Non zero flux solutions of kinetic equations

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Plan

1. Non zero flux solutions. Two examples:

   a.- Uehling Uhlenbeck equation.

   b.- Coagulation equation.

2. General context: weak turbulence theory.
3. Main result.
4. Sketch of the proof.

Kinetic description

Dilute set of particles of mass $m \geq 0$, momentum $p \in \mathbb{R}^3$ and energy $E \geq 0$.

Constantly moving and colliding. Binary collisions dominate (dilute).

Best known model:

Boltzmann equation : dilute gas of identical paticles (same mass).

• Elastic collisions: two particles P_1, P_2 collide, give rise to particles P_3, P_4 such that

$$\boxed{p_1 + p_2 = p_3 + p_4, \qquad E_1 + E_2 = E_3 + E_4}$$

• For nonrelativistic particles of mass $m = 1$: $E = |p|^2/2$.

Homogeneous Boltzmann Equation

$f \equiv f(p,t)$: density function of particles with momentum p at time t.

Homogeneous gas: the density function f is independent of the space variable x.

$$
\begin{aligned}
\frac{\partial f}{\partial t}(t,p) &= Q(f)(t,p), \qquad t>0,\ p \in \mathbb{R}^3. \\
Q(f)(t,p) &= \int\int\int_{\mathbb{R}^9} W\left(p,p_2,p_3,p_4\right) q(f) dp_2 dp_3 dp_4 \\
q\left(f\right) &= f_3 f_4 (1+\varepsilon f)(1+\varepsilon f_2) - f\, f_2 (1+\varepsilon f_3)(1+\varepsilon f_4) \\
W\left(p,p_2,p_3,p_4\right) &= \delta(p+p_2-p_3-p_4)\ \delta\left(|p|^2+|p_2|^2-|p_3|^2-|p_4|^2\right)
\end{aligned}
$$

If $\varepsilon = 0$: Boltzmann equation for classical particles 1872.

If $\varepsilon = 1$: Nordheim 1928, Uehling-Uhrenbeck 1933. Quantum particles+integer spin (bosons).

Conserved quantities

Due to the symetries of W : if f solves the equation then formallly, for all $t > 0$:

$$\frac{d}{dt}\int_{\mathbb{R}^3} f(t,p)\,dp = 0, \quad \text{conservation of the total number of particles } N(f)$$

$$\frac{d}{dt}\int_{\mathbb{R}^3} p\, f(t,p)\,dp = 0, \quad \text{momentum conservation } P(f)$$

$$\frac{d}{dt}\int_{\mathbb{R}^3} |p|^2\, f(t,p)\,dp = 0 \quad \text{energy conservation } E(f).$$

Formally? if the convergence of the integrals allows necessary manipulations: (multiplication by test function φ , integration, apply Fubini, etc...)

$$\frac{d}{dt}\int_{\mathbb{R}^3} \varphi(p)\, f(t,p)\,dp = \int_{\mathbb{R}^{12}} W(\cdots)\, f_3 f_4 (1+\varepsilon f)(1+\varepsilon f_2) \times$$
$$\times \left[\varphi(p) + \varphi(p_2) - \varphi(p_3) - \varphi(p_4)\right] dp_2 dp_3 dp_4$$

Equilibria

Maxwell's distributions ($\varepsilon = 0$)

Bose-Einstein distributions ($\varepsilon = 1$):

$$F(p) = \frac{1}{e^{\beta|p-p_0|^2-\mu} - \varepsilon}, \quad \beta > 0,\ p_0 \in \mathbb{R}^3,\ \mu \le 0,$$

$$F(p_3)\,F(p_4)\,(1+\varepsilon F(p_1))\,(1+\varepsilon F(p_2)) \equiv F(p_1)\,F(p_2)\,(1+\varepsilon F(p_3))\,(1+\varepsilon F(p_4))$$

whenever: $\quad p_1 + p_2 = p_3 + p_4 \quad$ and $\quad |p_1|^2 + |p_2|^2 = |p_3|^2 + |p_4|^2$

from where $q(F) \equiv 0$ and $Q(F)(p) = 0$ for all $p \in \mathbb{R}^3$.

