REGULARITY ANALYSIS FOR SYSTEMS OF REACTION-DIFFUSION EQUATIONS

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Abstract. This paper is devoted to the study of the regularity of solutions to some systems of reaction–diffusion equations. In particular, we show the global boundedness and regularity of the solutions in one and two dimensions. In addition, we discuss the Hausdorff dimension of the set of singularities in higher dimensions. Our approach is inspired by De Giorgi's method for elliptic regularity with rough coefficients. The proof uses the specific structure of the system to be considered and is not a mere adaptation of scalar techniques; in particular the natural entropy of the system plays a crucial role in the analysis.

Résumé. Ce travail est consacré à l'étude de la régularité des solutions de certains systèmes d'équations de réaction–diffusion. En particulier, nous montrons que les solutions peuvent être bornées et régulières en dimensions un et deux alors qu'en dimensions supérieures nous discutons la dimension de Hausdorff de l'ensemble des points singuliers. L'approche proposée ici s'inspire de la méthode de De Giorgi pour étudier la régularité de problèmes elliptiques avec des coefficients discontinus. La preuve exploite la stucture spécifique des systèmes considérés et n'est pas une simple adaptation de techniques scalaires. L'entropie associée naturellement au système joue un rôle crucial dans cette analyse.

Key words. Reaction-diffusion systems. Regularity of solutions.

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1 Introduction

This paper is devoted to the analysis of the following system of reaction-diffusion equations

$$\begin{aligned}
&\partial_t a_i - \nabla \cdot (D_i \nabla a_i) = Q_i(a), \qquad\qquad i \in \{1, \ldots, p\},\\
&Q_i(a) = (\mu_i - \nu_i) \left(k_{\rm f} \prod_{j=1}^{p} a_j^{\nu_j} - k_{\rm b} \prod_{j=1}^{p} a_j^{\mu_j} \right),\\
&a_{i\ |t=0} = a_i^0.
\end{aligned} \tag{1.1}$$

The equation holds for $t \geq 0$ and the space variable x lies in Ω where

- either $\Omega = \mathbb{R}^N$,

- or $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and the system is completed by imposing the Neumann boundary condition

$$D_i \nabla a_i \cdot \nu(x)_{|\partial\Omega} = 0,$$

where $\nu(x)$ stands for the outer normal vector at $x \in \partial\Omega$.
Throughout the paper, the symbol ∇ denotes the gradient operator with respect to the space variable x only. The matrices $D_i(x)$ are required to satisfy

$$\begin{aligned}
&D_i \in \big(L^\infty(\Omega)\big)^{N\times N},\\
&D_i(x)\xi \cdot \xi \geq \alpha |\xi|^2, \qquad \alpha > 0 \qquad \text{for any } \xi \in \mathbb{R}^N,\ x \in \Omega.
\end{aligned} \tag{1.2}$$

Let us comment this assumption:

- the analysis below is interesting when there are different diffusion matrices: assuming $D_i = D$, a common value, makes the problem easier;

- there is no regularity assumption on the coefficients;

- the standard uniform coercivity condition is assumed. The case of degenerate coefficients leads to specific difficulties which are beyond the scope of this paper.

Such a system is intended to describe e.g. the evolution of a chemical solution: the unknown a_i stands for the density of the species labelled by $i \in \{1, \ldots, p\}$ within the solution. The right hand side of (1.1) follows from the mass action principle applied to the reversible reaction

$$\sum_{i=1}^{p} \nu_i A_i \leftrightarrow \sum_{i=1}^{p} \mu_i A_i,$$

where the μ_i and ν_i's — the so-called stoichiometric coefficients — are integers. The (positive) coefficients $k_{\rm f}$ and $k_{\rm b}$ are the rates corresponding to the forward and backward reactions, respectively. According to the physical interpretation, the unknowns are implicitly non-negative quantities: $a_i \geq 0$. In fact, this property holds thanks to the structure of the system. Indeed, (1.1) can be written

$$\partial_t a_i - \nabla \cdot (D_i \nabla a_i) + L_i(a) a_i = G_i(a) \tag{1.3}$$

where the nonlinear functions G_i and L_i have the property: if the components a_k of a are non-negative then $G_i(a) \geq 0$ and $L_i(a) \geq 0$. Hence preservation of non-negativity, when starting from a non-negative initial data, can be considered among the a priori estimates of the problem (see appendix for more details). The main ingredients of our analysis rely on the following properties:

- The mass is conserved. The stoichiometric coefficients satisfy

$$\begin{array}{l}\text{There exists } (m_1,\ldots,m_p)\in\mathbb{N}^p,\ m_i\neq 0, \text{ such that}\\ \displaystyle\sum_{i=1}^p m_i\mu_i=\sum_{i=1}^p m_i\nu_i.\end{array} \tag{1.4}$$

  It implies the mass conservation

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^p\int_\Omega m_i\ a_i\,\mathrm{d}x=0.$$

- The entropy is dissipated. We set $K=k_{\mathrm{b}}/k_{\mathrm{f}}$, then

$$\sum_{i=1}^p Q_i(a)\ln(a_i/K^{1/(p(\mu_i-\nu_i))})=-k_{\mathrm{f}}\Big(\prod_{i=1}^p a_i^{\mu_i}-K\prod_{i=1}^p a_i^{\nu_i}\Big)\ln\left(\frac{\displaystyle\prod_{i=1}^p a_i^{\mu_i}}{K\displaystyle\prod_{i=1}^p a_i^{\nu_i}}\right)\le 0. \tag{1.5}$$

In order to simplify the notations, and without loss of generality, we restrict ourselves to the case

$$m_i=1,\qquad k_{\mathrm{f}}=1=k_{\mathrm{b}}.$$

A crucial role will be played by the quantity

$$\overline{\mu}=\sum_{i=1}^p\mu_i=\sum_{i=1}^p\nu_i,$$

where the coefficients μ_i and ν_i are still integers.

In our study of such systems restrictions on the space dimension N and the parameter $\overline{\mu}$ appear. One of the most interesting situations we are able to deal with is the following example corresponding to 4 species subject to the reactions

$$A_1+A_3\leftrightarrow A_2+A_4.$$

It leads to

$$Q_i(a)=(-1)^{i+1}(a_2a_4-a_1a_3). \tag{1.6}$$

We refer for a thorough introduction to the modeling issues and mathematical properties of such reaction diffusion systems to [11, 13, 14, 19, 20, 21, 23, 28, 31]. Information can also be found in the survey [6] with connection to coagulation-fragmentation models and in [24] for applications in biology. Let us also mention that (1.1) can be derived through hydrodynamic scaling from kinetic models, see [2].

In this contribution we are interested in the derivation of new L^∞ estimates and we investigate the regularity of the solutions of (1.1). Quite surprisingly, the question of global boundedness becomes trivial when the diffusion coefficients vanish. Indeed, consider $D_i=0$, and a bounded initial value. The property (1.4) implies that for each x fixed, the total mass $\sum_{i=1}^p m_ia_i(t,x)$ is time independent. Then, the non-negativity of the a_i's implies that each a_i is uniformly bounded. Conversely, certain reaction diffusion systems might exhibit blow up phenomena, see e.g. [22, 26], as it is also well known when considering nonlinear heat equations [15, 35]. Therefore global well-posedness and discussion of smoothing effects — that is gain of regularity of the solution compared to the initial data — is an issue.

Standard techniques can indeed be applied to show the existence of a smooth solution of (1.1) locally in time, with, say, initial data in $L^1 \cap L^\infty(\Omega)$. We sketch in the appendix the basic argument that proves the local existence of a smooth non negative solution. The challenging question consists in extending the result on arbitrarily large time intervals. Roughly speaking, this is due to a lack of estimates since the only natural bounds are provided by the mass conservation (1.4) and the entropy dissipation (1.5). In particular, the mass conservation only provides an estimate of the solution in L^1 which is not enough for the right hand side $Q_i(a)$ to make sense as a distribution! However, by using the tricky techniques introduced in [25, 26], it has been shown recently in [10] that the solutions of (1.1) in the quadratic case (1.6) are a priori bounded in $L^2((0,T) \times \Omega)$ so that the nonlinear reaction term makes sense at least in L^1. This non trivial estimate can be obtained by exploiting the entropy dissipation and the non degeneracy of the diffusion coefficients. In [10], using also the arguments introduced in [25], it allows to establish the global existence of weak solutions of (1.1), (1.6). Dealing with higher order nonlinearities or degenerate coefficients the difficulty might lead to introduce a suitable notion of renormalized solutions, see [10] again. We also mention the recent work [27] where the quadratic system is analyzed with diffusion acting only in one direction. The dissipation property (1.5) is also the basis for studying the asymptotic trend to equilibrium [8, 9] in the spirit of the entropy/entropy dissipation techniques which are presented e.g. in [34] (we refer also to [1] for further investigation of the large time behavior of nonlinear evolution systems using the entropy dissipation).

Our approach is inspired by De Giorgi's methods for studying the regularity of solution of diffusion equations without requiring the regularity of the coefficients, see [7]. The crucial step consists in establishing a L^∞ estimate on the solution. Regularity of the solution follows in a classical way (see appendix). This approach has been used in [33] to obtain an alternative proof to the regularity results for the Navier-Stokes equation [4, 17] and it also shares some features with the strategy introduced in [29, 30]. It has also been applied to study convection-diffusion equations [18] and regularity for the quasi-geostrophic equation [5]. Here, it is worth pointing out that the proof utilizes strongly the structure of the whole system and the argument is not a mere refinement of a scalar approach. As we shall see however, restrictions appear between the space dimension N and the degree of nonlinearity of the reaction term measured by means of $\overline{\mu}$. For this reason, the L^∞ estimates can be proved in two dimension for the quadratic operator (1.6) or in one dimension considering cubic terms.

Theorem 1.1 *We consider the quadratic operator* (1.6) *(or assume $\overline{\mu} = 2$). Let $N = 2$ and suppose that the diffusion coefficients fulfill* (1.2)*. Let $a_i^0 \geq 0$ satisfy*

$$\sum_{i=1}^{4} \int_\Omega a_i^0 \big(1 + |x| + |\ln(a_i^0)|\big)\,\mathrm{d}x = M_0 < \infty. \tag{1.7}$$

Then, (1.1) *admits a global solution such that for any $0 < T \leq T^\star < \infty$, a_i belongs to $L^\infty((T, T^\star) \times \Omega)$.*

Theorem 1.2 *Let $N = 1$ with $\overline{\mu} \leq 3$ and suppose that the diffusion coefficients fulfill* (1.2)*. Let $a_i^0 \geq 0$ satisfy* (1.7)*. Then,* (1.1) *admits a global solution such that for any $0 < T \leq T^\star < \infty$, a_i belongs to $L^\infty((T, T^\star) \times \Omega)$.*

We point out that these statements do not require any regularity property on the diffusion coefficients D_i which are only supposed to be bounded. As a byproduct, by using the new bound, a direct bootstrap argument shows the global regularity of the solution (see appendix).

Corollary 1.1 *Let the assumptions of Theorem 1.1 or 1.2 be fulfilled. Suppose moreover that the D_i's belong to $C^k(\Omega)$ with bounded derivatives up to order k. Then, for any $0 < T \leq T^\star < \infty$, the solution belongs to $L^\infty(T, T^\star; C^k(\Omega))$. Accordingly for C^∞ coefficients with bounded derivatives, the solution is C^∞ on $(T, T^\star) \times \Omega$.*

Such statements could be helpful for investigating the large time behavior: they can be the starting point to apply the strategy developed in [8, 9] and then this would lead to the proof of the convergence to the equilibrium state for large time, with an exponential rate. We do not discuss further this issue which requires a sharp estimate of the bound with respect to the final time $T^\star$. Instead, we consider the case of higher dimensions: the same method provides information on the Hausdorff dimension (definitions are recalled in Section 4) of the set of the singular points of the solutions.

