The Oort-Hulst-Safronov coagulation equation: finite speed of propagation and self-similar solutions

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Outline

1. The Oort-Hulst-Safronov (OHS) coagulation equation
2. The constant kernel: $a(x, x_*) = 1$
3. The sum kernel $a(x, x_*) = x^\lambda + x_*^\lambda$, $0 < \lambda < 1$
4. The multiplicative kernel: $a(x, x_*) = xx_*$

Outline

The OHS coagulation equation

Mean-field model for the aggregation of stellar objects in astrophysics [Oort & van de Hulst (1946), Safronov (1972)].
Size distribution function: $f(t, x) \geq 0$
($t > 0$=time, $x \in (0, \infty)$=size).

$$\begin{aligned}\partial_t f(t,x) &= -\partial_x \left(f(t,x) \int_0^x x_*\, a(x,x_*)\, f(t,x_*)\, dx_* \right) \\ &\quad - f(t,x) \int_x^\infty a(x,x_*)\, f(t,x_*)\, dx_* \,,\end{aligned}$$

where $a(x, x_*) = a(x_*, x) \geq 0$ denotes the coagulation kernel.
Example: $a(x, x_*) = x^\alpha x_*^\beta + x^\beta x_*^\alpha$, $0 \leq \alpha \leq \beta \leq 1$.

Links OHS/Smoluchowski coagulation equation

The growth rate of the size distribution function of particles of size $x > 0$

$$\int_0^x x_* \, a(x, x_*) \, f(x_*) \, dx_*$$

depends on the whole distribution of particles of size smaller than x through a weighted average and not on the details of the sizes of the particles coalescing to produce particles of size x as in the Smoluchowski coagulation equation.

There are connections between the OHS and Smoluchowski coagulation equations, either via discrete models [Dubovski (1999)] or continuous models [Lachowicz, L. & Wrzosek (2003)].

Weak solutions [Lachowicz, L. & Wrzosek (2003)]

Assume that a is locally Lipschitz continuous with

$$\partial_x a(x, x_*) \geq -a_0 \,, \qquad a_0 \geq 0.$$

Given $f_0 \in L^1(0, \infty, (1+x)dx)$, $f_0 \geq 0$, there is a weak solution f to the OHS equation with $f(0) = f_0$ if

- either $\sup\{a(x, x_*)/x_* \ : \ x \in (0, R)\} \longrightarrow 0$ as $x_* \to \infty$ for each $R > 0$ and then

$$M_1(t) := \int_0^\infty x\, f(t,x)\, dx \leq M_1(0)\,, \qquad t \geq 0\,,$$

- or $a(x, x_*) \leq K\,(1 + x + x_*)$ and then

$$M_1(t) = M_1(0)\,, \qquad t \geq 0\,.$$

Mass conservation/gelation [Lachowicz, L. & Wrzosek (2003)]

The inequality $M_1(t) \le M_1(0)$ cannot in general be improved to an equality when the assumption $a(x, x_*) \le K\,(1 + x + x_*)$ is not fulfilled. Define the gelation time

$$T_{\text{gel}} := \inf\{t \ge 0 \;:\; M_1(t) < M_1(0)\} \in [0, \infty]\,.$$

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- If $a(x, x_*) \le K\,(1 + x + x_*)$ then $T_{\text{gel}} = \infty$.
- If $a(x, x_*) \ge K\,(xx_*)^{\lambda/2}$ for $\lambda \in (1, 2]$ then $T_{\text{gel}} < \infty$.

Smoluchowski's equation: [Escobedo, Mischler & Perthame (2002)]

Finite speed of propagation

Assume that supp $f_0 \subset [0, R_0]$. There are $T_\star \in (0, \infty]$ and $R \in \mathcal{C}^1([0, T_\star))$ such that supp $f(t) \subset [0, R(t)]$ for $t \in [0, T_\star)$ with the alternative

$$T_\star = \infty \quad \text{OR} \quad T_\star < \infty \ \text{ and } \ \lim_{t \to T_\star} R(t) = \infty\,.$$