- Extensively studied: existence, uniqueness, entropy maxima, stability...

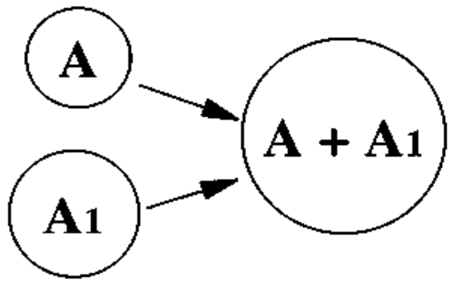
Coagulation equation

Dilute set of particles. Driven by externally given movement.

For example if the particles are in a gas: Brownian motion.

Non elastic collisions: aggregations.

Two particles of mass A and A_1 collide and form a single one of mass $A + A_1$



$f(t,x)$: density of particles of mass $x > 0$ at time $t \geq 0$.

Function f is independent of the velocity (all particles follow a Brownian motion).

$$\frac{\partial f}{\partial t}(t,x) = C(f)(t,x)$$

$$C(f)(t,x) = \frac{1}{2}\int_0^x W(x-y,y)f(t,x-y)f(t,y)dy -$$

$$-f(t,x)\int_0^\infty W(x,y)f(t,y)dy$$

$$W(x,y) = W(y,x), \text{ homogeneous function of degree } \lambda.$$

W: Depend on the type of movement of the particles and the particles themselves.

M. von Smoluchowski Z. Phys. 1916: $W = \left(x^{1/3} + y^{1/3}\right)^2$.

Equilibrium and conservations

- By the symmetries of W, if f is a solution then, formally:

$$\frac{d}{dt}\int_0^\infty x\, f(t,x)\, dx = 0$$

this is conservation of the total mass.

- The only equilibrium is: $f \equiv 0$.

Flux

For a given function $g(t,x)$, $x \in R^N$, the quantity:

$$\frac{d}{dt}\int_{B(x_0,R)} g(t,x)\,dx : \text{ flux of } g \text{ through } \partial B(x_0,R).$$

If g satisfies $\partial_t g = T(g)$ for some operator T, then:

$$\frac{d}{dt}\int_{B(x_0,R)} g(t,x)\,dx \equiv \int_{B(x_0,R)} T(g)(t,x)\,dp$$

For an arbitrary function we may still define the flux as:

$$J\left(g, \partial B(x_0,R)\right) = \int_{B(x_0,R)} T(g)(x)\,dx$$

The conservation laws for the U-U and the coagulation equation say that given a solutions f and if:

$$J_1(f) = \int_{\mathbb{R}^3} Q(f)\, dp, \qquad \text{total particle flux for U-U}$$

$$J_2(f) = \int_{\mathbb{R}^3} |p|^2\, Q(f)\, dp \qquad \text{total energy flux for U-U}$$

$$J(f) = \int_0^\infty xC(f)\, dx, \qquad \text{total particle flux for coagulation}$$

Then,

$$\begin{cases} J_1(f) = J_2(f) = 0 \\ \\ J(f) = 0 \end{cases}$$

Non zero flux solutions ?

- Coagulation equation. $(\partial_t f = C(f))$.

Function $G(x) = x^{-\frac{3+\lambda}{2}}$ is a weak stationary solution:

$$\int_0^{x/2} \left[W(x-y,y)G(x-y) - W(x,y)G(x)\right] G(y)dy -$$

$$-G(x)\int_{x/2}^{\infty} W(x,y)G(y)dy = 0.$$

Moreover, for every $R > 0$:

$$\int_0^R x\, C(G)\,(x)\, dx = -2\pi$$

The function G has a non zero flux at $+\infty$. (P.G.J. van Dongen, J. Phys. A '87.)