Theorem 1.3 *Let $N \geq 3$ and $\overline{\mu} = 2$. We suppose that the coefficients D_i are constant with respect to $x \in \Omega$. Let $a_i^0 \geq 0$ satisfy (1.7). We consider a solution of (1.1) on $(0, T) \times \Omega$. We call a singular point, any point (t, x) having a neighborhood on which one of the function a_i is not C^∞. Then, the Hausdorff dimension of the set of singular points of the solution a does not exceed $(N^2 - 4)/N$.*

In the next section, we briefly recall the fundamental estimate that follows from (1.5). This bound is used in Section 3 where we adapt De Giorgi's approach to the system (1.1). Section 4 is devoted to the estimate of the Hausdorff dimension of the set of singularities in higher space dimensions.

2 Entropy dissipation

In the following Sections, we adopt the viewpoint of discussing a priori estimates formally satisfied by the solutions of (1.1). As usual the derivation of such estimates relies on various manipulations such as integrations by parts, permutations of integrals and so on. Of course, such formulae apply to the smooth solutions of the problem that can be shown to exist on a small enough time interval by using classical reasoning for nonlinear parabolic equations (see the appendix). Moreover, these estimates also apply to solutions of suitable approximations of the problem (1.1). Such approximations should be defined so that the essential features of the system are preserved. Hence, let us reproduce the reasoning in [10]: by truncation and regularization we deal with an initial data

$$a_i^{0,\eta} \in C_c^\infty(\Omega), \qquad a_i^{0,\eta} \geq 0$$

which converges in $L^1(\Omega)$ to a_i^0 as $\eta > 0$ tends to 0 and such that

$$\sup_{\eta>0} \sum_{i=1}^{p} \int_\Omega a_i^{0,\eta}\big(1 + |x| + |\ln(a_i^{0,\eta})|\big)\,\mathrm{d}x \leq C_0 \ \sum_{i=1}^{p} \int_\Omega a_i^{0}\big(1 + |x| + |\ln(a_i^{0})|\big)\,\mathrm{d}x = C_0\ M_0 < \infty.$$

Next, let us consider a cut-off function $\zeta \in C_c^\infty(\mathbb{R})$ such that $0 \leq \zeta(s) \leq 1$, $\mathrm{supp}(\zeta) \subset B(0, 2)$ and $\zeta(s) = 1$ for $|s| \leq 1$. Then, in (1.1) we replace $Q_i(a)$ by

$$Q_i^\eta(a) = Q_i(a)\zeta(\eta|a|),$$

with $|a| = \sqrt{a_1^2 + ... + a_p^2}$. Accordingly, for any $\eta > 0$ fixed, and $a_i \in L^1(\Omega)$, $Q_i^\eta(a)$ belongs to $L^\infty(\Omega)$. We can show that the corresponding regularized problem admits a unique (non-negative) smooth solution, globally defined, see [16, 28]. Therefore, in what follows we discuss a priori estimates on solutions of (1.1): for the sake of simplicity we detail the arguments working directly on (1.1), but we keep in mind that the arguments apply to the regularized problem as well. In turn, we obtain bounds on the sequence a_i^η, which are uniform with respect to $\eta > 0$. Finally, existence of a global solution satisfying the estimates follows by performing the passage to the limit $\eta \to 0$; a detail that we skip here, referring for instance to [10].

We start by discussing the a priori estimates that can be naturally deduced from (1.4) and (1.5). The results here apply in full generality, without assumptions on $p, N, \overline{\mu}$.

Proposition 2.1 *Assume* (1.2), (1.4) *and* (1.5). *Let* $a_i^0 \geq 0$ *satisfy*

$$\sum_{i=1}^{p} \int_\Omega a_i^0 \big(1 + |x| + |\ln(a_i^0)|\big)\,\mathrm{d}x = M_0 < \infty. \tag{2.1}$$

We set

$$\mathfrak{D}(t,x) = \Big(\prod_{i=1}^{p} a_i^{\mu_i} - \prod_{i=1}^{p} a_i^{\nu_i}\Big) \ln\left(\frac{\displaystyle\prod_{i=1}^{p} a_i^{\mu_i}}{\displaystyle\prod_{i=1}^{p} a_i^{\nu_i}}\right)(t,x) \geq 0.$$

Then, for any $0 < T < \infty$, *there exists* $0 < C(T) < \infty$ *such that*

$$\sup_{0\leq t\leq T} \Big\{ \sum_{i=1}^{p} \int_\Omega a_i \big(1 + |x| + |\ln(a_i)|\big)(t,x)\,\mathrm{d}x + \sum_{i=1}^{p} \int_0^t \int_\Omega \big|\nabla\sqrt{a_i}\big|^2(s,x)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega \mathfrak{D}(s,x)\,\mathrm{d}x\,\mathrm{d}s \Big\} \leq C(T).$$

If Ω *is a bounded domain, this estimate holds for* $T = +\infty$.

Proof. As a consequence of (1.4) and (1.5), we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{p} \int_\Omega a_i\big(1 + \ln(a_i)\big)\,\mathrm{d}x + \sum_{i=1}^{p} \int_\Omega D_i \nabla a_i \cdot \frac{\nabla a_i}{a_i}\,\mathrm{d}x + \int_\Omega \mathfrak{D}\,\mathrm{d}x = 0.$$

Then, the coercivity condition (1.2) means that we can establish the following lower bound

$$\sum_{i=1}^{p} \int_\Omega D_i \nabla a_i \cdot \frac{\nabla a_i}{a_i}\,\mathrm{d}x \geq \alpha \sum_{i=1}^{p} \int_\Omega \frac{|\nabla a_i|^2}{a_i}\,\mathrm{d}x = 4\alpha \sum_{i=1}^{p} \int_\Omega \big|\nabla\sqrt{a_i}\big|^2\,\mathrm{d}x.$$

In the case when Ω is a bounded domain then the conclusion of the theorem follows as

$$\begin{aligned}\sum_{i=1}^{p} \int_\Omega a_i |\ln(a_i)|\,\mathrm{d}x &= \sum_{i=1}^{p} \int_\Omega a_i \ln(a_i)\,\mathrm{d}x - 2\sum_{i=1}^{p} \int_\Omega a_i \ln(a_i) \mathbb{1}_{0\leq a_i\leq 1}\,\mathrm{d}x \\ &\leq \sum_{i=1}^{p} \int_\Omega a_i \ln(a_i)\,\mathrm{d}x + p\frac{2}{e}|\Omega|,\end{aligned}$$

where here and below, $\mathbb{1}_{\mathcal{M}}$ denotes the characteristic function of the set $\mathcal{M}$. In the case when $\Omega = \mathbb{R}^N$, then the argument proceeds as follows. By using (1.4) and denoting by M the supremum norm of the diffusion coefficients, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{p}\int_\Omega a_i|x|\,\mathrm{d}x &= -\sum_{i=1}^{p}\int_\Omega D_i\nabla a_i\cdot\frac{x}{|x|}\,\mathrm{d}x\\ &\leq M\sum_{i=1}^{p}\int_\Omega |\nabla a_i|\,\mathrm{d}x = M\sum_{i=1}^{p}\int_\Omega \frac{|\nabla a_i|}{\sqrt{a_i}}\sqrt{a_i}\,\mathrm{d}x\\ &\leq \frac{\alpha}{2}\sum_{i=1}^{p}\int_\Omega \frac{|\nabla a_i|^2}{a_i}\,\mathrm{d}x + \frac{M^2}{2\alpha}\sum_{i=1}^{p}\int_\Omega a_i\,\mathrm{d}x,\end{aligned}$$

by using the standard inequality $|rs| \leq r^2/2 + s^2/2$. Hence, we arrive at

$$\begin{aligned}&\sum_{i=1}^{p}\int_\Omega a_i\big(1+|x|+\ln(a_i)\big)\,\mathrm{d}x + \frac{\alpha}{2}\sum_{i=1}^{p}\int_0^t\int_\Omega \frac{|\nabla a_i|^2}{a_i}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \mathfrak{D}\,\mathrm{d}x\,\mathrm{d}s\\ &\qquad\leq M_0 + \frac{M^2}{2\alpha}\sum_{i=1}^{p}\int_0^t\int_\Omega a_i\,\mathrm{d}x\,\mathrm{d}s\\ &\qquad\leq \big(1+tM^2/(2\alpha)\big)M_0.\end{aligned}$$

It remains to control the negative part of the $a_i\ln(a_i)$'s. To this end, we use the following classical argument:

$$\begin{aligned}\int_\Omega a_i|\ln(a_i)|\,\mathrm{d}x &= \int_\Omega a_i\ln(a_i)\,\mathrm{d}x - 2\int_\Omega a_i\ln(a_i)\big(\mathbb{1}_{0\leq a_i\leq e^{-|x|/2}} + \mathbb{1}_{e^{-|x|/2}\leq a_i\leq 1}\big)\,\mathrm{d}x\\ &\leq \int_\Omega a_i\ln(a_i)\,\mathrm{d}x + \frac{4}{e}\int_\Omega e^{-|x|/4}\,\mathrm{d}x + \int_\Omega |x|a_i\,\mathrm{d}x\end{aligned}$$

since $-s\ln(s) \leq \frac{2}{e}\sqrt{s}$ for any $0 \leq s \leq 1$. We conclude by combining together all the pieces. ∎

3 L^∞ bounds

In the spirit of the Stampacchia cut-off method, L^∞ bounds of solutions of certain PDEs can be deduced from the behavior of suitable nonlinear functionals. Here, such a functional is constructed in a way that uses the dissipation property (1.5). Let us consider the non-negative, C^1 and convex function

$$\Phi(z) = \begin{cases} (1+z)\ln(1+z) - z & \text{if } z \geq 0,\\ 0 & \text{if } z \leq 0.\end{cases}$$

Then, for $k \geq 0$, we are interested in the evolution of

$$\sum_{i=1}^{p}\int_\Omega \Phi(a_i - k)\,\mathrm{d}x.$$

Lemma 3.1 *There exists a universal constant C, such that for every $a = (a_1, \dots, a_p)$ solution of* (1.1)*, for any $k \geq 0$, and for any $0 \leq s \leq t < \infty$, we have*

$$\begin{aligned}
&\sum_{i=1}^p \int_\Omega \Phi(a_i - k)(t,x)\,\mathrm{d}x + 4\alpha \sum_{i=1}^p \int_s^t \int_\Omega \left|\nabla\sqrt{1+[a_i-k]_+}\right|^2(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leq \sum_{i=1}^p \int_\Omega \Phi(a_i-k)(s,x)\,\mathrm{d}x \\
&\qquad\qquad + C\sum_{i=1}^p \int_s^t \int_\Omega \left(1 + k^{\overline{\mu}} + (1+k)[a_i-k]_+^{\overline{\mu}-1}\right)\ \ln(1+[a_i-k]_+)(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau
\end{aligned}$$

where $[z]_+ = \max(0,z)$ denotes the non-negative part of z.

