Some stability problems in Kinetic Theory

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Aim

Understanding macroscopic stability-bifurcations scenarios at kinetic level.

Two examples:

- Phase transitions: Multiple equilibrium states of a Thermodynamic system when the temperature goes below some critical value. Minimizers of the free energy correspond to dynamically stable equilibrium solutions w.r.t. a kinetic evolution.
- Benard experiment: Convective motions of a fluid when the Rayleigh number crosses some critical value. Persistence of the stability scenario for small values of the Knudsen number.

References

- R. Esposito, Y. Guo and R. Marra, *Phase Transition in a Vlasov-Boltzmann Binary Mixture.* (2009) ArXiv 0904.0791v1 [math-ph].
- L. Arkeryd, R. Esposito, R. Marra and A. Nouri, *Stability of the laminar solution of the Boltzmann equation for the Benard problem.* Bull. Inst. Math. Academia Sinica, vol. 3, pp. 51-97 (2008).
- L. Arkeryd, R. Esposito, R. Marra and A. Nouri, *Stability for Rayleigh-Benard convective solutions of the Boltzmann equation.* (2008) ArXiv 0812.3720v1 [math-ph].

Outline

- Phase coexistence:
  - The model;
  - Stability and instability results;
  - Sketch the proof.
- Benard problem:
  - Background and Heuristics;
  - Results;
  - Main difficulties.

Phase coexistence

- Motivations: Marra's talk.
- Model: Binary fluid. Blue and red particles undergoing color blind elastic collisions and interacting via a repulsive self-consistent force between different colors
- Results: Stability–instability of the equilibrium solutions.
- Techniques: Energy-entropy inequalities; growing mode for the instability.

The model

$f^1 = f^{\text{red}}$ and $f^2 = f^{\text{blue}}$ probability distribution functions on the phase space $\Omega \times \mathbb{R}^3$, satisfying the evolution equations

$$\partial_t f^1 + v \cdot \nabla_x f^1 + F^1 \cdot \nabla_v f^1 = B(f^1, f^1) + B(f^1, \ f^2)$$

$$\partial_t f^2 + v \cdot \nabla_x f^2 + F^2 \cdot \nabla_v f^2 = B(f^2, f^2) + B(f^2, f^1)$$

Self-consistent forces F^1, F^2,

$$F^1(x,t) = -\nabla_x \int_\Omega dx' U(|x - x'|) \int_{\mathbb{R}^3} \mathrm{d}v f^2(x', v, t)$$

$U \geq 0$ smooth, finite range, bounded, $\int_\Omega U(|x|)dx = 1$, decreasing.

$$B(f,g) = \int_{\mathbb{R}^3} dv_* \int_{|\omega|=1|} d\omega |(v - v_*) \cdot \omega| [f(v')g(v'_*) - f(v)g(v_*)]$$

Equilibrium

For $\beta = T^{-1} > 0$ set $\mu_\beta = (\frac{\beta}{2\pi})^{\frac{3}{2}} \mathrm{e}^{-\beta \frac{v^2}{2}}$.

The equilibrium solutions are:

$$f^1(x,v) = \rho^1(x)\mu_\beta(v), \qquad f^2(x,v) = \rho^2(x)\mu_\beta(v),$$

$$\beta^{-1} \log \rho^1(x) + \int_\Omega dx' U(|x-x'|)\rho^2(x') = C^1,$$

$$\beta^{-1} \log \rho^2(x) + \int_\Omega dx' U(|x-x'|)\rho^1(x') = C^2.$$

Euler-Lagrange equations for the *free energy* functional

$$\mathcal{F}_\Omega[\rho^1,\rho^2] = \beta^{-1} \int_\Omega dx [\rho^1 \log \rho^1 + \rho^2 \log \rho^2$$
$$+ \int_\Omega dx \int_\Omega dx' U(|x-x'|)\rho^1(x)\rho^2(x').$$

Local free energy

$$\varphi(\rho^1, \rho^2) = \beta^{-1}[\rho^1 \log \rho^1 + \rho^2 \log \rho^2] + \rho^1 \rho^2,$$

$$\mathcal{F}_\Omega[\rho^1, \rho^2] = \int_\Omega dx \varphi(\rho^1(x), \rho^2(x))$$

$$+\frac{1}{2}\int_\Omega dx \int_\Omega dx' U(|x - x'|)(\rho^1(x) - \rho^1(x'))(\rho^2(x') - \rho^2(x)).$$

Set $\rho = \rho^1 + \rho^2$ and $m_\beta = \tanh(\frac{1}{2}\rho\beta m_\beta)$

$\rho\beta < 2$: Unique miminizer of φ: $\rho^1 = \rho^2$ (Mixed phase).

$\rho\beta > 2$: $m_\beta > 0$; $\rho^\pm = \frac{1}{2}\rho(1 \pm m_\beta)$

- Minimizers: $\rho^1 = \rho^+, \rho^2 = \rho^-$, (red rich phase); $1 \leftrightarrow 2$;
- Maximizer (local): $\rho^1 = \rho^2$

Phase coexistence

Set $\boxed{\rho = 2.}$

For $\beta > 1$, non spatially homogeneous solutions are possible: regions of red rich and blue rich phases separated by interfaces.