• **U-U equation.** $(\partial_t f = Q(f))$. Has no such solution but...

Look for "singular" or non bounded $f >> 1$. Then :

$$\begin{aligned}
q\,(f) &= f_3 f_4(1+f_1)(1+f_2) - f_1 f_2(1+f_3)(1+f_4) \\
&\approx f_3 f_4(f_1+f_2) - f_1 f_2(f_3+f_4) =: q_m\,(f) \\
\frac{\partial f}{\partial t}(t,p) &\approx Q_m(f)(t,p) \equiv \int\int\int W[\cdots]\, q_m(f)\, dp_2 dp_3 dp_4
\end{aligned}$$

Also studied by: Hasselman '62; Benney & Saffman '67; Zakharov '67... in the "Weak Turbulence" context.

We still have equilibria:

$$q_m\left(\widetilde{F}\right) = 0 \quad \text{for:} \quad \widetilde{F} = 1, \;\; \widetilde{F} = \frac{1}{|p|^2} = 0.$$

But moreover:

The function $G(p) = |p|^{-7/3}$, satisfies:

$$Q_m(G)(p) = \int\int\int_{\mathbb{R}^9} W\left(p_1, p_2, p_3, p_4\right) q_m(G) dp_2 dp_3 dp_4 = -C_* \delta_0$$

where $C_* > 0$ is an explicit constant. (V. E. Zakharov Sov. Physics JETP, 1967)

G is a fundamental solution of the non linear operator Q_m.

Then, for all $R > 0$: $$\int_{B(0,R)} Q_m(G)(p)\, dp = -C_*.$$

• The particle density $G(p)$ defines a stationary distribution of particles with non zero flux at the origin for the operator Q_m.

Also for Classical Boltzmann & More...

Same arguments used for Classical Boltzmann equation ($\varepsilon = 0$) by:

Katz, Kontorovich, Moiseev & Novikov (Sov. Phys. 1975)

for general homogeneous scattering probability of homogeneity $2m$ two solutions:

$$A_1 \, |p|^{-\frac{2m+7}{4}} \quad \text{and} \quad A_2 \, |p|^{-\frac{2m+6}{4}}$$

And also for Landau equation, relativistic Boltzmann equations etc...

Used to determine the electron distribution function formed under high-power laser radiation .

Back to U-U and Coagulation$\longrightarrow$

These two non zero flux solutions are related to two physical phenomena:

- The Bose Einstein condensation in gases of bosons (U-U equation).

- The gelation phenomena in aggregation processes (coagulation equation).

These are two examples in weak turbulence theory.

Developped by:

K. Hasselman, J. Fluid Mechanics 1962

V. E. Zakharov, V. S. Lvov & G. Falkovich, 1992

A. C. Newell, S. Nazarenko & L. Biven, Physica D 2001.

Weak turbulence

Development started in the 60's.

Gives equations which describe quantitatively the energy transfer between turbulent, weakly non linear and dispersive waves in fluids.

The key characteristics of the description by means of weak turbulence are:

- to derive a kinetic equation for the temporal evolution of the energy spectrum density of the wave field

- to obtain non zero flux solutions of these equations.

Has been applied to waves in atmosphere and ocean, semiconductor lasers and turbulent plasmas...

Problem

- The fundamental solution of the modified U-U equation: $G(p) = |p|^{-7/3}$.
- The stationary weak solution of the coagulation equation $G(x) = x^{-(3+\lambda)/2}$.

For none of them can the natural quantities of the equation be defined:

- For the modified Uehling Uhlenbeck it would be the total number of particles:

$$\int_{\mathbb{R}^3} G(p)\,dp = C\int_0^\infty G(p)\,|p|^2\,d|p| \quad (\text{ diverge as } |p| \to +\infty).$$

- For the coagulation equation it would be the total mass:

$$\int_0^\infty x\,G(x)\,dx \quad (\text{diverges at } x = 0).$$

Question

Existence of solutions to the Cauchy problems associated to these equations:

• with non zero flux

• for which the natural quantities are well defined ?