Remark 3.1 *Notice that the universal constant does not depend on the actual solution a nor on k. It is also worth noticing that, in order to make sense of this inequality we need only $a_i^{\overline{\mu}-1}\ln(1+a_i)$ to be integrable, although it is required to have $a_i^{\overline{\mu}}$ to be integrable to make sense of the equation (1.1). This point will be very important in the next section. Crucial to the analysis is the similarity of the function Φ and the natural entropy of the system (1.1).*

Proof. Multiplying (1.1) by $\Phi'(a_i - k)$ and summing yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^p \int_\Omega \Phi(a_i-k)\,\mathrm{d}x + \sum_{i=1}^p \int_\Omega D_i \nabla a_i \cdot \nabla a_i\ \Phi''(a_i-k)\,\mathrm{d}x = \sum_{i=1}^p \int_\Omega Q_i(a)\Phi'(a_i-k)\,\mathrm{d}x. \tag{3.1}$$

Then, we observe that (1.2) leads to

$$\begin{aligned}
\sum_{i=1}^p \int_\Omega D_i\nabla a_i \cdot \nabla a_i\ \Phi''(a_i-k)\,\mathrm{d}x &= \sum_{i=1}^p \int_\Omega D_i \nabla a_i \cdot \nabla a_i\ \frac{\mathbb{1}_{a_i \geq k}}{1+[a_i-k]_+}\,\mathrm{d}x \\
&= \sum_{i=1}^p \int_\Omega D_i \nabla(1+[a_i-k]_+)\cdot\nabla(1+[a_i-k]_+)\ \frac{\mathrm{d}x}{1+[a_i-k]_+} \\
&\geq \alpha \sum_{i=1}^p \int_\Omega \frac{\left|\nabla(1+[a_i-k]_+)\right|^2}{1+[a_i-k]_+}\,\mathrm{d}x \\
&\geq 4\alpha \sum_{i=1}^p \int_\Omega \left|\nabla\sqrt{1+[a_i-k]_+}\right|^2 dx.
\end{aligned}$$

Next, we rewrite the right hand side of (3.1) as

$$\begin{aligned}
&\sum_{i=1}^p \int_\Omega Q_i(a)\ \ln(1+[a_i-k]_+)\,\mathrm{d}x \\
&= \sum_{i=1}^p \int_\Omega \left(Q_i(a) - Q_i(1+[a-k]_+)\right)\ \ln(1+[a_i-k]_+)\,\mathrm{d}x \\
&\qquad + \sum_{i=1}^p \int_\Omega Q_i(1+[a-k]_+)\ln(1+[a_i-k]_+)\,\mathrm{d}x,
\end{aligned}$$

where (1.5) implies that the last term is non-positive. We are thus left with the task of estimating $\big(Q_i(a) - Q_i(1+[a-k]_+)\big)\ \ln(1+[a_i-k]_+)$.

To this end, let us consider the polynomial function $P:\mathbb{R}^p \to \mathbb{R}$ defined by $P(u) = \prod_{i=1}^p u_i^{\nu_i}$. Clearly, given $u, v \in \mathbb{R}^p$, we have

$$|P(u) - P(v)| = \Big| \int_0^1 \nabla P(u + s(v-u)) \cdot (u-v)\, \mathrm{d}s \Big| \le C\|u-v\| \int_0^1 \|\nabla P(u+s(v-u))\|\, \mathrm{d}s$$

where $\|\cdot\|$ represents for any norm on $\mathbb{R}^p$. As a matter of fact, since the μ_i's and ν_i's are non-zero integers, we have $\partial_j P(u) = \nu_j \prod_{i=1}^p u_i^{\nu'_{i,j}}$ where $\nu'_{i,j} = \nu_i$ if $i \ne j$ and $\nu'_{j,j} = \nu_j - 1$. In particular, note that $\sum_{i=1}^p \nu'_{i,j} = \overline{\mu} - 1$. Therefore, working with the ℓ^1 norm, we get

$$\|\nabla P(u)\| \le \sum_{j=1}^p \Big(\nu_j \prod_{i=1}^p |u_i|^{\nu'_{i,j}} \Big)$$

which yields, by using the convexity of the functions $z \mapsto z^{\nu'_{i,j}}$,

$$|P(u) - P(v)| \le \sum_{\ell=1}^p |u_\ell - v_\ell| \times \sum_{j=1}^p \nu_j \Big(\prod_{i=1}^p |u_i|^{\nu'_{i,j}} + \prod_{i=1}^p |v_i|^{\nu'_{i,j}} \Big).$$

Clearly, we have $\prod_{i=1}^p |u_i|^{\nu'_{i,j}} \le C\ \sum_{i=1}^p (1 + |u_i|^{\overline{\mu}-1})$ and finally we obtain

$$|P(u) - P(v)| \le C\ \sum_{\ell=1}^p |u_\ell - v_\ell| \times \sum_{i,j=1}^p \nu_j (1 + |u_i|^{\overline{\mu}-1} + |v_i|^{\overline{\mu}-1}).$$

We apply this inequality with $u_i = a_i$ and $v_i = 1 + [a_i - k]_+$ and we make use of the following simple remarks

$$\begin{cases} 0 \le (1+[a_i-k]_+)^{\overline{\mu}-1} \le C\ (1 + [a_i - k]_+^{\overline{\mu}-1}), \\ 0 \le a_i \le [a_i - k]_+ + k \text{ so that } 0 \le a_i^{\overline{\mu}-1} \le C([a_i-k]_+^{\overline{\mu}-1} + k^{\overline{\mu}-1}), \\ \big|a_i - (1 + [a_i - k]_+)\big| \le 1 + \big|a_i - [a_i-k]_+\big| \le 1 + k. \end{cases}$$

Applying the same reasoning with μ_i replacing ν_i, we arrive at

$$\big|Q_i(a) - Q_i(1+[a-k]_+)\big| \le C\ (1+k)\ \sum_{j=1}^p \big(1 + k^{\overline{\mu}-1} + [a_j - k]_+^{\overline{\mu}-1}\big),$$

where the constant C depends on $\overline{\mu}$ and $p < \infty$. Then, we end the proof by using the simple inequality: for any $u, v \ge 0$, $u^{\overline{\mu}-1}\ln(1+v) + v^{\overline{\mu}-1}\ln(1+u) \le 2\big(u^{\overline{\mu}-1}\ln(1+u) + v^{\overline{\mu}-1}\ln(1+v)\big)$. (As usual we have adopted the convention to keep the same notation C for a constant that does not depend on the solution, even when the value of the constant might change from one line to the other.) ∎

Remark 3.2 *We point out that the arguments above do not extend straightforwardly to situations where the unknown a is an infinite sequence, like e.g. for coagulation-fragmentation models (the constant C involves a sum over the reactants).*

Let $0 < T < T^\star < \infty$ and $0 < K < \infty$ be fixed. Set

$$0 < t_n = T(1 - 1/2^n) < T < T^\star, \qquad 0 < k_n = K(1 - 1/2^n) < K.$$

Let us denote

$$\mathcal{U}_n = \sup_{t_n \le t \le T^\star} \sum_{i=1}^{p} \int_\Omega \Phi(a_i - k_n)(t,x)\,\mathrm{d}x + 4\alpha \sum_{i=1}^{p} \int_{t_n}^{T^\star} \int_\Omega \left|\nabla \sqrt{1 + [a_i - k_n]_+}\right|^2(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau.$$

The aim is to show that, for a suitable choice of $K > 0$, $\mathcal{U}_n$ tends to 0 as $n \to \infty$ which will yield the L^∞ bound.

We start by making use of Lemma 3.1 with $0 \le t_{n-1} \le s \le t_n \le t \le T^\star$ and we average with respect to $s \in (t_{n-1}, t_n)$. Since $t_n - t_{n-1} = T/2^n$, we obtain

$$\begin{aligned}
&\frac{T}{2^n} \sum_{i=1}^{p} \int_\Omega \Phi(a_i - k_n)(t,x)\,\mathrm{d}x + 4\alpha \sum_{i=1}^{p} \int_{t_{n-1}}^{t_n} \int_s^t \int_\Omega \left|\nabla \sqrt{1 + [a_i - k_n]_+}\right|^2(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s \\
&\le \sum_{i=1}^{p} \int_{t_{n-1}}^{t_n} \int_\Omega \Phi(a_i - k_n)(s,x)\,\mathrm{d}x\,\mathrm{d}s + C \sum_{i=1}^{p} \int_{t_{n-1}}^{t_n} \int_s^t \int_\Omega \Gamma(k_n, a_i - k_n)(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s
\end{aligned}$$

with the short hand notation $\Gamma(k,u) = (1 + k^{\overline{\mu}} + (1+k)[u]_+^{\overline{\mu}-1}))\ \ln(1 + [u]_+)$. Since in the integration domain $s \ge t_{n-1}$ and $t \le T^\star$, the last integral can be dominated by

$$\sum_{i=1}^{p} \int_{t_{n-1}}^{t_n} \int_{t_{n-1}}^{T^\star} \int_\Omega \Gamma(k_n, a_i - k_n)(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s \le \frac{T}{2^n} \sum_{i=1}^{p} \int_{t_{n-1}}^{T^\star} \int_\Omega \Gamma(k_n, a_i - k_n)(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau.$$

Similarly $s \le t_n$ leads to the following bound from below

$$\begin{aligned}
&4\alpha \sum_{i=1}^{p} \int_{t_{n-1}}^{t_n} \int_s^t \int_\Omega \left|\nabla \sqrt{1 + [a_i - k_n]_+}\right|^2(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s \\
&\qquad \ge 4\alpha \sum_{i=1}^{p} \int_{t_{n-1}}^{t_n} \int_{t_n}^t \int_\Omega \left|\nabla \sqrt{1 + [a_i - k_n]_+}\right|^2(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}s \\
&\qquad \ge 4\alpha \frac{T}{2^n} \sum_{i=1}^{p} \int_{t_n}^t \int_\Omega \left|\nabla \sqrt{1 + [a_i - k_n]_+}\right|^2(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}$$

Hence, for any $t_n \le t \le T^\star$, we have

$$\begin{aligned}
&\sum_{i=1}^{p} \int_\Omega \Phi(a_i - k_n)(t,x)\,\mathrm{d}x + 4\alpha \sum_{i=1}^{p} \int_{t_n}^t \int_\Omega \left|\nabla \sqrt{1 + [a_i - k_n]_+}\right|^2(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\le \frac{2^n}{T} \sum_{i=1}^{p} \int_{t_{n-1}}^{t_n} \int_\Omega \Phi(a_i - k_n)(s,x)\,\mathrm{d}x\,\mathrm{d}s + C \sum_{i=1}^{p} \int_{t_{n-1}}^{T^\star} \int_\Omega \Gamma(k_n, a_i - k_n)(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}$$

Taking the supremum over $t_n \le t \le T^\star$, we obtain

$$\begin{aligned}
\mathcal{U}_n \le{}& \frac{2^n}{T} \sum_{i=1}^{p} \int_{t_{n-1}}^{t_n} \int_\Omega \Phi(a_i - k_n)(s,x)\,\mathrm{d}x\,\mathrm{d}s \\
&+ C \sum_{i=1}^{p} \int_{t_{n-1}}^{T^\star} \int_\Omega \left(1 + k_n^{\overline{\mu}} + (1+k_n)[a_i - k_n]_+^{\overline{\mu}-1}\right)\ \ln(1 + [a_i - k_n]_+)(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned} \tag{3.2}$$

The crucial step consists now in establishing the following nonlinear estimate, where restriction on both the space dimension N and $\overline{\mu}$ appear.

Proposition 3.1 *Suppose $N = 1$ or $N = 2$. There exists a constant $C > 0$ (which does not depend on the solution, nor on T, $T^\star$, K) such that*

$$\mathcal{U}_n \leq C\ (1+T^\star)\ \mathcal{K}(n,K,T)\ \mathcal{U}_{n-1}^{(N+2)/N}$$

where

$$\mathcal{K}(n,K,T) = \frac{Q_K}{T}\ 2^{n(2N+4)/N} + (1+K^{\overline{\mu}})S_K\ 2^{n(N+4)/N} + (1+K)R_K\ 2^{n((2N+4)/N-\overline{\mu})}$$

and $S_K = 2\max(1/K^{(N+4)/N}, 1/K^{(N+2)/N})$, $R_K = 2\max(1/K^{(2N+4)/N-\overline{\mu}}, 1/K^{2(N+1)/N-\overline{\mu}})$, $Q_K = S_K + 2\max(1/K^{4/N}, 1/K^{2/N})$.