Set $\boxed{\Omega = \mathbb{R}}$. Define

$$\hat{\rho}^1(x) = \begin{cases} \rho^-, & x < 0 \\ \rho^+, & x > 0 \end{cases}, \qquad \hat{\rho}^2(x) = \begin{cases} \rho^+, & x < 0 \\ \rho^-, & x > 0 \end{cases}.$$

Excess free energy:

$$\widehat{\mathcal{F}}[\rho^1, \rho^2] = \lim_{\ell \to \infty} \left[\mathcal{F}_{(-\ell,\ell)}[\rho^1, \rho^2] - \mathcal{F}_{(-\ell,\ell)}[\hat{\rho}^1, \hat{\rho}^2] \right]$$

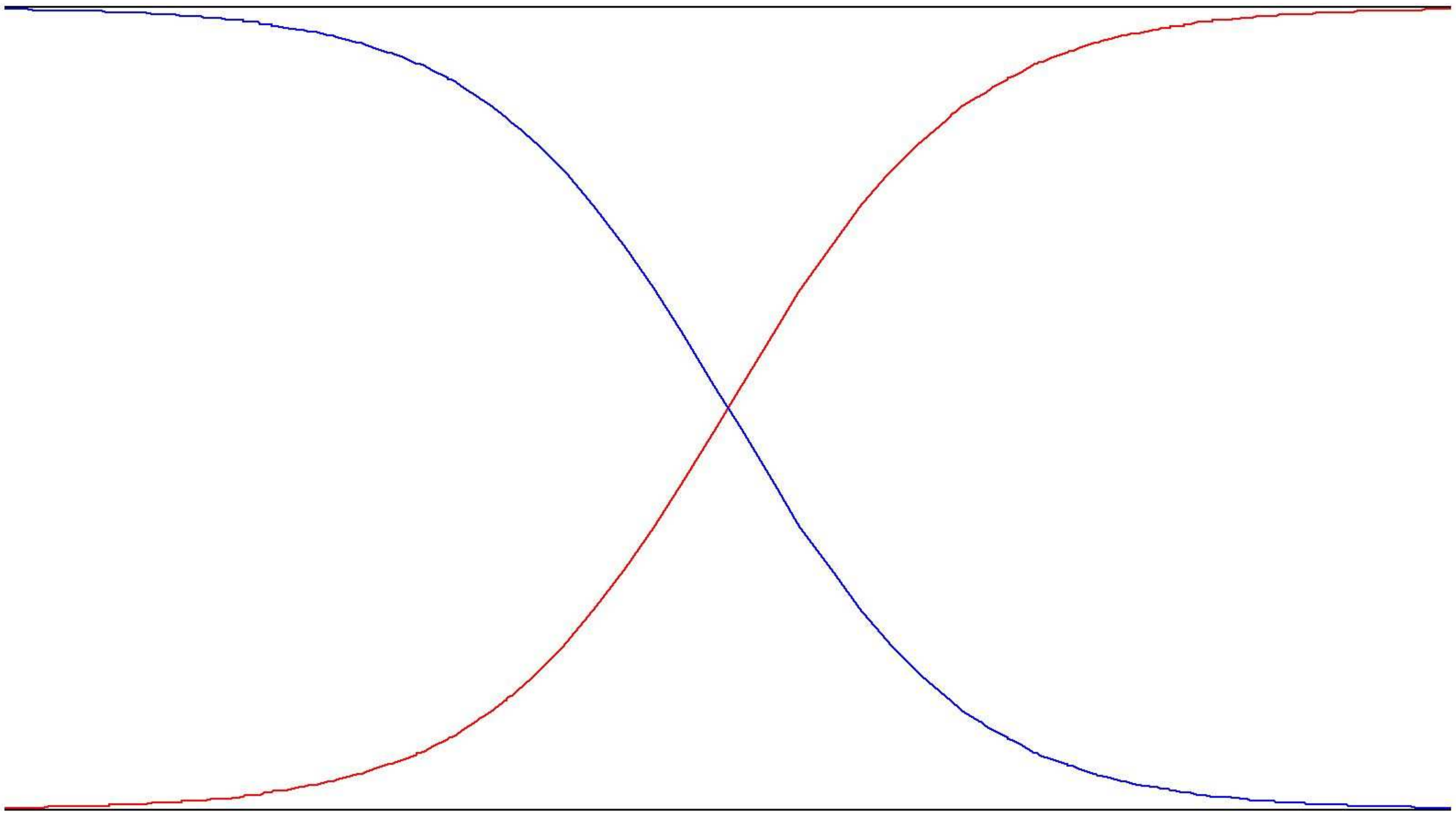
$\widehat{\mathcal{F}}[\rho^1, \rho^2]$ is not finite if $\lim_{x \to \pm\infty} \rho^1 \neq \rho^\mp$ or $\lim_{x \to \pm\infty} \rho^2 \neq \rho^\pm$.

Front solution

Theorem[Carlen, Carvalho, R.E., Lebowitz, Marra] *Let* $\beta > 1$. *There is a unique (up to translations) minimizer to the excess free energy* $\widehat{\mathcal{F}}$. *Let* $\bar\rho = (\bar\rho_1, \bar\rho_2)$ *be the one such that* $\rho_1(0) = \rho_2(0)$.