The answer in both cases is yes.

In order to simplify the presentation:

I present the precise result of the U-U equation.

A similar result holds for the coagulation equation.

Theorem. (U-U equation) Radial coordinates.

$$\frac{\partial f}{\partial t}(t,x) = Q_r(f)(t,x), \quad x \equiv |p|^2 > 0,\ t > 0, \quad \text{(U-U)}_r \text{ equation}$$
$$f(0,x) = f_0(x),\ x > 0.$$

Suppose that:

$$\begin{aligned}
\left| f_0(x) - A\,x^{-7/6} \right| &\leq \frac{B}{x^{7/6-\delta}}, \quad 0 \leq x \leq 1, \\
\left| f_0'(x) + \frac{7}{6} A\,x^{-13/6} \right| &\leq \frac{B}{x^{13/6-\delta}}, \quad 0 \leq x \leq 1, \\
f_0 \in \mathbf{C}(\mathbb{R}^+) \quad \text{y} \quad f_0(x) &\leq B\frac{e^{-Dx}}{x^{7/6}}, \quad x \geq 1
\end{aligned}$$

where A, B, C, D, δ positive constants.

Then, there exists:

- a solution $f(t,x)$ of (UU)$_r$, $f \in \mathbf{C}^{1,0}((0,T) \times (0,+\infty))$,
- a function $\lambda(t) \in \mathbf{C}[0,T] \cap \mathbf{C}^1(0,T)$,
- constants $L > 0$, $T > 0$ such that:

$$
\begin{aligned}
&(i) \quad 0 \le f(x,t) \le L \frac{e^{-Dx}}{x^{7/6}}, \quad \text{if } x > 0,\ t \in (0,T),\\
&(ii)\ |f(x,t) - \lambda(t)\, x^{-7/6}| \le L\, 6^{-7/6+\delta/2}, \quad x \le 1,\ t \in (0,T),\\
&(iii) \quad |\lambda(t)| \le L, \quad \text{if} \ \ t \in (0,T).
\end{aligned}
$$

(E., S. Mischler & J. J. L. Velázquez, Proc. Roy. Soc. Edinburgh 2008)

Remarks

1. The solution f is classical and radially symmetric with respect to p.

The equation is satisfied at each point (p,t), $p\in\mathbb{R}^3$, $t\in(0,T)$.

2. The solution behaves like:

$$f(t,p)\sim\lambda(t)\,|p|^{-7/3}\quad \text{if}\quad 0<|p|<1,\ t\in(0,T)$$

3. And:

$$\frac{d}{dt}\left(\int_{|p|^2\le R} f(t,p)\,dp\right)=\int_{|p|^2\le R} Q_r(f(t,p)\,dp=-C\lambda^3(t)+\mathcal{O}(R^{1/10}),$$

as $R\to 0$. Non zero flux at the origin.

Non Uniqueness ?

X. Lu in J. Stat. Phys. **116** (2004) proves global existence of weak radial solutions for the U-U equation.

Method of Lu's proof:
1. Solve a regularised equation with a "truncated kernel".
2. Uniform apriori estimates.
3. Pass to the limit and obtain a weak solution $\mathcal{F}$ such that:

- For all $t > 0$, $\sqrt{\cdot}\,\mathcal{F}(t,\cdot)$ is a non negative bounded measure in $\mathbb{R}^+$.
- The total mass is constant: $\dfrac{d}{dt}\displaystyle\int_0^\infty d\left(\mathcal{F}(t,x)\sqrt{x}\right) = 0.$

Our solution is classical, local (in time) and looses mass at $t > 0$.