Let us explain how the restrictions on N and $\overline{\mu}$ work. First of all, it will be crucial to remark that $\mathcal{K}(n,K,T)$ is bounded with respect to $K > 1$ provided $\overline{\mu} \leq 2(N+1)/N - 1 = (N+2)/N$ which means $\overline{\mu} = 2$ in dimension $N = 2$ and $\overline{\mu} = 2$ or 3 in dimension $N = 1$. Second of all, we go back to Lemma 3.1 and we shall exploit the dissipation term that comes from the diffusion. Indeed, we expect an estimate of $\Phi(a_i - k)$ in $L^\infty(0,T^\star; L^1(\Omega))$ together with an estimate of $(1+[a_i-k]_+)^{-1/2}\nabla(1+[a_i-k]_+)$ in $L^2((0,T^\star)\times\Omega)$. Combining these information would lead to $\nabla Z([a_i-k]_+) \in L^2(0,T^\star; L^1(\Omega))$ where

$$Z(u) = \int_0^u \sqrt{\frac{\Phi(z)}{1+z}}\ \mathrm{d}z = \int_0^u \sqrt{\ln(1+z) + \frac{1}{1+z} - 1}\ \mathrm{d}z.$$

Let us consider a non-negative function u defined on $[T,T^\star]\times\Omega$ such that $Z(u)$ belongs to $L^\infty(T,T^\star;L^1(\Omega))$ and $\nabla Z(u)$ belongs to $L^2(T,T^\star;L^1(\Omega))$ According to the Gagliardo-Nirenberg-Sobolev inequality (see [3], Th. IX.9, p. 162) the latter implies that

$$Z(u) \in L^2(T,T^\star; L^{N/(N-1)}(\Omega)).$$

We seek a homogeneous Lebesgue space with respect to the variables t, x. For $N \leq 2$ we can obtain:

$$Z(u) \in L^{(N+2)/N}((T,T^\star)\times\Omega).$$

Indeed, if $N = 2$ we have $(N+2)/N = N/(N-1) = 2$, and if $N = 1$:

$$\int_T^{T^\star}\int_\Omega |v|^3\,\mathrm{d}x\,\mathrm{d}t \leq \int_T^{T^\star} \|v(t)\|_{L^1(\Omega)}\|v(t)\|^2_{L^\infty(\Omega)}\,\mathrm{d}t \leq \|v\|_{L^\infty(T,T^\star;L^1(\Omega))}\|v\|^2_{L^2(T,T^\star;L^\infty(\Omega))}.$$

Eventually, we aim at comparing $Z(u)^{(N+2)/N}$ to $\psi(u)\ln(1+u)$ where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ has a polynomial behavior. Specifying the behavior of ψ will induce restrictions on $\overline{\mu}$ that depend on the space dimension. Of course, it suffices to discuss the comparizon as $u \to 0$ and $u \to \infty$. Since $\ln(1+z)+1/(1+z)-1 \sim_{z\to 0} z^2/2$ we first obtain that $Z(u)^{(N+2)/N} \geq u^{(2N+4)/N}/8$ for $u \in [0,\delta]$, $\delta > 0$ small enough. It follows that $\psi(u)\ln(1+u)$ can indeed be dominated by $Z(u)$ for bounded u's provided $\psi(u) \sim_{u\to 0} u^{(N+4)/N}$. Next, there exists $A > 0$ such that for $z \geq A$ large enough, we have $\ln(1+z)+1/(1+z)-1 \geq \frac{1}{2}\ln(1+z)$. Thus, for $u \geq 2A$ we get

$$Z(u) \geq \frac{1}{\sqrt{2}}\int_{u/2}^u \sqrt{\ln(1+z)}\,\mathrm{d}z \geq \frac{1}{2\sqrt{2}}u\sqrt{\ln(1+u/2)} \geq C_1 u\sqrt{\ln(1+u)}.$$

Hence $Z(u)^{(N+2)/N}$ dominates $\psi(u)\ln(1+u)$ provided $N \leq 2$ and $\psi(u) \sim_{u\to\infty} u^{(N+2)/N}$. Reasoning the same way, we also prove that there exists $C > 0$ such that $Z(u) \leq C\ \Phi(u)$ holds for any $u \geq 0$. Let us summarize the properties that we need to justify Proposition 3.1.

Lemma 3.2 *Let us set*

$$\psi(u) = u^{(N+4)/N} \mathbb{1}_{0\le u\le 1} + u^{(N+2)/N} \mathbb{1}_{u\ge 1}.$$

There exists a constant $C > 0$ such that

$$\psi(u)\ln(1+u) \le C\ Z(u)^{(N+2)/N}, \qquad \text{and} \qquad Z(u) \le C\ \Phi(u).$$

holds for any $u \ge 0$. Furthermore, for every non-negative function u defined on $[T, T^\star] \times \Omega$ we have:

$$\int_T^{T^\star} \left| \int_\Omega |\nabla Z(u)| \,\mathrm{d}x \right|^2 \mathrm{d}\tau \le \sup_{T\le\tau\le T^\star} \left(\int_\Omega \Phi(u)(\tau,x)\,\mathrm{d}x \right) \int_T^{T^\star} \int_\Omega \left|\nabla\sqrt{1+u}\right|^2(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau.$$

Proof of Proposition 3.1. The proof splits into two steps: firstly we modify (3.2) so that, secondly, we can make the dissipation terms appear by appealing to the Gagliardo-Nirenberg inequality.
Step 1. The first step consists in showing the following inequality:

$$\mathcal{U}_n \le C\ \mathcal{K}(n,K,T) \sum_{i=1}^p \int_{t_{n-1}}^{T^\star} \int_\Omega \psi(a_i - k_{n-1}) \ln(1 + [a_i - k_{n-1}]_+)\,\mathrm{d}x\,\mathrm{d}\tau, \tag{3.3}$$

where the auxiliary function ψ has been introduced in Lemma 3.2. We start by noting that if $a_i \ge k_n \ge k_{n-1}$, then $(a_i - k_{n-1})/(k_n - k_{n-1}) \ge 1$. Therefore we can write for any $\alpha, \beta \ge 0$,

$$\begin{aligned} \mathbb{1}_{a_i\ge k_n} &\le \left(\frac{[a_i - k_{n-1}]_+}{k_n - k_{n-1}}\right)^\alpha \mathbb{1}_{\{k_n\le a_i\le 1+k_{n-1}\}} + \left(\frac{[a_i - k_{n-1}]_+}{k_n - k_{n-1}}\right)^\beta \mathbb{1}_{a_i\ge 1+k_{n-1}} \\ &\le \frac{2^{n\alpha}}{K^\alpha}[a_i - k_{n-1}]_+^\alpha \mathbb{1}_{0\le a_i - k_{n-1}\le 1} + \frac{2^{n\beta}}{K^\beta}[a_i - k_{n-1}]_+^\beta \mathbb{1}_{a_i - k_{n-1}\ge 1}. \end{aligned}$$

By using these simple estimates with $\alpha = (N+4)/N$, $\beta = (N+2)/N$ and $\alpha = (N+4)/N - \overline{\mu} + 1$, $\beta = (N+2)/N - \overline{\mu} + 1$ respectively (note that in both case $\alpha \ge \beta$), we are led to

$$(1 + k_n^{\overline{\mu}}) \ln(1 + [a_i - k_n]_+) \le (1 + K^{\overline{\mu}})\ 2^{n(N+4)/N}\ S_K\ \psi(a_i - k_{n-1}) \ln(1 + [a_i - k_{n-1}]_+),$$

and

$$(1 + k_n)[a_i - k_n]_+^{\overline{\mu}-1} \ln(1 + [a_i - k_n]_+) \le (1 + K)\ 2^{n((2N+4)/N - \overline{\mu})}\ R_K\ \psi(a_i - k_{n-1}) \ln(1 + [a_i - k_{n-1}]_+).$$

Coming back to (3.2) yields

$$\begin{aligned} \mathcal{U}_n \le\ & \frac{2^n}{T} \sum_{i=1}^p \int_{t_{n-1}}^{t_n} \int_\Omega \Phi(a_i - k_n)\,\mathrm{d}x\,\mathrm{d}s \\ & + C\Big((1 + K^{\overline{\mu}}) S_K 2^{n(N+4)/N} + (1+K) R_K 2^{n((2N+4)/N - \overline{\mu})}\Big) \\ & \qquad \times \sum_{i=1}^p \int_{t_{n-1}}^{T^\star} \int_\Omega \psi(a_i - k_{n-1}) \ln(1 + [a_i - k_{n-1}]_+)\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned}$$

The first integral in the right hand side can be dominated in a similar way (using $\alpha = 4/N$, $\beta = 2/N$); precisely, we have

$$\begin{aligned}
&\frac{2^n}{T}\sum_{i=1}^{p}\int_{t_{n-1}}^{t_n}\int_\Omega \Phi(a_i - k_n)\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad \le \frac{2^n}{T}\sum_{i=1}^{p}\int_{t_{n-1}}^{t_n}\int_\Omega (1+[a_i-k_n]_+)\ \ln(1+[a_i-k_n]_+)\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad \le \frac{1}{T}2^{n(2N+4)/N}\ Q_K\ \sum_{i=1}^{p}\int_{t_{n-1}}^{T^\star}\int_\Omega \psi(a_i-k_{n-1})\ln(1+[a_i-k_{n-1}]_+)\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Therefore, we have proved from (3.2) that (3.3) holds.

Step 2. Now, we go back to Lemma 3.2 so that (3.3) becomes

$$\mathcal{U}_n \le C\ \mathcal{K}(n,K,T)\ \sum_{i=1}^{p}\int_{t_{n-1}}^{T^\star}\int_\Omega \big|Z([a_i-k_{n-1}]_+)\big|^{(N+2)/N}\,\mathrm{d}x\,\mathrm{d}\tau. \tag{3.4}$$

Let us distinguish depending on the dimension $N = 1$ or $N = 2$ how we conclude by using the Gagliardo-Nirenberg-Sobolev inequality.

For $N = 2$, using the Gagliardo-Nirenberg-Sobolev inequality and Lemma 3.2, we obtain

$$\begin{aligned}
\mathcal{U}_n \quad \le \quad C\ \mathcal{K}(n,K,T)\ \sum_{i=1}^{p}\Bigg[&\int_{t_{n-1}}^{T^\star}\left(\int_\Omega \big|\nabla Z([a_i-k_{n-1}]_+)\big|\,\mathrm{d}x\right)^2\mathrm{d}\tau \\
&+\int_{t_{n-1}}^{T^\star}\left(\int_\Omega \Phi([a_i-k_{n-1}]_+)\,dx\right)^2 ds\Bigg]
\end{aligned}$$

Then, we use the second statement in Lemma 3.2 to obtain

$$\begin{aligned}
\mathcal{U}_n \quad \le \quad & C\ \mathcal{K}(n,K,T) \\
&\times\sum_{i=1}^{p}\Bigg[\left(\sup_{t_{n-1}\le\tau\le T^\star}\int_\Omega \Phi(a_i-k_{n-1})(\tau)\,\mathrm{d}x\int_{t_{n-1}}^{T^\star}\int_\Omega \big|\nabla\sqrt{1+[a_i-k_{n-1}]_+}\big|^2\,\mathrm{d}x\,\mathrm{d}\tau\right) \\
&\qquad +\int_{t_{n-1}}^{T^\star}\left(\int_\Omega \Phi([a_i-k_{n-1}]_+)\,dx\right)^2 ds\Bigg] \\
\le \quad & C(1+T^\star)\ \mathcal{K}(n,K,T)\ \mathcal{U}_{n-1}^2.
\end{aligned}$$

For $N=1$, we proceed as follows

$$\begin{aligned}
\mathcal{U}_n &\leq C\ \mathcal{K}(n,K,T)\sum_{i=1}^{p}\int_{t_{n-1}}^{T^\star}\left(\|Z(a_i-k_{n-1})(t,\cdot)\|^2_{L^\infty(\Omega)}\ \int_\Omega Z(a_i-k_{n-1})(t,x)\,\mathrm{d}x\right)\mathrm{d}t\\
&\leq C\ \mathcal{K}(n,K,T)\ \sum_{i=1}^{p}\Bigg[\sup_{t_{n-1}\leq t\leq T^\star}\int_\Omega Z(a_i-k_{n-1})\,\mathrm{d}x\\
&\qquad\qquad\times\int_{t_{n-1}}^{T^\star}\left(\int_\Omega\big(|Z(a_i-k_{n-1})|+|\nabla Z(a_i-k_{n-1})|\big)\,\mathrm{d}x\right)^2\mathrm{d}t\Bigg]\\
&\leq C\ \mathcal{K}(n,K,T)\ \sum_{i=1}^{p}\Bigg[2T^\star\Big(\sup_{t_{n-1}\leq t\leq T^\star}\int_\Omega\Phi(a_i-k_{n-1})\,\mathrm{d}x\Big)^3\\
&\qquad\qquad+\left(\sup_{t_{n-1}\leq t\leq T^\star}\int_\Omega\Phi(a_i-k_{n-1})\,\mathrm{d}x\right)^2\int_{t_{n-1}}^{T^\star}\int_\Omega\left|\nabla\sqrt{1+[a_i-k_{n-1}]_+}\right|^2\mathrm{d}x\,\mathrm{d}t\Bigg]\\
&\leq C\ (1+T^\star)\ \mathcal{K}(n,K,T)\ \mathcal{U}_{n-1}^3.
\end{aligned}$$

This ends the proof of Proposition 3.1. ∎

Finishing the proof of the L^∞ bound needs the following elementary claim.