More precisely, $R(0) = R_0$ and R solves the ODE

$$\frac{dR}{dt}(t) = \int_0^\infty x\; a(R(t), x)\; f(t, x)\; dx\,, \quad t \in [0, T_\star)\,.$$

Clearly, $M_1(t) = M_1(0)$ for $t \in [0, T_\star)$ and thus $T_\star \leq T_{\text{gel}}$.
[Dubovski (1999), Lachowicz, L. & Wrzosek (2003)]

Improved criterion for mass conservation

Assume that $\mathrm{supp}\, f_0 \subset [0, R_0]$ and there is a positive and increasing function A such that

$$a(x, x_*) \le A(x) + A(x_*) \quad \text{and} \quad \int_1^\infty \frac{1}{A(s)}\, ds = \infty\,.$$

Then $T_\star = T_{\text{gel}} = \infty$.

Finite speed of propagation: estimates

- If $a(x, x_*) = x^\lambda + x_*^\lambda$, $\lambda \in [0, 1)$, then

$$c\, t^{1/(1-\lambda)} \le R(t) \le C\, t^{1/(1-\lambda)}\,, \quad t \ge 1\,.$$

- If $a(x, x_*) = x + x_*$, then

$$R_0\, e^{M_1(0)t} \le R(t) \le R_0\, e^{2M_1(0)t}\,, \quad t \ge 0\,.$$

Finite speed of propagation: estimates

- If $a(x, x_*) = (xx_*)^{\lambda/2}$, $\lambda \in [0, 1)$, then

$$R(t) \le C\, t^{1/(1-\lambda)}, \quad t \ge 1.$$

- If $a(x, x_*) = (xx_*)^{1/2}$, then

$$R(t) \le R_0\, e^{M_1(0)t}, \quad t \ge 0.$$

Dynamical scaling assumption

For homogeneous coagulation kernels

$$a(\xi x, \xi x_*) = \xi^\lambda \, a(x, x_*) , \quad \xi > 0 , \; x > 0 , \; x_* > 0 ,$$

it is expected that

$$f(t, x) \sim \frac{1}{s(t)^\omega} \, \psi\left(\frac{x}{s(t)}\right) \quad \text{as} \quad t \to T_{\text{gel}} ,$$

where ω, the mean size s ($s(t) \to \infty$ as $t \to T_{\text{gel}}$) and the scaling profile ψ are to be determined.

Mass Conservation $\longrightarrow T_{\text{gel}} = \infty$ and $\boxed{\omega = 2}$

[van Dongen & Ernst (1986), Leyvraz (2003)]

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Mass-conserving self-similar solutions

For $\varrho > 0$, define

$$\psi_\varrho(x) := \frac{2}{\varrho}\, \mathbf{1}_{[0,\varrho]}(x)\,, \quad x > 0\,.$$

Then

$$(t, x) \longmapsto \frac{1}{t^2}\, \psi_\varrho\left(\frac{x}{t}\right)$$

is a self-similar solution to the OHS equation with first moment equal to ϱ for all $t \geq 0$.

Convergence and decay estimates

Define the cumulative distribution function

$$F(t,x) := \int_x^\infty f(t,x_*)\, dx_* , \quad (t,x) \in [0,\infty) \times (0,\infty) .$$

Then,

- for every $p \in [1,\infty)$,

$$\lim_{t\to\infty} t^{(p-1)/p} \, \|F(t) - F_{M_1(0)}(t)\|_{L^p} = 0 .$$

- In addition, if $f_0 \in L^1(0,\infty,x^2dx)$ and $p \in [1,2]$, then

$$\|F(t) - F_{M_1(0)}(t)\|_{L^p} \le C\, t^{-p/2} , \quad t \ge 0 .$$

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Liapunov functionals in self-similar variables

Self-similar variables: define Φ by

$$F(t,x) = \frac{1}{1+t}\,\Phi\left(\ln(1+t), \frac{x}{1+t}\right), \quad (t,x) \in [0,\infty) \times (0,\infty),$$

and Φ_ϱ by $\Phi_\varrho(y) := (2/\varrho)\,(\varrho - y)_+$, $y \geq 0$. Then

- $t \longmapsto \|\Phi(t) - \Phi_{M_1(0)}\|_{L^1}$ is a non-increasing function.
- $$t \longmapsto \|\Phi(t) - \Phi_{M_1(0)}\|_{L^2}^2 + \frac{4}{M_1(0)} \int_0^\infty (x - M_1(0))_+\, \Phi(t,x)\, dx$$

  decays exponentially.