Proof

Based on the linearisation of the (U-U)$_r$ equation :

$$\begin{aligned}\frac{\partial f}{\partial t} &= \int\int_{\mathbb{R}^2} w\,(x_1,x_3,x_4)\,q(f)\,dx_3\,dx_4 \\ q\,(f) &= f_3 f_4(1+f_1)(1+f_2) - f_1 f_2(1+f_3)(1+f_4)\end{aligned}$$

around the initial data as follows:

$$f(t,x) = \lambda(t)\,f_0(x) + g(t,x).$$

We write:

$$q\,(f) = q\,(\lambda(t)\,f_0 + g) = \underbrace{\underbrace{\ell_2\,(\lambda(t)\,f_0, g)}_{\text{quadratic in } \lambda(t) f_0} + \underbrace{\ell_1\,(\lambda(t)\,f_0, g)}_{\text{linear in } \lambda(t) f_0}}_{\text{linear in g}} + q\,(\lambda(t)\,f_0) + R$$

$$q\left(f\right)=\ell_2\left(\lambda(t)\,f_0,g\right)+\ell_1\left(\lambda(t)\,f_0,g\right)+q\left(\lambda(t)\,f_0\right)+R$$

with the remaining term R quadratic with respect to g.

Using $f_0(x)\sim x^{-7/6}$ as $x\to 0$ we approximate:

$$\ell_2\left(\lambda(t)\,f_0,g\right)=\lambda(t)^2\,\ell_2\left(f_0,g\right)\approx\lambda(t)^2\,\ell_2\left(x^{-7/6},g\right)$$

This introduces a new term, absorbed in the term R.

Rescaling the time variable t we may write the equation as follows:

$$\begin{aligned}\frac{\partial g}{\partial\tau} &= \mathcal{L}(g)+\mathcal{R}\\ \mathcal{L}(g) &= \int\int_{\mathbb{R}^2} w\left(x_1,x_3,x_4\right)\ell_2\left(x^{-7/6},g\right)dx_3dx_4\end{aligned}$$

A similar calculation is also performed for the coagulation equation.

In both equations (U-U & coagulation) we arrive at:

$$\frac{\partial g}{\partial \tau} \quad = \quad \mathcal{L}(g) + \mathcal{R}$$

Usual form of a non linear equation associated to a linear unbounded operator $\mathcal{L}$.

The usual strategy is now:

1.- Precise estimates on the solutions of the linear homogeneous problem.

2.- Duhamel's formula and fixed point argument to treat the non linear problem.

First step: the linearised equation.

The fundamental solutions of $\partial_t g = \mathcal{L} g$ may be calculated explicitely.

By suitable changes of variables ($x = e^X$) and Fourier transform it is reduced to

The Carleman equation

To find function $\widehat{G}(t,\xi)$ such that, for all $t > 0$, $G(t,\cdot)$ is analytic in a strip:

$$S = \{\xi;\ \xi = u + iv,\ A_1 < v < A_2,\, u \in \mathbb{R}\}$$

and satisfies, for all $t > 0$ and $\xi \in S$:

$$\boxed{\partial_t \widehat{G}(t,\xi) = \widehat{G}\,(t, \xi + i\,h)\,\Phi\,(\xi + i\,h)\,; \quad \widehat{G}(0,\xi) = 1} \tag{1}$$

for some $h \in \mathbb{R}$, $\Phi : \mathbb{C} \to \mathbb{C}$ and $A_1 < A_2$ explicitly given by the kernel W.

In our two examples:

- $h = \dfrac{\lambda - 1}{2}; \quad \Phi(\xi) = -\dfrac{2\sqrt{\pi}\;\Gamma(i\xi + 1 + \frac{\lambda}{2})}{\Gamma(i\xi + \frac{\lambda+1}{2})}$ (Coagulation equation)

- $$\Phi(\xi) = -a + \sum_{j=0}^{\infty} \frac{A_1(j)}{(1 - 6i\xi + 12j)} + \sum_{j=0}^{\infty} \frac{A_2(j)}{(1 - 3i\xi + 3j)} + \sum_{j=0}^{\infty} \frac{A_3(j)}{(3 + 2i\xi + 2j)}$$
$$+ \sum_{j=0}^{\infty} \frac{A_4(j)}{(10 + 3i\xi + 6j)}; \qquad h = -\frac{1}{3}$$ (U-U equation)

Remark. From a classical PDE: $u_t = P(D)\,u$, with P a polynomial, we obtain the ordinary differential equation: $U_t(t,\xi) = P(\xi)\,U(t,\xi)$ ($\xi \in \mathbb{R}^3$, a parameter).