Lemma 3.3 *Let $\big(\mathcal{V}_n\big)_{n\in\mathbb{N}}$ be a sequence verifying*

$$\mathcal{V}_n\leq M^n\mathcal{V}_{n-1}^q$$

for some $M>0$, $q>1$. Then for any $n_0\in\mathbb{N}$, there exists ε, such that if $\mathcal{V}_{n_0}<\varepsilon$, then $\lim_{n\to\infty}\mathcal{V}_n=0$.

Proof. Without loss of generality we suppose $n_0=0$. Let us set $\mathcal{W}_n=\ln(\mathcal{V}_n)$. We have

$$\mathcal{W}_n\leq n\ln(M)+q\mathcal{W}_{n-1}$$

which yields

$$\mathcal{W}_n\leq\ln(M)\sum_{j=0}^{n}q^{n-j}j+q^n\mathcal{W}_0\leq q^n\ln(M^{1/(q(1-1/q)^2)}\ \mathcal{V}_0).$$

So, if $\mathcal{V}_0<M^{-1/(q(1-1/q)^2)}$, $\mathcal{W}_n$ converges to $-\infty$, and $\mathcal{V}_n$ converges to 0. ∎

Hence, it remains to check that the first term of the iteration can be made small choosing K large enough. Indeed, let us go back to Proposition 3.1. Picking $K>1$, we can summarize the obtained estimate as

$$\mathcal{U}_n\leq C(1+T^\star)(1+1/T)\ 2^{n(2N+4)/N}\ \mathcal{U}_{n-1}^{(N+2)/N}.$$

The keypoint is to remark that Q_K, KR_K and $K^{\overline{\mu}}S_K$ remain bounded for large K's so that the constant C above does not depend on K. Hence, we apply Lemma 3.3 to $\mathcal{V}_n=\big(C(1+T^\star)(1+1/T)\big)^{2/N}\ \mathcal{U}_n$, $q=1+2/N$ and $M=2^{(2N+4)/N}$.

Now, let us specialize (3.4) to the case $n=2$; we get (with C which still does not depend on K)

$$\mathcal{U}_2\leq C(1+1/T)\left[\sum_{i=1}^{p}\int_0^{T^\star}\left(\int_\Omega|\nabla Z(a_i-K/2)|\,\mathrm{d}x\right)^2\mathrm{d}t+\sum_{i=1}^{p}\int_0^{T^\star}\left(\int_\Omega|\Phi(a_i-K/2)\,\mathrm{d}x|\right)^2\mathrm{d}t\right]\quad(3.5)$$

in dimension $N=2$ and in dimension $N=1$ the same expression is multiplied by the quantity $\sup_{0\leq t\leq T^\star}\sum_{i=1}^{p}\int_\Omega\Phi(a_i-K/2)\,\mathrm{d}x$. This allows to establish the following statement.

Lemma 3.4 *Let $\epsilon > 0$. Then, there exists $K_\epsilon \geq 1$ such that for any $K \geq K_\epsilon$ we have $\mathcal{U}_2 \leq \epsilon$.*

Proof. The proof reduces to prove that the two integrals in the right hand side of (3.5) tend to 0 as $K \to +\infty$. As a matter of fact, there exists $C > 0$ such that for any $z \geq 0$ we have $(1+z)\ln(1+z) \leq C\ z(1+|\ln(z)|)$. Furthermore, there exists $C > 0$ such that for any $k > 1$ and $z \geq 0$, we have

$$[z-k]_+(1+|\ln([z-k]_+)|) \leq C\ z(1+|\ln z|).$$

Accordingly, we deduce that $\Phi(a_i - K/2)$ converges to 0 for a.e $(t,x) \in (0,T^\star) \times \Omega$ as K goes to infinity and it is dominated by $a_i(1+|\ln(a_i)|)$, which satisfies

$$\sum_{i=1}^p \int_0^{T^\star} \left(\int_\Omega a_i(1+|\ln(a_i)|)\,\mathrm{d}x| \right)^2 \mathrm{d}t < \infty$$

owing to Proposition 2.1. Applying the Lebesgue dominated convergence theorem then shows that

$$\lim_{K\to\infty} \left\{ \sum_{i=1}^p \int_0^{T^\star} \left(\int_\Omega \Phi(a_i - K/2)\,\mathrm{d}x \right)^2 \mathrm{d}t \right\} = 0.$$

Next, we simply write

$$\nabla Z(a_i - K/2) = \mathbb{1}_{a_i \geq K/2} \sqrt{\ln(1+[a_i - K/2]_+) + \frac{1}{1+[a_i-K/2]_+} - 1}\ \nabla a_i.$$

Then, we remark that $z \mapsto \ln(1+z) + 1/(1+z) - 1$ is non-decreasing which allows to establish

$$|\nabla Z(a_i - K/2)| \leq \mathbb{1}_{a_i \geq K/2} \sqrt{\ln(1+a_i) + \frac{1}{1+a_i} - 1}\ |\nabla a_i| = \mathbb{1}_{a_i\geq K/2}\ |\nabla Z(a_i)| \leq |\nabla Z(a_i)|.$$

Observe that $\mathbb{1}_{a_i \geq K/2}\ |\nabla Z(u)|$ decreases to 0 as $K \to \infty$ for a.e $(t,x) \in (0,T^\star) \times \Omega$. Furthermore, Lemma 3.2 yields

$$\sum_{i=1}^p \int_0^{T^\star} \left(\int_\Omega |\nabla Z(a_i)|\,\mathrm{d}x \right)^2 \mathrm{d}t \leq \sum_{i=1}^p \sup_{0\leq t\leq T^\star} \int_\Omega \Phi(a_i)\,\mathrm{d}x\ \int_0^{T^\star} \int_\Omega |\nabla\sqrt{1+a_i}|^2\,\mathrm{d}x\,\mathrm{d}t < \infty$$

by using the basic estimates in Proposition 2.1 again. We conclude by classical integration theory arguments. ∎

We can now finish the proof of Theorem 1.1. Let us emphasize the dependence with respect to K by denoting $\mathcal{U}_n^{(K)}$. We first fix K which makes $\mathcal{U}_2^{(K)}$ small enough (remark that K is more constrained as T is chosen small) so that we obtain by applying Lemma 3.3

$$\lim_{n\to\infty} \mathcal{U}_n^{(K)} = 0.$$

However, we clearly have

$$\mathcal{U}_n^{(K)} \geq \frac{1}{T^\star - t_n} \int_{t_n}^{T^\star} \int_\Omega \Phi(a_i - k_n)\,\mathrm{d}x\,\mathrm{d}t \geq 0.$$

Letting n go to infinity and applying the Fatou lemma, we deduce that

$$\frac{1}{T^\star - T} \int_T^{T^\star} \int_\Omega \Phi(a_i - K)\,\mathrm{d}x\,\mathrm{d}t = 0,$$

which implies that $0 \leq a_i(t,x) \leq K$ for a.e $(t,x) \in (T,T^\star) \times \Omega$.

Remark 3.3 *Since the initial data is required to satisfy* (1.7) *only and is not supposed to be bounded, it is clearly hopeless to extend Theorem 1.1 with $T = 0$. It appears clearly through the factor $1/T$ which appear in the estimates above.*

4 Hausdorff dimension of the set of singular points

In this section we study the Hausdorff dimension of the blow-up points of the solutions of (1.1). The derivation of the necessary estimates remains close to the strategy described in the previous section; again a restriction on the degree of nonlinearity appears. It turns out that relevant results can be obtained by this method in dimension $N \geq 3$ with $\overline{\mu} = 2$, while we are not able to reach improvements in direction of higher nonlinearities for lower dimensions. For the sake of simplicity, in what follows we assume that the diffusion coefficients D_i are constant with respect to the space variable (but they still depend on i, otherwise the problem becomes trivial by remarking that $\rho(t,x) = \sum_{i=1}^{p} a_i(t,x)$ satisfies the heat equation $\partial_t \rho - D\Delta_x \rho = 0$, with D the common value of the diffusion coefficients). Then, we shall prove Theorem 1.3.

To begin with, let us recall a few definitions about Hausdorff dimension. For a given nonempty set $A \subset \mathbb{R}^d$, $s \geq 0$, $\delta > 0$, we set

$$\mathcal{H}^s_\delta(A) = \inf\Big\{\frac{\Gamma(1/2)^s}{2^s\ \Gamma(s/2+1)} \sum_i \big(\mathrm{diam}(A_i)\big)^s,\ A \subset \bigcup_i A_i,\ \mathrm{diam}(A_i) \leq \delta\Big\},$$

and then $\mathcal{H}^s(A) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(A)$. The Hausdorff dimension of A is defined by

$$\dim_{\mathcal{H}}(A) = \inf\{s > 0,\ \mathcal{H}^s(A) = 0\} = \sup\{s > 0,\ \mathcal{H}^s(A) = +\infty\}.$$

We refer to [12] (p. 171) for more details.

The starting point of the proof is two-fold. Firstly, we use mass conservation and entropy dissipation to control the solution in a certain L^p space, identifying the highest exponent p for which such an estimate is possible. Secondly, we remark that the problem admits an invariant scaling. This is the purpose of the following claims.

Lemma 4.1 *Let $N > 2$ and $\Omega \subset \mathbb{R}^N$. There exists $C > 0$ such that for any $T > 0$ and for any non-negative function $u \in L^\infty(0,T;L^1(\Omega))$ verifying $\nabla\sqrt{u} \in L^2((0,T)\times\Omega)$, we have*

$$\int_0^T \int_\Omega |u|^{\frac{(N+2)}{N}}\,\mathrm{d}x\,\mathrm{d}t \leq C\ \|u\|^{2/N}_{L^\infty(0,T;L^1(\Omega))}\ \Big(T\|u\|_{L^\infty(0,T;L^1(\Omega))} + \|\nabla\sqrt{u}\|^2_{L^2((0,T)\times\Omega)}\Big). \tag{4.1}$$

Next, we introduce the function

$$\Psi(X) = X\mathbb{1}_{0\leq X\leq 1} + \sqrt{X}\mathbb{1}_{X\geq 1}.$$

There exists $C > 0$ such that for any $T > 0$ and for any non-negative function u verifying $\nabla\sqrt{u} \in L^2((0,T)\times\Omega)$, and $\Phi(u) \in L^\infty(0,T;L^1(\Omega))$ we have

$$\int_0^T \int_\Omega |\Psi(u)|^{\frac{2(N+2)}{N}}\,\mathrm{d}x\,\mathrm{d}t \leq C\ \|\Phi(u)\|^{2/N}_{L^\infty(0,T;L^1(\Omega))}\ \Big(T\|\Phi(u)\|_{L^\infty(0,T;L^1(\Omega))} + \|\nabla\sqrt{1+u}\|^2_{L^2((0,T)\times\Omega)}\Big). \tag{4.2}$$

We are concerned with weak solutions of (1.1), that is functions a_i that verifies (1.1) in the sense of distributions, together with the estimates in Proposition 2.1, deduced from the fundamental properties (1.4) and (1.5) of the system. We shall use the fact that the norm $L^{(N+2)/N}$ of such a solution is finite, as a consequence of (4.1). Another important ingredient relies on the invariant scaling of the equation.