"Linearization"

Consider again the cumulative distribution function

$$F(t,x) := \int_x^\infty f(t,x_*)\, dx_*\,, \quad (t,x) \in [0,\infty) \times (0,\infty)\,.$$

and let P be its inverse function (so that $F(t, P(t,y)) = y$ formally). Then

$$\partial_t P(t,y) = \frac{y^2}{2}\, \partial_y P(t,y) + \int_y^\infty P(t,y_*)\, dy_*\,.$$

Linear transport equation with a nonlocal reaction term.

Self-similar solutions with “fat tails”

We look for self-similar solutions of the form

$$\frac{1}{s(t)^\omega}\,\psi\left(\frac{x}{s(t)}\right)\,, \quad s(t) := ((\omega-1)t)^{1/(\omega-1)}\,.$$

Setting $m = (2-\omega)/(\omega-1)$ for $\omega \in (1,2)$,

$$P_\psi(y) = A\,(y-\omega)+\frac{B}{y^{m+1}}\sum_{k=0}^{\infty} a_k\left(\frac{y}{2(\omega-1)}\right)^k\,, \quad y \in [0, 2(\omega-1)]\,,$$

$$P_\psi(2(\omega-1)) = P'_\psi(2(\omega-1)) = 0\,, \quad a_k := (k-m)\left(\prod_{j=1}^{k}\frac{j-m-3}{j}\right)$$

Smoluchowski's eq. [Bertoin (2002), Menon & Pego (2004)]

Outline

Mass-conserving self-similar solutions

In that case, $s(t) = t^\gamma$ with $\gamma := 1/(1-\lambda)$ and ψ solves

$$\gamma \int_0^\infty \psi(x)\, (\varphi(x) - x\, \varphi'(x))\, dx$$
$$+ \int_0^\infty \int_0^x (x_*\, \varphi'(x) - \varphi(x_*))\, a(x, x_*)\, \psi(x_*)\, \psi(x)\, dx_* dx = 0\,.$$

Problem: for any $x_0 > 0$, $\gamma\, x_0^{-\lambda}\, \delta_{x_0}$ is a (measure-valued) weak solution to this equation.
We expect that there is also a “smooth” solution.

Existence [Bagland & L. (2007)]

There exists a non-negative function $\psi \in L^1(0,\infty;(x+x^\lambda)\,dx)$, $x_0>0$ and $q>0$ such that

$$\int_0^\infty x\,\psi(x)\,dx = 1\,, \quad M_\lambda := \int_0^\infty x^\lambda\,\psi(x)\,dx \in (1,\gamma)\,,$$

$$\psi \in \mathcal{C}((0,\infty)\backslash\{x_0\}) \quad \text{with} \quad \text{supp}\,(\psi)=[0,x_0] \quad \text{and} \quad \psi(x_0-)>0\,,$$

$$\lim_{x\to 0} x^\tau\,\psi(x) = q \quad \text{with} \quad \tau := 2-\frac{M_\lambda}{\gamma} \in (1,1+\lambda)\,,$$

$$\begin{aligned}&\left(\gamma\,x - \int_0^x \left(x^\lambda + x_*^\lambda\right)\,x_*\,\psi(x_*)\,dx_*\right)\,\psi(x)\\ &= \left(\gamma - \int_x^\infty x_*^\lambda\,\psi(x_*)\,dy_*\right)\,\int_x^\infty \psi(x_*)\,dx_*\,,\end{aligned}$$

$(t,x) \longmapsto t^{-2\gamma}\psi(xt^{-\gamma})$ self-similar solution to the OHS equation.