A non linear transformation

$$\widehat{G}(t,\xi) = C_1 \int_{\Im m y=\beta_0} \frac{V(y)}{V(\xi)} t^{-\frac{y-\xi}{h}\,i}\,\Gamma\left(\frac{y-\xi}{h}\,i\right)\,dy$$
$$A_1 < \beta_0 < A_2$$

$(C_1 \in \mathbb{C}$ constant) reduces the problem to a stationary equation:

$$V \text{ analytic in } \Im m\,\xi \in \left(A_1,\, \widetilde{A}_2\right) \text{ and:}$$
$$V\,(\xi) = -V\,(\xi + i\,h)\,\Phi\,(\xi + i\,h)\,; \quad A_1 < \Im m\,\xi < \widetilde{A}_2. \tag{2}$$

Remark. Equation (2) posses infinitely many solutions:

$$V(\xi) = V_{part}(\xi)\,P(\xi); \quad P(\xi) = P\,(\xi + i\,h)$$

A particular solution V_{part} is given by the Cauchy integral:

$$V_{part}(\xi) = \exp\left[-\frac{i}{h}\int_{\Im m y=\beta} \ln\left(\Phi(y)\right)\left(\frac{1}{1-e^{\frac{2\pi}{h}(\xi-y)}} - \frac{1}{1-A\,e^{-\frac{2\pi}{h}y}}\right) dy\right]$$

with β such that $\Phi(\cdot)$ has no zeros nor poles on the line: $\Im m\xi = \beta$

and $A \in \mathbb{C}$ arbitrary such that $1 - A\,e^{-\frac{2\pi}{h}y} \neq 0$ as $\Im m y = \beta$.

Even to define V_{part} precisely we need:

- to specify β
- specify the branch of $\ln\left(\Phi(\cdot)\right)$

How to chose one solution? Why do we need to choose one?

Why ?

• For each $t > 0$, $\widehat{G}(t,\cdot)$ has to be the Fourier transform of a tempered distribution G

$$G(t,X) = \frac{1}{\sqrt{2\pi}} \int e^{iX\xi} \widehat{G}(t,\xi) d\xi$$

• For each $t > 0$, $\widehat{G}(t,\cdot)$ has to be analytic in the strip $\Im m\xi \in (A_1, A_2)$ in order to have necessary behaviour of $G(t,X)$ as $X \to \pm\infty$.

How ?

• By a suitable choice of β, the branch of $\ln(\Phi(\cdot))$ and the function $P(\xi)$.

Differences between the different cases appear here:

The problem is very sensitive to the behaviour of $\Phi(\xi)$ as $\Re e\xi \to \pm\infty$:

- For U-U:

$$\Phi(\xi) \sim -a + \frac{b_1}{\xi^{1/6}} + \frac{b_2}{\xi} + \cdots, \quad |\xi| \to +\infty$$

- For the coagulation equation , if $|\xi| \to +\infty$:

$$\Phi(\xi) \sim -\sqrt{2\pi}(1+iQ)\sqrt{Q\,\xi} + \cdots, \qquad Q \equiv Q(\xi) = \mathrm{sgn}(\Re e[\xi])$$

Consequences:

- Although the original problems seem to be very similar, the mathematical arguments may strongly vary from one example to another.

- Different regularity properties of the solutions.

Regularity properties of the solutions:

- No regularizing effects in the U-U equation.