Lemma 4.2 *Let a be a solution of* (1.1). *Let $t_0 > 0$ and $x_0 \in \Omega$. Then, for any $0 < \varepsilon \ll 1$*

$$a_\varepsilon(t,x) = \varepsilon^{2/(\overline{\mu}-1)}\ a(t_0 + \varepsilon^2 t, x_0 + \varepsilon x)$$

satisfies (1.1).

Lemma 4.2 is straightforward. Let us sketch the proof of Lemma 4.1.

Proof of Lemma 4.1. There exists a constant $C > 0$ such that for any $X \geq 0$

$$\Psi(X) \leq C\sqrt{\Phi(X)}, \qquad \Psi(X) \leq C(\sqrt{1+X} - 1).$$

Moreover, Ψ is a Lipschitzian function verifying

$$0 \leq \Psi'(X) \leq 2\sqrt{2}\frac{\mathrm{d}}{\mathrm{d}X}(\sqrt{1+X} - 1).$$

Hence, we get

$$\|\Psi(u)\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C\|\Phi(u)\|_{L^\infty(0,T;L^1(\Omega))}, \qquad \|\nabla\Psi(u)\|_{L^2(0,T;L^2(\Omega))} \leq C\|\nabla\sqrt{1+u}\|_{L^2(0,T;L^2(\Omega))}.$$

Since $2 < 2(N+2)/N < 2N/(N-2)$, the Hölder inequality yields

$$\int_0^T \int_\Omega |\Psi(u)|^{\frac{2(N+2)}{N}}\,\mathrm{d}x\,\mathrm{d}t \leq \int_0^T \left(\int_\Omega |\Psi(u)|^2\,\mathrm{d}x\right)^{2/N} \left(\int_\Omega |\Psi(u)|^{2N/(N-2)}\,\mathrm{d}x\right)^{(N-2)/N}\,\mathrm{d}t.$$

Therefore the Sobolev embedding $H^1(\Omega) \subset L^{2N/(N-2)}(\Omega)$ leads to (4.2) since

$$\begin{aligned}
\int_0^T \int_\Omega |\Psi(u)|^{\frac{2(N+2)}{N}}\,\mathrm{d}x\,\mathrm{d}t &\leq C\left(\sup_{0\leq t\leq T}\int_\Omega |\Psi(u)|^2\,\mathrm{d}x\right)^{2/N} \int_0^T\int_\Omega \left(|\Psi(u)|^2 + |\nabla\Psi(u)|^2\right)\mathrm{d}x\,\mathrm{d}t. \\
&\leq C\|\Phi(u)\|^{2/N}_{L^\infty(0,T;L^1(\Omega))}\left(T\|\Phi(u)\|_{L^\infty(0,T;L^1(\Omega))} + \int_0^T\int_\Omega |\nabla\sqrt{1+u}|^2\,\mathrm{d}x\,\mathrm{d}t\right).
\end{aligned}$$

We obtain (4.1) with a similar combination of the Hölder inequality and the Sobolev embedding. ■

Remark 4.1 *We shall use the inequality* (4.2) *with a sequence of balls $B(0,1) \subset \mathcal{B}_n \subset B(0,2)$ as space domain. Since the proof of* (4.2) *involves the Sobolev embedding, the constant C thus depends on the parameter n. However, we can estimate it uniformly. Indeed, the Sobolev embedding on $\Omega = B(0,1)$ reads*

$$\left(\int_{B(0,1)} |u(x)|^{2N/(N-2)}\,\mathrm{d}x\right)^{(N-2)/N} \leq C_1\left(\int_{B(0,1)} |u(x)|^2\,\mathrm{d}x + \int_{B(0,1)} |\nabla u(x)|^2\,\mathrm{d}x\right),$$

with C_1 the Sobolev constant on $B(0,1)$. We apply it with $u(x) = \lambda^{(N-2)/2}u(\lambda x)$, $\lambda > 0$. By using the change of variable $y = \lambda x$, it follows that

$$\left(\int_{B(0,\lambda)} |u(y)|^{2N/(N-2)}\,\mathrm{d}y\right)^{(N-2)/N} \leq C_1\left(\lambda^{-2}\int_{B(0,\lambda)} |u(y)|^2\,\mathrm{d}y + \int_{B(0,\lambda)} |\nabla u(y)|^2\,\mathrm{d}y\right).$$

Hence, the Sobolev constant on $B(0,\lambda)$ for any $\lambda > 1$ is dominated by $2C_1$.

Keeping in mind Lemma 4.2 we consider now solutions of (1.1) that are defined for negative times. Let us set

$$k_n = 1 - 1/2^n, \qquad t_n = 1 + 1/2^n \qquad \mathcal{B}_n = B(0, t_n), \qquad \mathcal{Q}_n = (-t_n, 0) \times \mathcal{B}_n.$$

Note that $\mathcal{B}_n \subset \mathcal{B}_{n-1}$ and $\mathcal{Q}_n \subset \mathcal{Q}_{n-1}$. We introduce a cut-off function

$$\begin{cases} \zeta_n : \mathbb{R}^N \to \mathbb{R}, & 0 \le \zeta_n(x) \le 1, \\ \zeta_n(x) = 1 \text{ for } x \in \mathcal{B}_n, & \zeta_n(x) = 0 \text{ for } x \in \complement \mathcal{B}_{n-1}, \\ \displaystyle\sup_{i,j \in \{1,\dots,N\},\ x \in \mathbb{R}^N} |\partial^2_{ij} \zeta_n(x)| \le C\ 2^{2n}. \end{cases}$$

We define

$$\mathcal{U}_n = \sup_{-t_n \le t \le 0} \sum_{i=1}^p \int_{\mathcal{B}_n} \Phi(a_i - k_n)\,\mathrm{d}x + \sum_{i=1}^p \iint_{\mathcal{Q}_n} \left|\nabla \sqrt{1 + [a_i - k_n]_+}\right|^2 \mathrm{d}x\,\mathrm{d}s.$$

Multiplying (1.1) by $\zeta_n(x)\Phi'(a_i - k_n)$ we obtain the following localized version of (3.1)

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^p \int_\Omega \Phi(a_i - k_n)\ \zeta_n\,\mathrm{d}x + \sum_{i=1}^p \int_\Omega D_i \nabla a_i \cdot \nabla a_i\ \Phi''(a_i - k_n)\ \zeta_n\,\mathrm{d}x \\ &\quad = \sum_{i=1}^p \int_\Omega Q_i(a)\Phi'(a_i - k_n)\zeta_n\,\mathrm{d}x + \sum_{i=1}^p \int_\Omega D_i : D^2\zeta_n\ \Phi(a_i - k_n)\,\mathrm{d}x, \end{aligned} \tag{4.3}$$

where $D^2\zeta_n$ stands for the hessian matrix of ζ_n and $A : B = \sum_{k,l=1}^N A_{kl}B_{kl}$. Remark that $0 \le \mathbb{1}_{\mathcal{B}_n}(x) \le \zeta_n(x) \le \zeta_{n-1}(x) \le 1$ and $|\partial^2_{kl}\zeta_n(x)| \le 2^{2n}\mathbb{1}_{\mathcal{B}_{n-1}}(x)$. Then, reproducing the proof of Lemma 3.1 and (3.2) we obtain

$$\begin{aligned} \mathcal{U}_n \le\ & C2^{2n} \sum_{i=1}^p \iint_{\mathcal{Q}_{n-1}} \Phi(a_i - k_n)(s,x)\,\mathrm{d}x\,\mathrm{d}s \\ & + C \sum_{i=1}^p \iint_{\mathcal{Q}_{n-1}} \left(1 + k_n^{\overline{\mu}} + (1 + k_n)[a_i - k_n]_+^{\overline{\mu}-1}\right)\ \ln(1 + [a_i - k_n]_+)(\tau, x)\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{4.4}$$

From this relation we are able to establish the following statements.

Proposition 4.1 *Let $N > 2$ and $\overline{\mu} = 2$. The following relation holds*

$$\mathcal{U}_n \le C\ 2^{4n(N+1)/N}\ \mathcal{U}_{n-1}^{1+2/N}$$

for any $n \ge 1$. Accordingly, if $\mathcal{U}_1$ is small enough then $\lim_{n\to\infty} \mathcal{U}_n = 0$.

Corollary 4.1 *There exists a universal constant $\eta_\star > 0$ such that any solution of* (1.1) *satisfying*

$$\sum_{i=1}^p \int_{-2}^0 \int_{B(0,2)} |a_i|^{(N+2)/N}\,\mathrm{d}x\,\mathrm{d}\tau \le \eta_\star$$

is such that for any $i \in \{1, \dots, p\}$ we have

$$0 \le a_i(t, x) \le 1 \qquad a.\ e.\ in\ (-1, 0) \times B(0, 1).$$

Proof of Proposition 4.1. There exists $C_N > 0$ such that

$$\begin{aligned}\Phi(a_i - k_n) &\leq (1 + [a_i - k_n]_+)\ \ln\big(1 + [a_i - k_n]_+\big)\\ &\leq C_N \left(\mathbb{1}_{a_i \geq k_n} + \big|\Psi(a_i - k_n)\big|^{\frac{2(N+2)}{N}}\right).\end{aligned}$$

Similarly, assuming $0 \leq \overline{\mu} - 1 < 1 + 2/N$, we can find $C_{N,\overline{\mu}} > 0$ such that

$$[a_i - k_n]^{\overline{\mu}-1} \ln\big(1 + [a_i - k_n]_+\big) \leq C_{N,\overline{\mu}} \left(\mathbb{1}_{a_i \geq k_n} + \big|\Psi(a_i - k_n)]\big|^{\frac{2(N+2)}{N}}\right).$$

For $N \geq 2$, this restricts to the case $\overline{\mu} = 2$. Together with (4.4), this gives

$$\mathcal{U}_n \leq C 2^{2n} \sum_{i=1}^{p} \iint_{\mathcal{Q}_{n-1}} \left(\mathbb{1}_{a_i \geq k_n} + \big|\Psi(a_i - k_n)\big|^{\frac{2(N+2)}{N}}\right) \mathrm{d}x\, \mathrm{d}s.$$

We note that $0 \leq k_n \leq 1$, and $k_n \geq k_{n-1}$. Consequently we have $0 \leq \Psi(a_i - k_n) \leq \Psi(a_i - k_{n-1})$. Moreover, we remark that

$$\mathbb{1}_{a_i \geq k_n} = \mathbb{1}_{1 \geq a_i - k_{n-1} \geq k_n - k_{n-1}} + \mathbb{1}_{a_i - k_{n-1} \geq 1},$$

with

$$\begin{aligned}&\mathbb{1}_{1 \geq a_i - k_{n-1} \geq k_n - k_{n-1}} \leq \left(\frac{a_i - k_{n-1}}{k_n - k_{n-1}}\right)^{\frac{2(N+2)}{N}} \mathbb{1}_{0 \leq a_i - k_{n-1} \leq 1} \leq 2^{\frac{2n(N+2)}{N}} (a_i - k_{n-1})^{\frac{2(N+2)}{N}} \mathbb{1}_{0 \leq a_i - k_{n-1} \leq 1},\\ &\mathbb{1}_{1 \leq a_i - k_{n-1}} \leq (a_i - k_{n-1})^{\frac{N+2}{N}} \mathbb{1}_{a_i - k_{n-1} \geq 1} \leq 2^{\frac{2n(N+2)}{N}} (a_i - k_{n-1})^{\frac{N+2}{N}} \mathbb{1}_{a_i - k_{n-1} \geq 1}.\end{aligned}$$

