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The Markov Sequence Problem for Jacobi Polynomials

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Markov sequence problem

$$(X, \mathcal{S}, \mu) \text{ probability space}$$

Selfadjoint Markov operator

$$T : L^2(\mu) \to L^2(\mu)$$

linear, positivity preserving, $T1 = 1$. selfadjoint

$$\{f_n\}_{n=0}^{\infty}\ ,\ f_0 = 1$$

unit orthonormal basis

The selfadjoint Markov operators that have $\{f_n\}_{n=0}^{\infty}$ as eigenfunctions form a convex set.

Problem: Find the extreme points. Equivalently, find the extreme points of $\mathcal{M}$, the set of all Markov sequences that are the eigenvalues of a selfadjoint Markov operator that have $\{f_n\}_{n=0}^{\infty}$ as eigenfunctions.

0.1 THEOREM (Bakry–Huet (2006)). *Let X be a closed interval in $\mathbb{R}$, and let μ be a regular Borel probability measure whose support is X. Let $\{f_n\}_{n\geq 0}$ be a unit orthonormal basis for $L^2(\mu)$ consisting of real valued functions. Suppose there exists some $x_0 \in X$ such that for each $x \in X$,*

$$\lambda_n(x) := \frac{f_n(x)}{f_n(x_0)}$$

is a Markov sequence for $\{f_n\}_{n\geq 0}$. Then $\{\lambda_n\}_{n\geq 0} \in \mathcal{M}$, the set of all Markov sequences for $\{f_n\}_{n\geq 0}$, if and only if there exists a Borel probability measure ν so that

$$\lambda_n = \int_X \frac{f_n(x)}{f_n(x_0)} \mathrm{d}\nu(x) \ . \tag{0.1}$$

Finally, if $\{f_n\}_{n\geq 0}$ is a sequence of bounded continuous functions whose finite linear combinations are dense in $\mathcal{C}_b(X)$, then the probability measure ν in (0.1) is unique, so that $\mathcal{M}$ is a simplex and the $\{f_n(x)/f_n(x_0)\}_{n\geq 0}$ are its extreme points.

The elements of (0.1) are the eigenvalues of

$$K := \int_X K_z \mathrm{d}\nu(z)$$

where

$$K_z(x,y) = \sum_{n=0}^{\infty} \lambda_n(z) f_n(x) f_n(y) \ .$$

Conversely if K is Markov and selfadjoint then

$$K(x,y) := \sum_{n=0}^{\infty} \lambda_n f_n(x) f_n(y)$$

and

$$\begin{aligned}
\int_X f_k(x)\mathrm{d}\nu(x) &= \int_X f_k(x)\left[\sum_{n=0}^{\infty} \lambda_n f_n(x) f_n(x_0)\mathrm{d}\mu\right] \\
&= \sum_{n=0}^{\infty}\left[\int_X f_k(x) f_n(x)\mathrm{d}\mu\right] \lambda_n f_n(x_0) \\
&= \lambda_k f_k(x_0) \ ,
\end{aligned} \tag{0.2}$$

where

$$\mathrm{d}\nu = K(x, x_0)\mathrm{d}\mu$$

is a probability measure.

Note the 'Hyper group property'

$$\sum_{n=0}^{\infty} \frac{f_n(z)f_n(x)f_n(y)}{f_n(x_0)} \geq 0$$

Set

$$\mu_{x,y}(dz) = \sum_{n=0}^{\infty} \frac{f_n(z)f_n(x)f_n(y)}{f_n(x_0)}$$

$$F_n(x) = \frac{f_n(x)}{f_n(x_0)}$$

Then

$$F_n(x)F_n(y) = \int F_n(z)\mu_{x,y}(dz)$$

λ, ν measures. Define the convolution

$$\lambda \star \nu := \int \mu_{x,y}(dz)\lambda(dx)\nu(dy)$$

Commutative Banach Algebra.

