1(k)}$$

- $\omega^\lambda_{\text{ren}}(k) = \omega(k) + \lambda R_0 + \lambda^2 \Gamma_2(k)$
- $2\Gamma_1(k) \geq 0$ coincides with the loss term of the linearization of $\mathcal{C}_{\text{NL}}$ around $W$

  $\Rightarrow$ the correlation decays exponentially, as dictated by $\mathrm{e}^{-|\lambda^2 t|\Gamma_1(k)}$.

- Nearest neighbor couplings satisfy all of our assumptions ($\omega_{\text{nn}}(k) = c - \sum_{\nu=1}^{d} \cos(2\pi k^\nu)$)

# Assumptions 1



## $\ell_1$-clustering of the equilibrium measure

- There exists $c_0 > 0$ such that for all sufficiently small $\lambda$ and for all $n \geq 4$ the fully truncated correlation functions (*cumulants*) satisfy

$$\sup_{\Lambda, \sigma \in \{\pm 1\}^n} \sum_{x \in \Lambda^n} \delta_\Lambda(x_1) \left| \mathbb{E}\left[ \prod_{i=1}^{n} a_0(x_i, \sigma_i) \right]^{\text{trunc}} \right| \leq \lambda c_0^n n!$$

- In addition, for $n = 2$:

$$\limsup_{L \to \infty} \sum_{\|x\|_\infty \leq L/2} \left| \mathbb{E}[\psi_0(0)^* \psi_0(x)] - \mathbb{E}[\psi_0(0)^* \psi_0(x)]_{L=\infty}^{\lambda=0} \right| \leq \lambda 2 c_0^2$$

- The expressions can be studied by cluster expansions. For nn-interactions, they hold by Abdesselam, *et al.*, 2009.

# Assumptions 2



## Sufficient dispersivity of the free evolution

1. $\omega$ is real-analytic and $\omega(-k) = \omega(k)$.
2. (*$\ell_3$-dispersivity*). Let $p_t(x)$ denote the *free propagator*. We assume that there are $C, \delta > 0$ such that

$$\|p_t\|_3^3 = \sum_{x \in \mathbb{Z}^d} |p_t(x)|^3 \leq C(1+|t|)^{-1-\delta}\,.$$

3. (*crossing bounds*) (Similar conditions involving propagators of dispersions of the type $k \mapsto \sum_{i=1}^{3} c_i \omega(k + k_i')$.)
4. (*constructive interference*) $M^{\rm sing} \subset \mathbb{T}^d$ is a union of finitely many closed, *one-dimensional* submanifolds, for which

$$\left| \int_{\mathbb{T}^d} \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}t(\omega(k) \pm \omega(k-k_0))} \right| \leq \frac{C}{d(k_0, M^{\rm sing})} (1+|t|)^{-1}\,.$$

where $d(k_0, M^{\rm sing})$ is the distance of $k_0$ from $M^{\rm sing}$.

Comments on the second assumptions:

- Condition 2 ($\ell_3$-dispersivity) guarantees that $\Gamma(k)$ is well-defined.
- $M^{\text{sing}} \neq \emptyset$, since it must contain at least 0.
- The "bad set" $S$ used in the definition of the cutoff function $\Phi_0^\lambda$ consists of all $(k_1, k_2, k_3)$ such that $k_i + k_j \in M^{\text{sing}}$ for some $i, j$.
- Checking the assumptions can be tricky, in general (harmonic analysis?)

# Outline of the proof



1. Show that it is enough to prove the result assuming $t > 0$
2. Iterate a Duhamel formula $N_0(\lambda)$ times to expand $a_t$ into a perturbation sum (we choose $N_0! \approx \lambda^{-p}$, for a small $p$)
3. There are two types of terms in the expansion:

   **Main terms** These will contain a finite monomial of $a_0$ whose expectation can be evaluated using the "moments to cumulants formula".