- $\bar\rho$ *is smooth;* $\rho^- < \bar\rho_i(x) < \rho^+$;
- $\bar\rho_1$ *is increasing and* $\bar\rho_2$ *is decreasing;*
- $\beta^{-1}\log\bar\rho_1 + U * \bar\rho_2 = C = \beta^{-1}\log\bar\rho_2 + U * \bar\rho_1$;
- $\beta^{-1}\bar\rho_1' + \bar\rho_1 U * \bar\rho_2' = 0 = \beta^{-1}\bar\rho_2' + \bar\rho_1 U * \bar\rho_1'$;
- $\bar\rho_1(x) = \bar\rho_2(-x)$, $\bar\rho_1'(x) = -\bar\rho_2'(-x)$;
- $|\bar\rho_1(x) - \rho^\pm| e^{\alpha|x|} \to 0, x \to \mp\infty$;
  $|\bar\rho_2(x) - \rho^\mp| e^{\alpha|x|} \to 0, x \to \mp\infty$.

Front solution



Results

Theorem[R. E., Guo, Marra]: *Assume* $\rho = 2$*.*

- $\beta < 1$*: The unique equilibrium* $(f_1, f_2) = (\mu_\beta, \mu_\beta)$ *is stable.*
- $\beta > 1$*:*
  - *the homogeneous equilibria* $(f_1, f_2) = (\rho^+ \mu_\beta, \rho^- \mu_\beta)$ *and* $(f_1, f_2) = (\rho^- \mu_\beta, \rho^+ \mu_\beta)$ *are stable;*
  - *the equilibrium* $(f_1, f_2) = (\bar{\rho}^1(x)\mu_\beta, \bar{\rho}^2(x)\mu_\beta)$ *is stable w.r.t. symmetric perturbations;*
  - *the homogeneous equilibrium* $(f_1, f_2) = (\mu_\beta, \mu_\beta)$ *is unstable.*

Here stability and instability are in $L^\infty(\mathbb{R} \times \mathbb{R}^3)$ and in $H^1(\mathbb{R} \times \mathbb{R}^3)$. *Symmetric perturbation* means $h_1(x, v) = h_2(-x, Rv)$, where $Rv = (-v_1, v_2, v_3)$.

Remarks

- No convergence to the equilibrium is stated. This has to be compared with the Vlasov-Fokker-Plank case where there is an algebraic rate of convergence. No instability result for VFP.
- In order to have phase transitions: force not small. Treating the force terms as perturbations does not work. Strategy based on entropy-energy arguments: L^2 estimates promoted to L^∞ by analysis of the characteristics. Crucial step: spectral gap for the second derivative of the free energy.
- The instability is based on the construction of a growing mode for the linear collisionless case, perturbation arguments and persistence of the gorwing mode at non linear level.

Spectral gap

Given $\rho = (\rho_1, \rho_2)$, define the operator A on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ by

$$\langle u, Au \rangle = \frac{1}{2}\frac{d^2}{ds^2}\hat{\mathcal{F}}(\rho + su)\Big|_{s=0}.$$

Whenever ρ is a minimizer for the excess free energy, A is non negative. Let $\mathcal{P}$ be the projector on the null space of A

Lemma.[CCELM] *There is $\delta > 0$ such that*

$$\langle u, Au \rangle \geq \delta \|(1-\mathcal{P})u\|^2.$$

If $\rho = (\bar\rho_1, \bar\rho_2)$, then the null space of A is $\{c(\bar\rho_1', \bar\rho_2'),\ c \in \mathbb{R}\}$.

The null space of the analog of A is trivial if $\rho = (\rho^+, \rho^-)$ or $\rho = (\rho^-, \rho^+)$ (case $\beta > 1$) or $\rho = (1, 1)$ (case $\beta < 1$).

Entropy-energy functional

Given the equilibrium state $(M_1, M_2) = (\rho_1\mu_\beta, \rho_2\mu_\beta)$, let $g = (g_1, g_2)$ with $g_i = \dfrac{f_i - M_i}{\sqrt{M_i}}$ be the deviation from the equilibrium. Define: $\mathcal{M}_i(g) = \int_{\mathbb{R}} dx \int_{\mathbb{R}^3} dv \sqrt{M_i} g_i(x,v)$,

$$H(g) = \sum_{i=1}^{2} \int_{\mathbb{R}} dx \int dv \Big[f_i \log f_i - M_i \log M_i \Big],$$

$$\mathcal{E}(g) = \sum_{i=1}^{2} \int_{\mathbb{R}} dx \int dv \frac{v^2}{2} g_i \sqrt{M_i}$$

$$+ \int_{\mathbb{R}\times\mathbb{R}} dx dy U(|x-y|) \Big(\rho_{f_1}(x)\rho_{f_2}(y) - \rho_1(x)\rho_2(y) \Big),$$

$$\rho_{f_i} = \int dv f_i(x,v).$$

Entropy-energy functional

The energy-entropy functional is

$$\mathcal{H}(g) = H(g) + \beta\mathcal{E}(g) - \left(C + 1 + \log\left(\frac{\beta}{2\pi}\right)^{3/2}\right)\sum_{i=1}^{2}\mathcal{M}_i(g),$$

The energy-entropy functional does not increase:

$\mathcal{H}(g(t)) \leq \mathcal{H}(g(0))$ for any $t > 0$.