$$\int F_n \lambda \int F_n \nu = \int F_n \lambda \star \nu$$

The ultraspherical polynomials $\{p_n^{(\gamma)}\}_{n\geq 0}$ are the *orthonormal polynomials* on $[-1,1]$, for the measure $\mu^{(\gamma)}$

$$\mathrm{d}\mu^{(\gamma)}(t) = c_\gamma(1-t^2)^{\gamma-1/2}\mathrm{d}t$$

where

$$c_\gamma = \frac{1}{\sqrt{\pi}}\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1/2)}$$

For each $\gamma > 0$, and $a \in (-1,1)$, define an operator K_a on $L^2(\mu^{(\gamma)})$ by

$$K_a f(t) = \int_{-1}^{1} f\left(at + s\sqrt{1-a^2}\sqrt{1-t^2}\right)\mathrm{d}\mu^{(\gamma-1/2)} \ .$$

0.2 THEOREM. *(Bochner (1956)) For any* $\gamma > 0$*, the sequence* $\{\lambda_n\}_{n\geq 0}$ *is a Markov sequence for* $\{p_n^{(\gamma)}\}_{n\geq 0}$ *if and only if there is a probability measure* ν *on* $[-1,1]$ *such that*

$$\lambda_n = \int_{-1}^{1} \frac{p_n^{(\gamma)}(t)}{p_n^{(\gamma)}(1)}\mathrm{d}\nu(t) \ .$$

For each such Markov sequence $\{\lambda_n\}_{n\geq 0}$*, the measure* ν *is unique. In other words, for each* t*,* $\{p_n^{(\gamma)}(t)/p_n^{(\gamma)}(1)\}_{n\geq 0}$ *is a Markov sequence for* $\{p_n^{(\gamma)}\}_{n\geq 0}$*, and these are the extreme points of the set* $\mathcal{M}$ *of all such Markov sequences.*

• $K_a 1 = 1$, K_a is selfadjoint, with respect to $\mathrm{d}\mu^{(\gamma)}(t)$ in fact

$$\langle K_a f, g\rangle_{L^2(\mu^{(\gamma)})} =$$
$$c_\gamma c_{\gamma-1/2} \int_{-1}^{1}\int_{-1}^{1} g(t) f(u) \frac{(1-a^2-u^2-t^2+2atu)_+^{\gamma-1}}{(1-a^2)^{\gamma-1/2}} \mathrm{d}u\mathrm{d}t$$

• K_a preserves polynomials, hence the ultraspherical polynomials are eigenfunctions.

$$\lambda_n p_n^{(\gamma)}(t) = \int_{-1}^{1} p_n^{(\gamma)}\left(at + s\sqrt{1-a^2}\sqrt{1-t^2}\right) \mathrm{d}\mu^{(\gamma-1/2)}$$

• Evaluation formula: Take $t \to 1$ and obtain

$$\lambda_n p_n^{(\gamma)}(1) = p_n^{(\gamma)}(a)$$

or

$$\frac{p_n^{(\gamma)}(a) p_n^{(\gamma)}(t)}{p_n^{(\gamma)}(1)} = \int_{-1}^{1} p_n^{(\gamma)}\left(at + s\sqrt{1-a^2}\sqrt{1-t^2}\right) \mathrm{d}\mu^{(\gamma-1/2)}$$

the Gegenbauer product formula (1875) and Bochner's starting point.

Thus according to Bakry–Huet K_a are the extremal self-adjoint Markov operators the eigenvalues of a selfadjoint Markov operator whose eigenfunctions are the Ultraspherical polynomials are given as convex combination of

$$\frac{p_n^{(\gamma)}(a)}{p_n^{(\gamma)}(1)}$$

As a consequence

$$\sum_n \frac{p_n^{(\gamma)}(a)p_n^{(\gamma)}(t)p_n^{(\gamma)}(u)}{p_n^{(\gamma)}(1)} = \frac{((1-a^2)-(u^2+t^2-2atu))_+^{\gamma-1}}{(1-a^2)^{\gamma-1/2}}$$

As a corollary we have Laplace's formula for the ultraspherical polynomials

$$\frac{p_n^{(\gamma)}(t)}{p_n^{(\gamma)}(1)} = \int_{-1}^{1}(t+is\sqrt{1-t^2})^n \mathrm{d}\mu^{(\gamma-1/2)}$$

Where does the K_a come from?