   **Error terms** These will involve also $a_s$ for some $s > 0$. The expection is estimated by a Schwarz bound and stationarity of the equilibrium measure
   $\Rightarrow$ The bound involves again only finite moments of $a_0$.

4. Each cumulant induces linear dependencies between the wave vectors. These can be encoded in "Feynman graphs".
5. This results in a sum with roughly $(N_0!)^2$ non-zero terms. However, most of these vanish in the limit $\lambda \to 0$, due to oscillating phase factors.
6. Careful classification of graphs: we use a special resolution of the wave vector constraints which allows an estimation based on identifying, and iteratively estimating, certain *graph motives*.
7. Only a small fraction of the graphs (*leading graphs*) will remain. These consist of graphs obtained by iterative addition of one of the 20 *leading motives*.
8. The limit of the leading graphs is explicitly computable, and their sum yields the result in the main theorem.

# Main ingredients I

## The Duhamel formula



### Duhamel formula for monomials

$$
\begin{aligned}
\prod_{i=1}^{n} \widehat{a}_t(k_i,\sigma_i) &= \prod_{i=1}^{n} \widehat{a}_0(k_i,\sigma_i) \\
&\quad - \mathrm{i}\lambda \sum_{j=1}^{n} \sigma_j \int_0^t \mathrm{d}s \int_{(\Lambda^*)^3} \mathrm{d}k'\, \delta_\Lambda(k_j - k_1' - k_2' - k_3') \mathrm{e}^{-\mathrm{i}s\Omega(k',\sigma_j)} \\
&\qquad \times \prod_{i=1; i\neq j}^{n} \widehat{a}_s(k_i,\sigma_i) \Big\{ \Phi_1^\lambda(k') \widehat{a}_s(k_1',-1) \widehat{a}_s(k_2',\sigma_j) \widehat{a}_s(k_3',1) \\
&\qquad\qquad + \Phi_0^\lambda(k') \widehat{\mathcal{P}}\big[\widehat{a}_s(k_1',-1) \widehat{a}_s(k_2',\sigma_j) \widehat{a}_s(k_3',1)\big] \Big\}
\end{aligned}
$$

# Main ingredients II

## Moments to cumulants formula



### Cumulant expansion

For any index set $I$,

$$\mathbb{E}\Big[\prod_{i\in I}\widehat{a}_0(k_i,\sigma_i)\Big] = \sum_{S\in\pi(I)}\prod_{A\in S}\Big[\delta_\Lambda\Big(\sum_{i\in A}k_i\Big)C_{|A|}(k_A,\sigma_A)\Big],$$

where the sum runs over all partitions $S$ of the index set $I$.

- Assumption 1 $\Rightarrow$ uniform bound on the *cumulant functions*

$$C_n(k,\sigma) := \sum_{x\in\Lambda^n}\delta_\Lambda(x_1)\mathrm{e}^{-\mathrm{i}2\pi\sum_{i=1}^n x_i\cdot k_i}\mathbb{E}\Big[\prod_{i=1}^n a_0(x_i,\sigma_i)\Big]^{\text{trunc}}.$$

- Thus the cumulant expansion encodes all singularities of the moments.

# Main ingredients III

## Upper bounds of error terms



- All error terms are of the form

$$\int_0^{\tau\lambda^{-2}} \mathrm{d}s\, \mathbb{E}\Big[ \langle \widehat{f}, \widehat{a}_0 \rangle^* F_s[\widehat{a}_s] \Big] \ ,$$

where $F_s$ contains only a finite moment of the fields $\widehat{a}_s$.