Quadratic approximation. The coefficients have been chosen to cancel the linear part. For some $\tilde{f}_i$:

$$\mathcal{H}(g) = \sum_{i=1}^{2}\int_{\mathbb{R}} dx \int_{\mathbb{R}^3} dv \frac{(f_i(t) - M_i)^2}{2\tilde{f}_i}$$

$$+\beta\int_{\mathbb{R}} dx \int_{\mathbb{R}} dy U(|x-y|)(\rho_{f_1}(t,x) - \rho_1(x))(\rho_{f_2}(t,y) - \rho_2(y)).$$

Entropy-energy functional

Lemma. *If* $u_i = \rho_{f_i} - \rho_i$ *s.t.* $\mathcal{P}(u_1, u_2) = 0,$ *then there are* $\alpha > 0$ *and* κ *sufficiently small so that*

$$\alpha \sum_{i=1,2} \int_{\mathbb{R}} dx \int_{\mathbb{R}^3} dv \left\{ \frac{(f_i(t) - M_i)^2}{M_i} \mathbf{1}_{\{|f_i(t) - M_i| \le \kappa M_i\}} \right.$$

$$\left. + |f_i(t) - M_i| \mathbf{1}_{\{|f_i(t) - M_i| \ge \kappa M_i\}} \right\} \le \mathcal{H}(g(0)).$$

Remark: It is crucial that the initial perturbation is orthogonal to the null space of A. This is trivial for the spatially homogeneous equilibrium, while it is ensured by the symmetry condition for the phase coexisting equilibrium, where an orbital instability is possible.

Main proposition

Theorem: *Let* $w(v) = (\Sigma + |v|^2)^\gamma$, *with* Σ *sufficiently large and* $\gamma > \frac{3}{2}$. *If* $\|wg(0)\|_\infty + \sqrt{\mathcal{H}(g(0))} < \delta$ *for* δ *sufficiently small, then there is* $T_0 >$ *such that*

$$\|wg(T_0)\|_\infty \leq \frac{1}{2}\|wg(0)\|_\infty + C_{T_0}\sqrt{\mathcal{H}(g(0))}.$$

The stability follows by iteration on the time interval.

Growing mode

Idea: Without collisions there is a growing mode.

Collisions do not destroy it.

The linearization of the equation for the perturbation is:

$$\partial_t g + \mathcal{L} g = 0,$$

Notation: $\mu = \mu_\beta$; ξ first component of the velocity v,

$$(\mathcal{L} g)_i = \xi \partial_x g_i - \beta F(\sqrt{\mu} g_{i+1}) \xi \sqrt{\mu} - \alpha L_i g,$$

$$\alpha = 1 \text{ and } L_i g = \frac{1}{\sqrt{\mu}} \Big(B(\sqrt{\mu} g_i, 2\mu) + B(\mu, \sqrt{\mu}(g_1 + g_2)) \Big).$$

Seek for a growing mode of the form $g_1 = g_2 = \mathrm{e}^{\lambda t} \mathrm{e}^{\mathbf{i} k x} q(v)$.

Eigenvalue problem

$$\{\lambda + \mathbf{i}\xi k\}q - \beta k \mathbf{i}\hat{U}(k)\left\{\int_{\mathbb{R}^3} q\sqrt{\mu}dv\right\}\xi\sqrt{\mu} - \alpha Lq = 0.$$

Proposition 1: *Let $\beta > 1$. There exists sufficiently small $\alpha > 0$ such that there is an eigenfunction $q(v)$ and the eigenvalue λ with $\Re\lambda > 0$.*

Proof. First assume $\alpha = 0$. λ is found by Penrose criterion,

$$\beta\int_{\mathbb{R}^3}\frac{\xi^2\hat{U}(k)k^2\mu(v)}{\lambda^2 + k^2\xi^2}dv = 1.$$

Indeed, since $\beta\hat{U}(0) > 1$, by continuity there is $k_0 > 0$ such that $\beta\hat{U}(k_0) > 1$ and hence a $\lambda > 0$ so that this is satisfied.

Use Kato perturbation theorem to extend to $\alpha > 0$ small.

Eigenvalue problem

Proposition 2: *Let α_0 be the supremum of the α's such that Proposition 1 is true. Then $\alpha_0 = +\infty$.*

Proof. Indeed, if $\alpha_0 < \infty$ then, $\lambda_0 = \lim_{\alpha \to \alpha_0} \lambda_\alpha$ exists (up to subsequences) and is a purely imaginary eigenvalue. It can be shown that the corresponding eigenfunction must be in the null space of L and this implies $\lambda = 0$. Moreover, collisions disappear for such an eigenfunction and we can use again the Penrose criterion which implies $\beta \hat{U}(k_0) = 1$. This is in contradiction with the definition of k_0.

This provides a linear growing mode for any $\alpha > 0$.

Remark: It is crucial that L is a bounded perturbation. It does not work with the Fokker-Plank operator which is unbounded.

Non linear analysis. Bootstrap argument. Very technical.