Kac' collision model

$$Q\phi(\vec{v}) = \left(\begin{array}{c} N \\ 2 \end{array}\right)^{-1} \sum_{i<j} \frac{1}{2\pi}\int_0^{2\pi} \phi(R_{i,j,\theta}(\vec{v}))d\theta.$$

$$\exp(N(Q-I)t)$$

Gap Δ_N satisfies the recursion relation

$$\Delta_N \geq \delta_N \Delta_{N-1}$$

(Carlen-Carvalho-Loss (2000)) where δ_N is the gap of the operator defined by the bilinear form

$$\int_{S^{N-1}} f(x_1)g(x_2)\mathrm{d}\sigma_N$$

$\mathrm{d}\sigma_N$ is the uniform normalized surface measure of $\mathbb{S}^{N-1}$.

$$\langle K_a f, g\rangle_{L^2(\mu^{((N-2)/2)})} = \int_{\substack{\mathbb{S}^{N-1} \\ \widehat{u}_1 \cdot \widehat{u}_2 = a}} f(x\cdot\widehat{u}_1)g(x\cdot\widehat{u}_2)\mathrm{d}\sigma_N \ .$$

Clearly, K_a is selfadjoint and a simple calculation leads to the previous explicit formula for K_a for the special value $\gamma = (N-2)/2$.

Likewise the Kac model with three dimensional collisions (CCL(2003), Carlen-Geronimo-Loss (2008)) has a gap Δ_N that satisfies the same recursive inequality, i.e.,

$$\Delta_N \geq \Delta_{N-1}\delta_N$$

where this time δ_n is the gap of another operator $\mathcal{K}_a$. We replace 3 by m in what follows.

$$[x] = [x_1, \dots, x_N] \ , \ x_i \in \mathbb{R}^m$$

$$[x] : \mathbb{R}^N \to \mathbb{R}^m \ , \ u \to [x]u$$

usual matrix product. If $(x_1, \cdots x_N) \in \mathbb{S}^{mN-1}$ then $[x]u \in B^m$, the unit ball in $\mathbb{R}^m$.

For any $N > 2$ and $m > 1$, and any $-1 < a < 1$, define

$$\langle f, \mathcal{K}_a g\rangle_{L^2(B^m,\nu_{m,N})} = \int_{\mathbb{S}^{mN-1}} f([x]\widehat{u}_1)g([x]\widehat{u}_2)\mathrm{d}\sigma_{mN} \ ,$$

$$\widehat{u}_1 \cdot \widehat{u}_2 = a \ .$$

A straightforward calculation yields

$$\mathcal{K}_a f(v) = \int_B f\left(av + \sqrt{1-a^2}\sqrt{1-|v|^2}y\right)\mathrm{d}\nu_{m,N-1}(y)$$

where

$$\mathrm{d}\nu_{m,N}(v) = \frac{|S^{m(N-1)-1}|}{|S^{mN-1}|}(1-|v|^2)^{(m(N-1)-2)/2}\mathrm{d}v$$

Jacobi Polynomials $p_n^{(\alpha.\beta)}$ on $[-1,1]$, orthogonal with respect to the measure

$$\mu^{\alpha,\beta}(\mathrm{d}x) = c_{\alpha,\beta}(1-x)^\alpha(1+x)^\beta\mathrm{d}x$$

$\mathcal{K}_a$ commutes with rotations.

$$(K_{a,0}h)(2|v|^2-1) := (\mathcal{K}_a f)(v) \qquad \text{where} \qquad f(v) := h(2|v|^2-1) \ .$$

$$\alpha = (m(N-2)-2)/2 \quad \text{and} \quad \beta = (m-2)/2$$

$$(K_{a,0}h)(t) = \int_0^1 \int_0^\pi dm_{\alpha,\beta}(r,\theta)$$
$$h\left[(a^2(1+t)-1)+b^2(1-t)r^2+2abr(1-t^2)^{1/2}\cos\theta\right]$$

$$\begin{aligned} \mathrm{d}m_{\alpha,\beta}(r,\theta) \;&=\; \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\beta+1/2)\Gamma(\alpha-\beta)} \\ &\times\; (1-r^2)^{\alpha-\beta-1} r^{2\beta+1} \sin^{2\beta}\theta \mathrm{d}r\mathrm{d}\theta \end{aligned}$$

For $\alpha > \beta > -1/2$

- $K_{a,0}$ is selfadjoint on $L^2(\mu^{\alpha,\beta})$

This is fairly obvious for the special values of α and β.