Hence, we have

$$\mathbb{1}_{a_i \geq k_n} \leq 2^{\frac{2n(N+2)}{N}} \big|\Psi(a_i - k_{n-1})\big|^{\frac{2(N+2)}{N}}.$$

We are thus led to

$$\mathcal{U}_n \leq C 2^{\frac{4n(N+1)}{N}} \sum_{i=1}^{p} \iint_{\mathcal{Q}_{n-1}} \big|\Psi(a_i - k_{n-1})\big|^{\frac{2(N+2)}{N}} \mathrm{d}x\, \mathrm{d}s. \tag{4.5}$$

Finally, applying Lemma 4.1 (see also Remark 4.1) we obtain

$$\mathcal{U}_n \leq C 2^{\frac{4n(N+1)}{N}} \mathcal{U}_{n-1}^{1+2/N}.$$

Coming back to Lemma 3.3 finishes the proof of Proposition 4.1. ∎

Proof of Corollary 4.1. We are thus now left with the task of discussing the smallness of $\mathcal{U}_1$. Note that $\Psi(X) \leq \sqrt{X}$ for all $X > 0$. Hence, from (4.5) with $n = 1$, we find

$$\mathcal{U}_1 \leq C 2^{\frac{4(N+1)}{N}} \sum_{i=1}^{p} \iint_{\mathcal{Q}_0} |a_i|^{\frac{N+2}{N}} \mathrm{d}x\, \mathrm{d}s = C_N \sum_{i=1}^{p} \|a_i\|_{L^{(N+2)/N}((-2,0)\times B(0,2))}^{\frac{N+2}{N}}.$$

Hence, for $\sum_{i=1}^{p} \|a_i\|_{L^{(N+2)/N}((-2,0)\times B(0,2))}^{(N+2)/N}$ small enough, we have $\mathcal{U}_1$ small so that $\lim_{n\to\infty} \mathcal{U}_n = 0$, by Proposition 4.1. We conclude by reproducing the arguments at the end of the proof of Theorem 1.1. We have

$$0 \leq \int_{-t_n}^{0} \int_{\mathcal{B}_n} \Phi(a_i - k_n)\, \mathrm{d}x\, \mathrm{d}t \leq \mathcal{U}_n.$$

Hence letting n go to ∞ yields, by using the Fatou lemma,

$$\lim_{n\to\infty}\int_{-t_n}^{0}\int_{\mathcal{B}_n}\Phi(a_i-k_n)\,\mathrm{d}x\,\mathrm{d}t=0=\int_{-1}^{0}\int_{B(0,1)}\Phi(a_i-1)\,\mathrm{d}x\,\mathrm{d}t.$$

It implies that $0\le a_i(t,x)\le 1$ holds a. e. on $(-1,0)\times B(0,1)$. ∎

Now, these statements allow us to deduce some property of the solution of the original Cauchy problem. To this end, we go back to the scaling argument in Lemma 4.2. Indeed, we notice that

$$\int_{-2}^{0}\int_{B(0,2)}|a_\varepsilon(\tau,x)|^{(N+2)/N}\,\mathrm{d}x\,\mathrm{d}\tau=\frac{1}{2}\ \varepsilon^{2(N+2)/N-(N+2)}\int_{t_0-2\varepsilon^2}^{t_0+2\varepsilon^2}\int_{|y-x_0|\le 2\varepsilon}|a(s,y)|^{(N+2)/N}\,\mathrm{d}y\,\mathrm{d}s$$

holds (recall that we are dealing with the case $\overline{\mu}=2$ only). We deduce the following statement.

Lemma 4.3 *Let $N\ge 3$ and $\overline{\mu}=2$. Then there exists a universal constant $\eta_\star>0$ such that for any a solution of* (1.1)*, any $t_0>0$, $x_0\in\Omega$ and $0<\varepsilon\ll 1$, we have the following property. If:*

$$\sum_{i=1}^{p}\frac{1}{\varepsilon^{N+2}}\int_{t_0-2\varepsilon^2}^{t_0+2\varepsilon^2}\int_{|y-x_0|\le 2\varepsilon}|a(s,y)|^{(N+2)/N}\,\mathrm{d}y\,\mathrm{d}s\le\eta_\star\ \varepsilon^{-2(N+2)/N}$$

then a_i satisfies $0\le a_i(t,x)\le 1/\varepsilon^2$ on $|t-t_0|\le\varepsilon^2$, $|x-x_0|\le\varepsilon$ and a_i is C^∞ on this set.

Notice that it is enough to show the boundedness of the a_i's on the neighborhood of (t_0,x_0). Then the full regularity on the (possibly smaller) neighborhood is obtained by induction, using classical theory of parabolic equations (see appendix).

We start by localizing: namely, we consider $(0,T)\times B(0,R)$, $0<T,R<\infty$. We set

$$\mathcal{S}=\{(t,x)\in(0,T)\times B(0,R),\ u\text{ is not }C^\infty\text{ on a neighborhood of }(t,x)\}.$$

We cover $\mathcal{S}$ by rectangles with step size ε^2 in the time direction and ε in the space directions, centered at points $(t,x)\in\mathcal{S}$. By the Vitali covering lemma (see [32], p. 9) there exists a countable family denoted $\{C_j,\ j\in\mathbb{N}\}$, with C_j centered at $(t_j,x_j)\in\mathcal{S}$, made of such rectangles and such that

$$C_j\cap C_\ell=\emptyset\quad\text{for } j\neq\ell,\qquad\text{and}\qquad\mathcal{S}\subset\bigcup_{j\in\mathbb{N}}\widetilde{C_j}$$

where $\widetilde{C_j}$ stands for the rectangle centered at (t_j,x_j) with step size $2\varepsilon^2$ in the time direction and 2ε in the space directions. Since (t_j,x_j) does not satisfy the conclusion of Lemma 4.3, we have

$$\sum_{i=1}^{p}\frac{1}{\varepsilon^{N+2}}\iint_{\widetilde{C_j}}|a_i(s,y)|^{(N+2)/N}\,\mathrm{d}y\,\mathrm{d}s\ge\eta_\star\ \varepsilon^{-2(N+2)/N}.$$

We introduce the function

$$F_{\mathcal{S}}(t,x)=\sum_{j\in\mathbb{N}}\mathbb{1}_{\widetilde{C_j}}(t,x)\ \sum_{i=1}^{p}\frac{1}{\varepsilon^{N+2}}\iint_{\widetilde{C_j}}|a_i(s,y)|^{(N+2)/N}\,\mathrm{d}y\,\mathrm{d}s.$$

Hence, denoting by $\mathcal{L}$ the Lebesgue measure, we have the following estimate

$$\begin{aligned}\mathcal{L}\Big(\bigcup_{j\in\mathbb{N}} C_j\Big) &\leq \mathcal{L}\Big(\big\{(t,x)\in(0,T)\times B(0,R),\ F_{\mathcal{S}}(t,x)\geq \eta_\star/\varepsilon^{2(N+2)/N}\big\}\Big)\\ &\leq \frac{\varepsilon^{2(N+2)/N}}{\eta_\star}\int_0^T\int_\Omega F_{\mathcal{S}}(t,x)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

as a consequence of the Tchebyschev inequality. It yields by direct evaluation

$$\begin{aligned}\mathcal{L}\Big(\bigcup_{j\in\mathbb{N}} C_j\Big) &\leq \frac{\varepsilon^{2(N+2)/N}}{\eta_\star}\sum_{i=1}^p\sum_{j\in\mathbb{N}}\left(\iint_{\widetilde{C_j}}|a_i|^{(N+2)/N}\,\mathrm{d}y\,\mathrm{d}s\times 2^{N+2}\,\frac{\displaystyle\int_0^T\int_\Omega \mathbb{1}_{\widetilde{C_j}}(t,x)\,\mathrm{d}x\,\mathrm{d}t}{\mathcal{L}(\widetilde{C_j})}\right)\\ &= 2^{N+2}\,\frac{\varepsilon^{2(N+2)/N}}{\eta_\star}\sum_{i=1}^p\|a_i\|^{(N+2)/N}_{L^{(N+2)/N}((0,T)\times\Omega)}.\end{aligned}$$

Since the Lebesgue measure of the C_j's is proportional to ε^{N+2}, we deduce that the cardinality of the covering is of order $\mathcal{O}(\varepsilon^{2(N+2)/N-(N+2)} = \varepsilon^{-(N^2-4)/N})$. Furthermore, the $\widetilde{C}_j$'s realize a covering of $\mathcal{S}$ with sets of diameter ε; we conclude that the Hausdorff dimension of $\mathcal{S}$ is dominated by $(N^2-4)/N$. ∎

Remark 4.2 *It is not obvious that we can improve this estimate, which is in the spirit of [29, 30] for the Navier-Stokes equations, up to a sharp result as in [4, 17]. A difficulty is related to the fact that we are dealing with diffusion coefficients that depend on the component of the system, which prevents from using regularizations by a common heat kernel.*

A Appendix

In this appendix, we sketch the proofs of classical results on regularity and small time existence for quasi-linear parabolic systems. The first statements are concerned with the higher regularity of bounded solutions.

Proposition A.1 *Let $T, r > 0$ and $x_0 \in \mathbb{R}^N$. Let u be a bounded solution on $[0,T]\times B(x_0,r)$ of*

$$\partial_t u - \nabla\cdot(D\nabla u) = f(t,x,u,\nabla u).$$

with a diffusion matrix D verifying for some $\alpha > 0$

$$D(t,x)\xi\cdot\xi \geq \alpha|\xi|^2,$$

for any $(t,x)\in[0,T]\times B(x_0,r)$ and $\xi\in\mathbb{R}^N$. The function f lies in $C^\infty([0,T]\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N)$ and verifies for any $t\in[0,T]$, $x\in B(x_0,r)$, $|u|\leq M<\infty$ and $p\in\mathbb{R}^N$, $|f(t,x,u,p)|\leq C_{M,r}(1+|p|^2)$. Assume in addition that $D\in C^k([0,T]\times B(x_0,r))$ for an integer $k\geq 1$. Then for every $0<t<T$, we have

$$u\in L^\infty(t,T;C^k(B(x_0,r/2))).$$

In particular, if $D\in C^\infty([0,T]\times B(x_0,r))$, then for every $t>0$ we have also $u\in C^\infty([t,T]\times B(x_0,r/2))$.

This proposition proves Corollary 1.1 from Theorem 1.1 and Theorem 1.2. It is an easy application of the following result (see [16] Theorem 1.1 pp. 419–420 & Theorem 3.1, pp. 437–438, and, considering systems, Lemma 6.2 p. 592).

Theorem A.1 *Let $T > 0$ and Ω be a bounded domain in $\mathbb{R}^N$. Let $D \in C^1([0,T] \times \Omega)$ verify the coercivity condition: there exists $\alpha > 0$ such that for any $(t,x) \in [0,T] \times \Omega$ and $\xi \in \mathbb{R}^N$, we have*

$$D(t,x)\xi \cdot \xi \geq \alpha|\xi|^2.$$

Let $M > 0$. Let $f \in C^\infty([0,T] \times \Omega \times [-M,M] \times \mathbb{R}^N)$ be such that $|f(t,x,u,p)| \leq C_M(1 + |p|^2)$. Consider a bounded weak solution $u \in L^\infty([0,T] \times B)$, $|u(t,x)| \leq M$, to the quasi-linear equation

$$\partial_t u - \nabla \cdot (D\nabla u) = f(t,x,u,\nabla u).$$

Then, for any $0 < t < T$ and any ball B strictly included in Ω, u and ∇u are continuous on $[t,T] \times B$ with $|\nabla u|$ bounded on $[t,T] \times B$. The bound depends only on t, the distance of B to Ω, M, the constant C_M, the coercivity constant α, and the Lipschitz norm of D.

The result of [16] is actually slightly more general (it includes Hölderian regularity of u and ∇u), but this statement is enough for our purpose.