Instability theorem

Theorem. *Assume $\beta > 1$. There exist constants $k_0 > 0$, $\theta > 0$, $C > 0$, $c > 0$ and a family of initial $\frac{2\pi}{k_0}-$periodic data $f_i^\delta(0) = \mu + \sqrt{\mu} g_i^\delta(0) \geq 0$, with $g^\delta(0)$ satisfying*

$$\|\nabla_{x,\xi} g^\delta(0)\|_{L^2} + \|w g^\delta(0)\|_{L^\infty} \leq C\delta,$$

for δ sufficiently small, but the solution $g^\delta(t)$ satisfies

$$\sup_{0\leq t\leq T^\delta} \|w g^\delta(t)\|_{L^\infty} \geq c \sup_{0\leq t\leq T^\delta} \|g^\delta(t)\|_{L^2} \geq c\theta > 0.$$

Here the escape time is $T^\delta = \frac{1}{\Re\lambda} \ln\frac{\theta}{\delta}$,

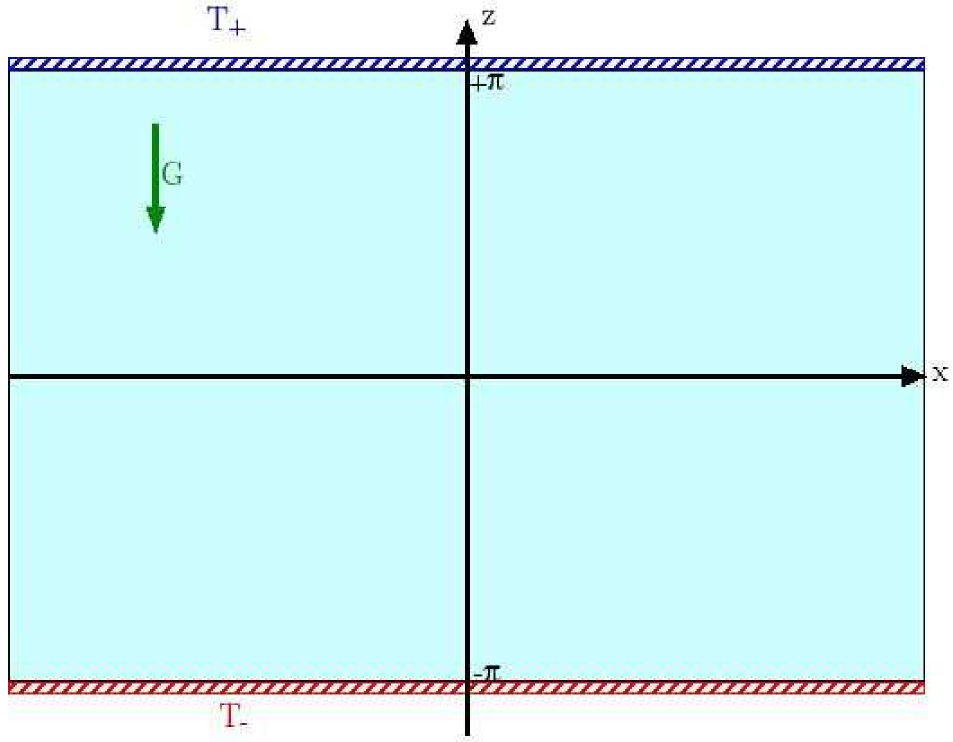
Note that the growing mode can be chosen symmetric.

The instability does not depend on the absence of symmetry.

Benard problem

The physical situation we want to study is the Benard experiment, which is about the behavior of a viscous heat conducting fluid under the action of the gravity and heated from below.

$$T_- > T_+$$



Convection for $T_- - T_+$ sufficiently large.

Boltzmann equation

Macroscopic space-time scale: $r = \dfrac{q}{L}$, $\tau = \dfrac{\tau_{\text{micr}}}{L}$;
Microscopic size of the spatial domain Ω is $\mathcal{O}(L)$.

$$\partial_\tau f + v \cdot \nabla_r f + F \cdot \nabla_v f = \frac{1}{\varepsilon} Q(f, f).$$

1. $f(r, v, \tau)$ for any $\tau \in \mathbb{R}^+$ is a normalized positive probability density on the phase space: $(r, v) \in \Omega \times \mathbb{R}^3$, $\Omega \subset \mathbb{R}^d$;
2. F is a conservative force; The mass is set to 1.
3. $\varepsilon > 0$ the Knudsen number i.e. the mean free path ℓ in macroscopic units: $\varepsilon = \dfrac{\ell}{L}$.
4. $Q(f, g)$ the Boltzmann symmetric collision kernel.

Diffusive scaling, low Mach number.

Write the solution to the Boltzmann equation as

$$f^{\varepsilon} = M(\rho_{\varepsilon}, u_{\varepsilon}, T_{\varepsilon}; v) + \varepsilon f_1 + \dots$$

Assume:

- $\tau = \varepsilon^{-1} t$, $F = \varepsilon F_1 = -\varepsilon \nabla_r U$;
- $u_{\varepsilon} = \varepsilon u_1$. (Mach number $\mathcal{M} = \mathcal{O}(\varepsilon)$)
- $\rho_{\varepsilon} = \bar{\rho} + \varepsilon \rho_1$, $T_{\varepsilon} = \bar{T} + \varepsilon T_1$, $\bar{T}$ and $\bar{\rho}$ positive constants.

Incompressible Navier-Stokes-Fourier system (INSF)

$$\nabla_r \cdot u_1 = 0; \quad \nabla_r(\bar{\rho} T_1 + \bar{T} \rho_1 + \bar{\rho} U) = 0,$$

$$\bar{\rho}\left(\partial_t u_1 + u_1 \cdot \nabla_r u_1\right) + \nabla_r p = \eta \Delta_r u_1 + \rho_1 \nabla_r U,$$

$$\frac{5}{2}\bar{\rho}(\partial_t T_1 + u_1 \cdot \nabla_r T_1) = \kappa \Delta_r T_1.$$

Kinetic Benard problem.