- Preserves the space of polynomials of fixed degree

- $\lim_{t\to 1} K_{a,0}h(t) = h(2a^2-1)$

As a consequence we have

0.3 THEOREM (Gasper (1971/1972)). *For $\alpha \geq \beta$ with $\beta > -1/2$ or $\alpha > \beta$ with $\beta = -1/2$ the sequence $\{\lambda_n\}_{n\geq 0}$ is a Markov sequence for $\{p_n^{(\alpha,\beta)}\}_{n\geq 0}$, if and only if there is a probability measure ν on $[-1,1]$ such that*

$$\lambda_n = \int_{-1}^{1} \frac{p_n^{(\alpha,\beta)}(x)}{p_n^{(\alpha,\beta)}(1)} \mathrm{d}\nu(x) \ .$$

For each such Markov sequence $\{\lambda_n\}_{n\geq 0}$, the measure ν is unique. In other words, for each t, $\{p_n^{(\alpha,\beta)}(t)/p_n^{(\alpha,\beta)}(1)\}_{n\geq 0}$ is a Markov sequence for $\{p_n^{(\gamma)}\}_{n\geq 0}$, and these are the extreme points of the set $\mathcal{M}$ of all such Markov sequences.

Selfadjointness follows from

$$\begin{aligned} &(K_{a,0}h_1, h_2) = \\ &b^{-2\alpha} \int_0^1 \int_0^1 \int_0^\pi h_1(2s^2-1) h_2(2\rho^2-1) \\ \times\ &(b^2 - s^2 - \rho^2 + 2a\rho s \cos\phi)_+^{\alpha-\beta-1} \rho^{2\beta+1} \\ &s^{2\beta+1} \sin^{2\beta}\phi \mathrm{d}\phi \, \mathrm{d}\rho \, \mathrm{d}s \,, \end{aligned}$$

Define

$(K_{a,\ell}h)(2|v|^2-1)H(v) = (\mathcal{K}_a f)(v)$ where $f(v) := h(2|v|^2-1)H(v)$

$H(v)$ harmonic polynomial of degree ℓ.

0.4 THEOREM. *For all $\alpha > \beta > -1/2$, and all non negative integers ℓ,*

$$a^\ell \frac{p_n^{(\alpha,\beta+\ell)}(t)}{p_n^{(\alpha,\beta+\ell)}(1)} p_n^{(\alpha,\beta+\ell)}(2a^2-1)$$
$$= \int_0^1 \int_0^\pi p_n^{\alpha,\beta+\ell} \left(\left[a^2(1+t) + b^2(1-t)r^2 + 2ab\sqrt{1-t^2} r \cos\theta \right] \right.$$
$$\times \left[\sum_{j=0}^{\ell} \binom{\ell}{j} a^{\ell-j} (br)^j \left(\frac{1-t}{1+t} \right)^{j/2} P_j^{(\beta)}(\cos\theta) \right] \mathrm{d}m_{\alpha,\beta}(r,\theta) \ ,$$

where $b = \sqrt{1-a^2}$ and $P_\ell^{(\beta)}$ is the ultraspherical polynomial with the normalization $P_\ell^{(\beta)}(1) = 1$.

0.5 COROLLARY. *Consider any $\alpha > \beta > -1/2$ and any integer $\ell \geq 0$. Then for all $t \in [-1,1]$,*

$$\frac{p_n^{(\alpha,\beta+\ell)}(t)}{p_n^{(\alpha,\beta+\ell)}(1)} = \int_0^1 \int_0^\pi \left[\frac{(1+t)-(1-t)r^2}{2} + i\sqrt{1-t^2} r \cos\theta \right]^n$$
$$\times \left[\sum_{k=0}^{\ell} \binom{\ell}{k} \left(\frac{1-t}{1+t} \right)^{k/2} (ir)^k P_k^{(\beta)}(\cos\theta) \right]$$
$$\times \mathrm{d}m_{\alpha,\beta}(r,\theta) \ .$$

where $\mathrm{d}m_{\alpha,\beta}$ is given as before.