- By Schwarz inequality, ($t = \tau\lambda^{-2}$)

$$\Big| \int_0^t \mathrm{d}s\, \mathbb{E}\Big[ \langle \widehat{f}, \widehat{a}_0 \rangle^* F_s[\widehat{a}_s] \Big] \Big|^2 \le t \mathbb{E}[|\langle \widehat{f}, \widehat{a}_0 \rangle|^2] \int_0^t \mathrm{d}s\, \mathbb{E}\big[ |F_s[\widehat{a}_s]|^2 \big]$$

- By stationarity of the initial measure,

$$\mathbb{E}\big[ |F_s[\widehat{a}_s]|^2 \big] = \mathbb{E}\big[ |F_s[\mathrm{e}^{\mathrm{i}s\omega^\lambda} \widehat{\psi}_0]|^2 \big]$$

# Main ingredients IV

## From phases to resolvents



### Representation of oscillating phase factors

Let $\gamma_i \in D$, $i \in I$, with $D \subset \mathbb{C}$ compact. For any $A \subset I$, non-empty, define $A' = \{*\} \cup I \setminus A$. Then for any path $\Gamma_D$ going once anticlockwise around $D$, we have

$$\int_{(\mathbb{R}_+)^I} \mathrm{d}s\, \delta\Big(t - \sum_{i\in I} s_i\Big) \prod_{i\in I} \mathrm{e}^{-\mathrm{i}\gamma_i s_i}$$

$$= -\oint_{\Gamma_D} \frac{\mathrm{d}z}{2\pi} \int_{(\mathbb{R}_+)^{A'}} \mathrm{d}s\, \delta\Big(t - \sum_{i\in A'} s_i\Big) \prod_{i\in A'} \mathrm{e}^{-\mathrm{i}\gamma_i s_i}\Big|_{\gamma_* = z} \prod_{i\in A} \frac{\mathrm{i}}{z - \gamma_i}$$

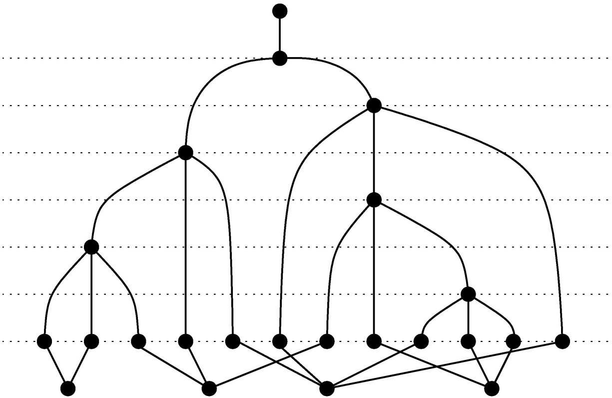
- Application: Take an absolute value inside the second integrals $\Rightarrow$ Removes all phase factors with indices $i \notin A$.

# Main ingredients V

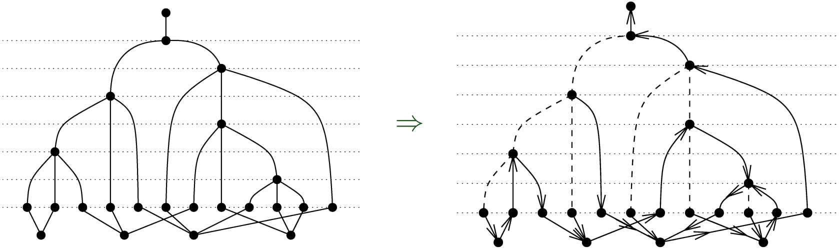
## Momentum graphs



An example of a momentum graph:



Resolution of the momentum constraints:



# Comments



- Infinite volume limit: can start from $\mathbb{Z}^d$ instead of $\Lambda$.
- $d < 4$? ($d = 3$ likely OK, how about $d = 1$??)
- Which interactions satisfy the assumptions? ($\ell_1$-clustering and oscillatory integrals)
- Decay of *energy-type* correlations: $\frac{1}{|\Lambda|}\langle H(0)H(t)\rangle$?
  $\Rightarrow$ Full linearized BE
- Non-stationary initial state? $\Rightarrow$ Nonlinear BE

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