Notation: $\Omega = \{(x,z) \,|\, z \in [-\pi,\pi], x \in \mathbb{R},\ x \mod 2\alpha\pi\}$

for some $\alpha > 0$ to be specified later.

Boltzmann equation in non-dimensional variables,

with gravity along z ($F_1 = (0,0,-G)$):

$$\partial_t f + \frac{1}{\varepsilon}(v_x\partial_x f + v_z\partial_z f) - G\partial_{v_z} f = \frac{1}{\varepsilon^2}Q(f,f)$$

$$Q(f,g)(v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}_2} d\omega |(v-v_*)\cdot\omega| \{f'g'_* + f'_*g' - fg_* - f_*g\},$$

$$f = f(v), \quad f_* = f(v_*), \quad f' = f(v'), \quad f'_* = f(v'_*),$$

$$v' = v - \omega(\omega\cdot(v-v_*)), \quad v'_* = v_* + \omega(\omega\cdot(v-v_*)).$$

Boundary conditions.

Diffuse reflection: incoming data are Maxwellians:

$$v_z > 0 : f(t, x, -\pi, v) = M_-(v) j_f^-(x),$$

$$v_z < 0 : f(t, x, +\pi, v) = M_+(v) j_f^+(x),$$

$$M_-(v) = \frac{1}{2\pi} \mathrm{e}^{-v^2/2}; \qquad M_+ = \frac{1}{2\pi(1-2\varepsilon\lambda)^2} \exp\left(-\frac{v^2}{2(1-2\varepsilon\lambda)}\right),$$

$$\int_{v_z>0} v_z M_\pm(v) = 1, \quad T_- = 1, \quad \lambda = \frac{T_- - T_+}{2\varepsilon T_-}, \quad T_+ = 1 - 2\varepsilon\lambda.$$

$$j_f^\pm(x) = \int_{\{w\in\mathbb{R}^3, w_z \lessgtr 0\}} dw |w_z| f(t, x, \pm\pi, w).$$

$$\Longrightarrow \qquad \int_{\mathbb{R}^3} dv v_z f(t, x, \pm\pi, v) = 0.$$

Rayleigh number.

The Rayleigh number is $\boxed{\mathcal{R} = 32G\lambda}$.

It is finite independently of ε.

As the Rayleigh number increases, convective phenomena arise in the hydrodynamic equations.

The simplest case: bifurcation of the stable purely conductive solution into stable convective solutions at a critical value $\mathcal{R}_c$ of $\mathcal{R}$.

We want to discuss this behavior in terms of the Boltzmann equation, in a suitably small neighborhood of $\mathcal{R}_c$.

Numerical analysis of kinetic equations for finite ε and asymptotic analysis for small ε by Kyoto group.

Convective motions may not be seen for ε sufficiently large.

Hydrodynamics.

Oberbeck-Boussinesq equations (O-B): ($\hat{e}_z$ unit vector in the direction of z, $\mathcal{P} = \frac{\eta}{\kappa}$ Prandtl number; $z \in (-\pi, \pi)$, $x \in \mathbb{R}$.)

$$
\begin{aligned}
&\operatorname{div} u = 0, \quad \partial_t u + u \cdot \nabla u + \nabla p = \mathcal{P} \Delta u + \mathcal{R}\,\mathcal{P}\,\theta \hat{e}_z; \\
&\frac{5}{2}(\partial_t \theta + u \cdot \nabla \theta) = \Delta \theta, \\
&u(t, x, \pm\pi) = 0, \quad \theta(t, x, -\pi) = 0, \quad \theta(t, x, \pi) = -2\lambda.
\end{aligned}
$$

Stationary conductive solution:

$$
\boxed{u_\ell = 0, \quad \theta_\ell(x, z) = -\frac{\lambda}{\pi}(z + \pi).}
$$

Hydrodynamical stability.

Theorem 1 (Stability of the Laminar solution): *There is $\mathcal{R}_c > 0$ such that for $\mathcal{R} < \mathcal{R}_c$ the stationary conductive solution is asymptotically stable for the O-B equations.*

Theorem 2 (Existence and Stability of the Convective solution): *There is $\delta > 0$ such that, if $0 < \mathcal{R} - \mathcal{R}_c < \delta$, there is α_c and a stationary solution (u_s, θ_s), periodic in x of period $2\alpha_c\pi$ differing from (u_ℓ, θ_ℓ) for $\mathcal{O}(\delta)$. Moreover it is asymptotically stable with respect to sufficiently small perturbations with the same period.*

The critical value $\mathcal{R}_c$ is computed by the linear analysis.

Huge literature on the subject.

The first reference on existence and stability the nonlinear convective solutions we are aware of is [Yudovich 1967].

Kinetic solution

We construct the solution by means of a truncated expansion in ε with a remainder.
The main difficulty is in the estimate of the remainder, but also the construction of the terms of the expansion requires some care (boundary layer terms).