Due to Koornwinder for $\ell = 0$.

The product formula for $\ell \neq 0$ has appeared in different form in a paper by Koornwinder and Schwartz. It has as a consequence the product formula for the orthogonal polynomials on the parabolic biangle.

$$B = \{(x_1, x_2) : 0 \leq x_1 \leq 1, 0 \leq x_2^2 \leq x_1\}$$

$$d\nu^{\alpha,\beta} = C_{\alpha,\beta}(1 - x_1)^{\alpha}(x_1 - x_2^2)^{\beta} dx_1 dx_2$$

The polynomials, orthogonal in this inner product, are denoted by

$$R_{n,k}^{\alpha,\beta}(x_1, x_2)$$

$$R_{n,k}^{\alpha,\beta}(x_1, x_2) = P_{n-k}^{(\alpha,\beta+k+1/2)}(2x_1 - 1)x_1^{k/2} P_k^{(\beta)}(x_1^{-1/2} x_2)$$

where $P_n^{\alpha,\beta}(x)$ as well as $P_n^{\beta}(x)$ are normalized, Jacobi, resp. Ultraspherical polynomials $P_n^{\alpha,\beta}(1) = 1, P_n^{\beta}(1) = 1$.

0.6 THEOREM. *(Koornwinder and Schwartz (1997))Let $\alpha \geq \beta + 1/2 \geq 0$. Then*

$$\begin{aligned} & R_{n,k}^{\alpha,\beta}(x_1^2, x_2) R_{n,k}^{\alpha,\beta}(y_1^2, y_2) \\ = & \int_{I \times J^3} R_{n,k}^{\alpha,\beta}(E^2, EG) d\nu^{\alpha,\beta}(r_1, \psi_1, \psi_2, \psi_3) \quad (0.3) \end{aligned}$$

Here

$$\begin{aligned} & d\nu^{\alpha,\beta}(r_1, \psi_1, \psi_2, \psi_3) \\ = & \; dm^{\alpha,\beta+1/2}(r_1, \psi_1) dm^{\beta-1/2}(\psi_2) dm^{\beta-1/2}(\psi_3) \end{aligned}$$

$$dm^{\beta}(\psi) = c_{\beta}(\sin \psi)^{2\beta+1} d\psi$$

$$dm^{\alpha,\beta}(r,\psi) = c_{\alpha,\beta}(1-r^2)^{\alpha-\beta-1}r^{2\beta+1}drdm^{\beta-1/2}(\psi)$$

and

$$\begin{aligned} E &= E(x_1,y_1;r,\psi)^2 \\ &= (x_1^2y_1^2 + (1-x_1^2)(1-y_1^2)r^2 \\ &+ 2x_1y_1(1-x_1^2)^{1/2}(1-y_1^2)^{1/2}r\cos\psi \end{aligned}$$

and

$$\begin{aligned} G &= G(x_1,x_2,y_1,y_2;r_1,\psi_1,\psi_2,\psi_3)) \\ &= D(C, D(\frac{x_2}{x_1},\frac{y_2}{y_1};1,\psi_2);1,\psi_3)\ , \end{aligned}$$

$$C = \frac{D(x_1,y_1;r_1,\psi_1)}{E(x_1,y_1;r_1,\psi_1)}$$

where generally

$$D(x,y;r,\psi) = xy + (1-x^2)^{1/2}(1-y^2)^{1/2}r\cos\psi\ .$$

Define

$$(T_{y_1,y_2}f)(x_1,x_2) := \int_{I\times J^3} f(E^2,EG)d\nu^{\alpha,\beta}(r_1,\psi_1,\psi_2,\psi_3)$$

- T_{y_1,y_2} is selfadjoint on $L^2(d\nu^{\alpha,\beta})$, $T_{y_1,y_2}1 = 1$
- T_{y_1,y_2} preserves polynomials of a given degree.
- $\lim_{x_2\to 1}(T_{y_1,y_2}f)(x_1,x_2) = f(1,1)$