Proof

Existence proof by a dynamical approach: construction of a compact and convex invariant subset for an evolution equation + Tychonov-Schauder fixed point theorem $\Longrightarrow$ existence of a steady state. [Gamba, Panferov & Villani (2004), Escobedo, Mischler & Rodriguez Ricard (2005), Fournier & L. (2005)]

Change of unknown functions:

- self-similar variables: $f \longrightarrow g$,
- Indefinite integral of g vanishing at infinity: $g \longrightarrow G$,
- Pseudo-inverse of G: $G \longrightarrow P$,
- Modification of the kernel: $a \longrightarrow a + 2\delta$.

Outline

Self-similar solutions with infinite mass

We look for self-similar solutions of the form

$$(t, x) \longmapsto \frac{1}{s(t)^\omega}\, \psi\left(\frac{x}{s(t)}\right) ,$$

where ω, the mean size s and the scaling profile ψ are to be determined with $s(t) \to \infty$ as $t \to T$.

$$s(t) = ((3 - \omega)(T - t))^{-1/(3-\omega)} , \quad t \in [0, T) .$$

Smoluchowski's equation: $\omega \in [5/2, 3)$ [Menon & Pego (2004)]

Self-similar solutions with infinite mass

Let ω_c ($\omega_c \sim 2.255$) be the unique real number in $(2, 3)$ such that

$$(\omega_c - 1)\ \ln\left(\frac{\omega_c - 1}{\omega_c - 2}\right) = 2\,.$$

For $\omega \in [\omega_c, 3)$ the function g_ω defined by

$$g_\omega(z) := 2z - (\omega - 1)\ \ln\left(1 + \frac{z}{\omega - 2}\right), \quad z \geq 0\,,$$

has a unique positive zero $r(\omega) \in ((3 - \omega)/2, 1]$ with $r(\omega_c) = 1$.

Self-similar solutions with infinite mass

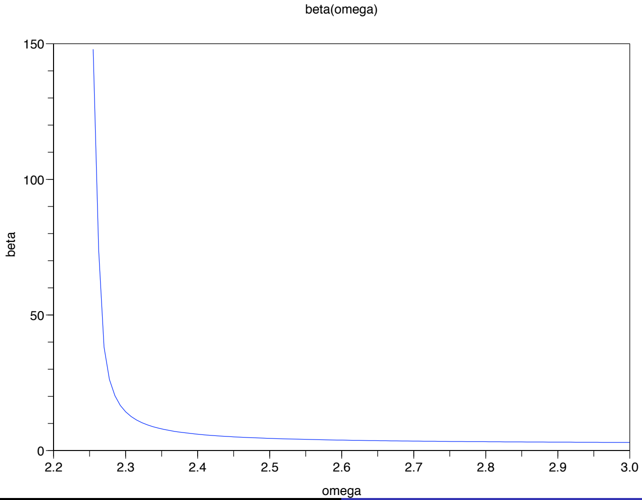
Given $\omega \in [\omega_c, 3)$, there is a self-similar solution to the OHS equation and the profile ψ enjoys the following properties:

- $\psi \in \mathcal{C}^1((0, X_\omega))$ with $X_{\omega_c} < \infty$ and $\psi(X_{\omega_c}-) > 0$ while $X_\omega = \infty$ for $\omega \in (\omega_c, 3)$,
- $\psi(x) \sim C_0(\omega)\ x^{-\omega}$ as $x \to 0$,
- if $\omega \in (\omega_c, 3)$, then $\psi(x) \sim C_\infty(\omega)\ x^{-\beta(\omega)}$ as $x \to \infty$ with

$$\beta(\omega) := \frac{\omega - r(\omega)}{1 - r(\omega)} \in (3, \infty)\,.$$

Remark: the second moment of ψ is finite but its first moment is not.

Behaviour as $x \longrightarrow \infty$



Proof

Change of unknown functions:

- Indefinite integral of $y \mapsto y\psi(y)$ vanishing at infinity: $\psi \longrightarrow \Psi$,
- Pseudo-inverse of Ψ: $\Psi \longrightarrow P$,
- Indefinite integral of P vanishing at ∞: $P \longrightarrow \mathcal{P}$.

Then

$$y\,\frac{d\mathcal{P}}{dy}(y) = g_\omega(\mathcal{P}(y))\,, \quad \mathcal{P}(0) = r(\omega)\,,$$

with the constraints: $\mathcal{P}$ is decreasing, non-negative, and convex.