We fix $M = (2\pi)^{-\frac{3}{2}} \mathrm{e}^{-v^2/2}$, and write $f^\varepsilon = M + \varepsilon M \Phi^\varepsilon$ (which takes care of the small Mach numbers conditions).

$$\partial_t \Phi^\varepsilon + \frac{1}{\varepsilon}\left(v_x \partial_x \Phi^\varepsilon + v_z \partial_z \Phi^\varepsilon - M^{-1} G \partial_{v_z} M\right)$$

$$-M^{-1} G \partial_{v_z}(M\Phi^\varepsilon) = \frac{1}{\varepsilon^2} L\Phi^\varepsilon + \frac{1}{\varepsilon} J(\Phi^\varepsilon, \Phi^\varepsilon),$$

$$Lf = 2M^{-1}Q(M, Mf), \quad J(f,f) = M^{-1}Q(Mf, Mf).$$

Kinetic solution

The boundary conditions for Φ^ε are:

$$v_z > 0 : \ \Phi^\varepsilon(t, x, -\pi, v) = \frac{1}{\sqrt{2\pi}} j^-_{\Phi^\varepsilon}(x),$$

$$v_z < 0 : \ \Phi^\varepsilon(t, x, \pi, v) = \frac{M_+(v)}{M(v)} j^+_{\Phi^\varepsilon}(x).$$

Φ^ε is expanded as:

$$\Phi^\varepsilon = \Phi_H + R = \sum_{n=1}^{k} \varepsilon^{n-1} \Phi_n + R.$$

Kinetic solution

The bulk parts of the Φ_n's are computed by using the Hilbert method.
In particular, Φ_1 has to be in the null space of L, which is spanned by 1, v_x, v_y, v_z and $|v|^2$. The coefficients are u and θ solving the $O - B$ equations:

$$\Phi_1 = -G(\pi + z) + u \cdot v + \frac{1}{2}\theta(v^2 - 5).$$

For $n > 1$ the bulk part of Φ_n is computed by the Hilbert procedure and thus depends on u and θ and their derivatives.
Remark: the bulk parts of the Φ_n's do not satisfy the diffusive boundary conditions. Boundary layer correction terms are to be included to restore the boundary conditions.

Kinetic solution

The conclusion from the Hilbert+boundary layer expansion is that the Φ_n's are smooth as consequence of the smoothness of u and θ and inherit the smallness and decay properties of (u, θ).

Given (u, θ) sufficiently smooth, we denote by

$$\Phi_{H,k}(u, \theta) = \Phi_H$$

the Hilbert expansion associated to (u, θ) up to the order k.

Remainder

The remainder R satisfies the equation

$$\partial_t R + \frac{1}{\varepsilon}(v_x\partial_x R + v_z\partial_z R) - M^{-1}G\partial_{v_z}(MR) =$$
$$\frac{1}{\varepsilon^2}LR + \frac{1}{\varepsilon}J(\Phi_H, R) + \frac{1}{\varepsilon}J(R,R) + A,$$

$$R(t,x,-\pi,v) = \frac{1}{\sqrt{2\pi}}j_R^-(x) + \psi_-(x,v), \qquad v_z > 0,$$
$$R(t,x,\pi,v) = \frac{M_+(v)}{M(v)}j_R^+(x) + \psi_+(x,v), \qquad v_z < 0.$$

Remainder

The inhomogeneous terms A and $\psi_\pm$ are computed in terms of u, θ and their derivatives.

$\psi_\pm$ is exponentially small as $\varepsilon \longrightarrow 0$

A is of order ε^m, the exponent m depending on the order k of truncation of the Hilbert expansion.

The number of terms to be kept in the Hilbert expansion, k, depends on the estimates one can obtain for R. $R = \mathcal{O}(\varepsilon^4)$.

Estimates not uniform in time for the remainder were given in [R.E., Lebowitz and Marra (1998)], where the stationary conductive solution was also constructed under more restrictive assumptions on the parameters. Here we need a good control of the long time behavior to prove the stability.([AN] for Couette).

Result.

Theorem [Arkeryd, R. E., Marra, Nouri]:

- Conductive case:
  - *Suppose* $\mathcal{R} < \mathcal{R}_c$*. Then there are* G_0 *and* ε_0 *such that, if* $G < G_0$ *and* $\varepsilon < \varepsilon_0$ *there is a positive stationary solution* f_ℓ *to the Boltzmann equation, corresponding to the conductive solution* (u_ℓ, θ_ℓ)*,*
  - *If* f_0^ε *is suitably chosen, there is a unique positive solution to the initial boundary value problem for the Boltzmann equations and it converges to* f_ℓ *as* $t \to +\infty$ *in* $L_2(\Omega \times \mathbb{R}^3, M(v)dxdzdv)$.

Result

- Convective case:
  - *Suppose $\mathcal{R} > \mathcal{R}_c$. Then there are $\alpha_c > 0$ and $\delta > 0$ such that, for all $\mathcal{R} < \mathcal{R}_c + \delta$ the following happens:*

    There are G_0 and ε_0 such that, if $G < G_0$ and $\varepsilon < \varepsilon_0$ there is a stationary solution f_s to the Boltzmann equation, corresponding to the convective solution in $\Omega = (-\alpha_c\pi, \alpha_c\pi) \times (-\pi, \pi)$, $(u_s(x,z), \theta_s(x,z))$, periodic in x with period $2\pi\alpha_c$.
  - *If f_0 is in a suitably chosen, there is a unique positive solution to the initial boundary value problem for the Boltzmann equations, and it converges to f_s as $t \to +\infty$ in $L_2(\Omega \times \mathbb{R}^3, M(v)dxdzdv)$.*

Result

Remark: While there is a restriction on the values of G, there is no limitation on the values of λ, so that it is possible to reach values of $\mathcal{R}$ in $(\mathcal{R}_c, \mathcal{R}_c + \delta)$. Remind that $\mathcal{R} \approx G\lambda$. This was not possible with the results of [ELM 1998].