Proof of Proposition A.1. Consider

$$t_j = t(1 - 2^{-j}), \qquad r_j = \frac{r}{2}(1 + 2^{-j}).$$

We show by induction for $1 \leq j \leq k$ that $u \in L^\infty(t_j, T; C^j(B(x_0, r_j)))$. Theorem A.1 implies that $u \in L^\infty(t_1, T; C^1(B(x_0, r_1))$. Assume that the result holds for $j \in \{1, ..., k-1\}$. Let α be a multi–index in $\mathbb{N}^N$ with length $j+1$. Then, $v = \partial^\alpha u$ is solution to

$$\partial_t v - \nabla \cdot (D\nabla v) = \tilde{f}(t,x,v,\nabla v)$$

where the function $\tilde{f}$ verifies the assumption of Theorem A.1, the associated constant C_M depending on t_j, r_j, $\|u\|_{L^\infty([t_j,T;C^j(B(x_0,r_j)))}$, α, and $\|D\|_{C^{j+1}}$. Applying Theorem A.1 again gives the estimate with $j+1$.

When $D \in C^\infty([0,T] \times B(x_0,r))$, once it has been proved that $\partial^\alpha u$ is continuous and belongs to $L^\infty((t,T) \times B(x_0, r/2))$ for any $\alpha \in \mathbb{N}^N$, we establish iteratively the regularity with respect to the time variable. ∎

Next, for the sake of completeness, we give a proof of the existence of smooth and bounded solutions of (1.1) on a small enough time interval.

Proposition A.2 *Let $a^0 \in [L^\infty(\Omega)]^p$ be such that $a_i^0 \geq 0$ for all $i \in \{1, ..., p\}$. Then there exists $T_0 > 0$ and $a \in [L^\infty([0,T_0] \times \Omega)]^p$ solution to (1.1). Moreover this solution is unique, regular on $[t,T_0] \times \Omega$ for any $0 < t < T_0$ (as long as D is smooth) and verifies $a_i(t,x) \geq 0$.*

Proof. Consider $y(t)$ solution to the ODE

$$\dot{y} = y^{\overline{\mu}}, \qquad y(0) = \|a^0\|_{L^\infty}.$$

Let $0 < T_b < \infty$ be such that

$$y(t) \leq 2\|a^0\|_{L^\infty}, \qquad 0 \leq t \leq T_b.$$

Set $a_i^{(0)}(t,x) = 0$. We construct, by induction for $j \geq 1$, the solutions $a^{(j)}$ on $[0,T_b] \times \Omega$ to the following linear parabolic system

$$\partial_t a_i^{(j)} - \nabla \cdot (D_i \nabla a_i^{(j)}) + L_i(a^{(j-1)}) a_i^{(j)} = G_i(a^{(j-1)}), \qquad i \in \{1, ..., p\},$$
$$a_i^{(j)}(0,x) = a_i^0(x),$$

where L_i and G_i are defined as in (1.3). We show also that $a^{(j)}$ is smooth and verifies

$$0 \leq a_i^{(j)}(t,x) \leq y(t), \qquad (t,x) \in [0,T_b] \times \Omega, \tag{A.1}$$

and for any $0 \leq t < T_b$, $j \geq 2$

$$\|a^{(j)}(t) - a^{(j-1)}(t)\|_{L^\infty(\Omega)} \leq 2p\overline{\mu}(2\|a^0\|_{L^\infty})^{\overline{\mu}-1} t \|a^{(j-1)} - a^{(j-2)}\|_{L^\infty([0,T_b]\times\Omega)}. \tag{A.2}$$

Clearly, (A.1) holds for $j = 1$ and $j = 2$ and (A.2) holds for $j = 2$. Assume that we have constructed $a^{(k)}$ for $k \in \{1, ..., j\}$ and that (A.1), (A.2) hold for those functions. Note that, for j fixed, the system is decoupled (the definition of $a_i^{(j)}$ does not depend on $a_m^{(j)}$ for $i \neq m$). The existence of a smooth solution $a^{(j+1)}$ follows from the classical theory of linear parabolic equations. For i fixed, 0 is a subsolution to the equation satisfied by a_i^{j+1} and y is a supersolution. The maximum principle gives the bounds (A.1) for $a^{(j+1)}$. We remark that, for any $i \in \{1, ..., p\}$

$$|G_i(a^{(j)}) - G_i(a^{(j-1)})| \leq 2\overline{\mu}[\sup(a^{(j)}, a^{(j-1)})]^{\overline{\mu}-1} |a^{(j)} - a^{(j-1)}|,$$
$$|L_i(a^{(j)}) - L_i(a^{(j-1)})| \leq 2(\overline{\mu}-1)[\sup(a^{(j)}, a^{(j-1)})]^{\overline{\mu}-2} |a^{(j)} - a^{(j-1)}|.$$

Moreover, we have

$$\partial_t (a_i^{(j+1)} - a_i^{(j)}) - \nabla \cdot (D_i \nabla (a_i^{(j+1)} - a_i^{(j)})) = G_i(a^{(j)}) - G_i(a^{(j-1)}) - (L_i(a^{(j)}) a^{(j+1)} - L_i(a^{(j-1)}) a^{(j)}),$$
$$(a^{(j+1)} - a^{(j)})(0,x) = 0, \qquad x \in \Omega.$$

By a comparison principle, we get (A.2) at the rank $j+1$. Hence the induction hypotheses (A.1), (A.2) are satisfied for any $j \geq 2$. Consider $T_0 = \inf(T_b, [4p\overline{\mu}(2\|a^0\|_{L^\infty})^{\overline{\mu}-1}]^{-1})$. Let $\mathcal{S}$ be the operator defined from $\left[L^\infty([0,T_0]\times\Omega)\right]^p$ to itself by $\mathcal{S}(a^{(j)}) = a^{(j+1)}$. Then (A.2) ensures that $\mathcal{S}$ is a strict contraction. So, by the Banach fixed point theorem, $a^{(j)}$ converges in $L^\infty([0,T_0]\times\Omega)$ to a function a. Passing to the limit in the equation, we get that a is solution to (1.1). Uniform bounds on $a_i^{(j)}$ gives that a_i is non-negative and uniformly bounded by $2\|a^0\|_{L^\infty}$ on $[0,T_0]\times\Omega$. Finally, Proposition A.1 proves the regularity of a. ■

References

[1] J. M. Ball. On the asymptotic behavior of generalized processes, with applications to nonlinear evolution equations. *J. Differential Equations*, 27(2):224–265, 1978.

[2] M. Bisi and L. Desvillettes. From reactive Boltzmann equations to reaction–diffusion systems. *J. Stat. Phys.*, 124(2–4):881–912, 2006.

[3] H. Brezis. *Analyse fonctionnelle. Théorie et applications.* Collection Mathématiques Appliquées pour la Maîtrise. Masson, Paris, 1983.

[4] L. Caffarelli, R. Kohn, and L. Nirenberg. Partial regularity of suitable weak solutions of the Navier-Stokes equations. *Comm. Pure Appl. Math.*, 35(6):771–831, 1982.

[5] L. Caffarelli and A. Vasseur. Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. *Annals of Math.*, to appear.

[6] J.-F. Collet. Some modelling issues in the theory of fragmentation-coagulation systems. *Commun. Math. Sci.*, 2(suppl. 1, F. Golse, T. Goudon, F. Lagoutière Eds.):35–54, 2004.

[7] E. De Giorgi. Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3)*, 3:25–43, 1957.

[8] L. Desvillettes and K. Fellner. Exponential decay toward equilibrium via entropy methods for reaction-diffusion equations. *J. Math. Anal. Appl.*, 319(1):157–176, 2006.

[9] L. Desvillettes and K. Fellner. Entropy methods for reaction-diffusion equations: Slowly growing a-priori bounds. *Rev. Mat. Iberoamericana*, to appear.

[10] L. Desvillettes, K. Fellner, M. Pierre, and J. Vovelle. About global existence for quadratic systems of reaction-diffusion. *J. Advanced Nonlinear Studies*, 7(3):491–511, 2007.

[11] P. Erdi and J. Tóth. *Mathematical models of chemical reactions.* Nonlinear Science: Theory and Applications. Manchester Univ. Press, 1989.

[12] H. Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[13] W. Feng. Coupled system of reaction–diffusion equations and applications in carrier facilitated diffusion. *Nonlinear Anal.*, 17:285–311, 1991.

[14] P. C. Fife. *Mathematical aspects of reacting and diffusing systems*, volume 28 of *Lecture Notes in Biomath.* Springer, Berlin, 1979.

[15] Y. Giga and R. V. Kohn. Asymptotically self-similar blow-up of semilinear heat equations. *Comm. Pure Appl. Math.*, 38(3):297–319, 1985.

[16] O. A. Ladyzenskaia, V. A. Solonnikov, and N. N. Uralceva. *Linear and quasi-linear equations of parabolic type*, volume 23 of *Transl. Math. Monographs.* AMS, Providence, 1968.

[17] F. Lin. A new proof of the Caffarelli–Kohn–Nirenberg theorem. *Comm. Pure and Appl. Math.*, 51:241–257, 1998.

[18] A. Mellet and A. Vasseur. L^p estimates for quantities advected by a compressible flow. *J. Math Anal. Appl.*, to appear.

[19] J. Morgan. Global existence for semilinear parabolic systems. *SIAM J. Math. Anal.*, 20(5):1128–1144, 1989.

[20] J. Morgan. Global existence for semilinear parabolic systems via Lyapunov type methods. In *Nonlinear semigroups, partial differential equations and attractors (Washington, DC, 1987)*, volume 1394 of *Lecture Notes in Math.*, pages 117–121. Springer, Berlin, 1989.

[21] J. Morgan. Boundedness and decay results for reaction-diffusion systems. *SIAM J. Math. Anal.*, 21(5):1172–1189, 1990.

[22] J. Morgan. On a question of blow-up for semilinear parabolic systems. *Differential Integral Equations*, 3(5):973–978, 1990.

[23] J. Morgan and S. Waggonner. Global existence for a class of quasilinear reaction-diffusion systems. *Commun. Appl. Anal.*, 8(2):153–166, 2004.

[24] J. D. Murray. *Mathematical biology*, volume 17 & 18 of *Interdisciplinary Applied Math.* Springer, New-York, 2003.

[25] M. Pierre. Weak solutions and supersolutions in L^1 for reaction-diffusion systems. *J. Evol. Equ.*, 3(1):153–168, 2003. Dedicated to Philippe Bénilan.

[26] M. Pierre and D. Schmitt. Blowup in reaction-diffusion systems with dissipation of mass. *SIAM Rev.*, 42(1):93–106 (electronic), 2000.

[27] M. Pierre and R. Texier-Picard. Global existence for degenerate quadratic reaction-diffusion systems. *Ann. IHP Anal. Non Linéaire*, to appear.

[28] F. Rothe. *Global solutions of reaction–diffusion systems*, volume 102 of *Lecture Notes in Math.* Springer, Berlin, 1984.

[29] V. Scheffer. Partial regularity of solutions to the Navier-Stokes equations. *Pacific J. Math.*, 66(2):535–552, 1976.

[30] V. Scheffer. Hausdorff measure and the Navier-Stokes equations. *Comm. Math. Phys.*, 55:55–97, 1977.

[31] L. W. Somathilake and J. M. J. J. Peiris. Global solutions of a strongly coupled reaction-diffusion system with different diffusion coefficients. *J. Appl. Math.*, 1:23–36, 2005.

[32] E. M. Stein. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[33] A. Vasseur. A new proof of partial regularity of solutions to Navier-Stokes equations. *NoDEA Nonlinear Differential Equations Appl.*, 14(5-6):753–785, 2007.

[34] C. Villani. Hypocoercive diffusion operators. In *Proceedings of the International Congress of Mathematicians*, Madrid, 2006.

[35] F. B. Weissler. An L^∞ blow-up estimate for a nonlinear heat equation. *Comm. Pure Appl. Math.*, 38(3):291–295, 1985.