Strategy of the proof.

Estimates for the remainder. Iterative procedure: one needs to study a linear equation of the type: ($R = R^{(n)}$)

$$\partial_t R + \frac{1}{\varepsilon}(v_x \partial_x R + v_z \partial_z R) - M^{-1} G \partial_{v_z}(MR) = \frac{1}{\varepsilon^2} LR$$

$$+\frac{1}{\varepsilon} J(\Phi_H, R) + B^{(n)}, \left(B^{(n)} = A + \varepsilon^{-1} J(R^{(n-1)}, R^{(n-1)}) \right)$$

$$R(t, x, -\pi, v) = \frac{1}{\sqrt{2\pi}} j_R^-(x) + \psi_-(x, v), \quad v_z > 0,$$

$$R(t, x, \pi, v) = \frac{M_+(v)}{M(v)} j_R^+(x) + \psi_+(x, v), \quad v_z < 0.$$

Strategy of the proof.

Main difficulties:
1) The operator $\varepsilon^{-2}L$ is non positive on $L_2(\mathbb{R}^3, M dv)$, but has a non trivial null space $\mathrm{Null}(L)$. One needs control of the component of R on $\mathrm{Null}(L)$.
2) The linear operator $\varepsilon^{-1}J(\Phi_H, \cdot\,)$ has a smaller factor in front, but has no sign and it is the main contribution when $R \in \mathrm{Null}(L)$.
3) The diffuse reflection boundary conditions require careful estimates of the solution at the boundary.
4) The control of the nonlinearity requires $L_\infty(\Omega)$-estimates, which are more intricate by the presence of the force.

More in Nouri's talk.

Strategy of the proof

Null(L) is five-dimensional and is spanned by $\psi_j(v)$, $j = 0, \ldots, 4$ with ψ_j's obtained by ortho-normalizing χ_j's ($\chi_0 = 1$, $\chi_1 = v_x$, $\chi_2 = v_y$, $\chi_3 = v_z$, $\chi_4 = v^2$) with respect to the inner product of $L^2_M = L^2(\mathbb{R}^3, M(v)dv)$.

Let P be the projector on Null(L) and $P^\perp = 1 - P$.

The operator L is symmetric in L^2_M and Range$(L) =$ Null$(L)^\perp$.

Spectral inequality: There is $c > 0$ such that

$$(f, Lf)_M \geq -c((1-P)f, \nu_0(1-P)f)_M,$$

with $\nu_0(v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}_2} d\omega |(v - v_*) \cdot \omega| M(v_*)$.

Strategy of the proof

On the other hand, for any $f \in L^2_M$,

$$|(f, J(\Phi_H, Pf))_M| \leq C\|Pf\|_M\|(1-P)f\|_M.$$

Using this (and ignoring the boundary) one gets the differential inequality

$$\frac{1}{2}\frac{d}{dt}|||f|||^2 \leq C|||f|||^2 + \int_\Omega |(B,f)_M|,$$

with $||| \cdot |||$ the norm in $L^2(\Omega \times \mathbb{R}^3, Mdxdzdv)$.
This produces bounds growing exponentially in time.

Not good enough!

Strategy of the proof

To avoid this we need to get a spectral inequality for the operator

$$L_J(f) = Lf + \varepsilon J(\Phi_H, Pf).$$

The null space of L_J, $\mathrm{Null}(L_J)$ is spanned by

$$\psi_j^\varepsilon = \psi_j - \varepsilon L^{-1} J(\Phi_H, \psi_j), \quad j = 0, \dots, 4.$$

Note that $L^{-1}J(f,g)$ makes sense because $J(f,g) \in \mathrm{Null}(L)^\perp$. Let P_J be the projector on $\mathrm{Null}(L_J)$. Then we can prove:

Proposition 1: *There are $\varepsilon_0 > 0$ and $c > 0$ such that, for any $\varepsilon < \varepsilon_0$*

$$\boxed{(f, L_J f)_M \le -c((1-P_J)f, \nu_0(1-P_J)f)_M.}$$

A similar inequality holds also for the adjoint L_J^*.

Strategy of the proof

Using this inequality we obtain

$$\frac{d}{dt}|||R|||^2 + \frac{1}{\varepsilon}\|R^{out}\|_{\sim}^2 + \frac{1}{\varepsilon^2}|||\nu^{1/2}(1-P_J)R|||^2 \leq$$
$$\int_\Omega dxdz|(B,R)_M| + \|\psi_\pm\|_\sim^2,$$

$$\|f\|_\sim^2 := \int_{-\alpha_c\pi}^{\alpha_c\pi} dx \int_{v_z>0} v_z M(v) \mid f(x,-\pi,v) \mid^2 dv +$$
$$\int_{-\alpha_c\pi}^{\alpha_c\pi} dx \int_{v_z<0} \mid v_z \mid M(v) \mid f(x,\pi,v) \mid^2 dv\Big)$$

The End

Thanks!