The fundamental solution of the linearized equation around $x^{-7/6}$:

$$\begin{cases} \dfrac{\partial f}{\partial t} = \mathcal{L}_{UU} f, \;\; t > 0 \\ \\ f(0, x) = \delta_{x=1} \end{cases}$$

is, for all $t > 0$ and $x \in (0, 2)$ such that:

$$f(t, x) \sim e^{-a\,t}\delta_{x=1} + \sigma(t)\, x^{-7/6} + \text{smaller terms}$$

• Regularising effects in the coagulation equation:

The fundamental solution of the linearised problem around $x^{-(3+\lambda)/2}$

$$\begin{cases} \dfrac{\partial g}{\partial t} = \mathcal{L}_{coag}(g) \\ \\ g(0,x) = \delta_{x=1} \end{cases}$$

is such that: $$g(t,x,1) = \begin{cases} t^{-2}\Psi\left(\frac{x-1}{t^2}\right) + \mathcal{O}\left(t^{-2}\right) \quad \text{for } x = 1 + \mathcal{O}(t^2) \\ \\ \mathcal{O}\left(\dfrac{t^{1-2\delta}}{|x-1|^{\frac{3}{2}-\delta}}\right) \quad \text{for } t^2 < |x-1| < \frac{1}{2} \end{cases}$$

$$\Psi(\chi) = \begin{cases} \dfrac{2}{\pi} e^{-\frac{\pi}{\chi^{3/2}}}, \quad \text{for all } \chi \geq 0, \\ 0 \quad \text{for all } \chi < 0. \end{cases}$$

The Dirac measure is immediately regularized and the fundamental solution is a regular function.

Another strong dependence on Φ:

Once $\widehat{G}(t,\xi)$ is known then:

$$G(t,X) = \frac{1}{\sqrt{2\pi}} \int e^{iX\xi} \widehat{G}(t,\xi) d\xi \qquad \text{for all } t > 0 \text{ and } X \in R.$$

The behaviour of $G(t,X)$ for $X \to \pm\infty$ is given by the singularities of $\widehat{G}(t,\xi)$.

The singularities of $\widehat{G}(t,\xi)$ are given by the zeros and poles of the function Φ.

That is what fixes the precise values of the power law behaviour of the solutions of the kinetic equations:

- $7/6$ for U-U
- $(3+\lambda)/2$ for the coagulation equation.

$$\Phi(\xi) \quad = \quad -\frac{2\sqrt{\pi}\ \Gamma(i\xi + 1 + \frac{\lambda}{2})}{\Gamma(i\xi + \frac{\lambda+1}{2})} \quad \text{(coagulation)}$$

$$\Phi(\xi) \quad = \quad -a + \sum_{j=0}^{\infty} \frac{A_1(j)}{(1 - 6i\xi + 12j)} + \sum_{j=0}^{\infty} \frac{A_2(j)}{(1 - 3i\xi + 3j)} + \sum_{j=0}^{\infty} \frac{A_3(j)}{(3 + 2i\xi + 2j)}$$

$$+ \sum_{j=0}^{\infty} \frac{A_4(j)}{(10 + 3i\xi + 6j)} \quad \text{(U-U equation)}$$

where $a > 0$, $A_i(j)$ explicitely known constants.

For U-U, zeros are not known except $7\,i/6$ and $13\,i/6$. It is possible to:

estimate some zeros

estimate some regions of the complex plane without zeros

numerically, using the argument principle.

Conclusions.

• Have obtained non zero flux solutions for U-U and coagulation equations for which the natural quantities are well defined.

• These are two examples of equations obtained by the weak turbulence theory.

• Describe same problem in two different physical situations.

• Treated using the same techniques.

• Nevertheless some variety in the mathematical tools and in the results.

• Much more cases to study. The differences make difficult to forecast the precise final result:

Other examples

1.-Weak turbulence in capillary water-waves.

$$\begin{aligned}
\frac{\partial n}{\partial t}(p,t) &= \int_{\mathbb{R}^6} |V_{p,1,2}|^2 f_{p,1,2}\delta(p-p_1-p_2)\delta(\omega_p-\omega_1-\omega_2)dp_1dp_2 + \\
&\quad +2\int_{\mathbb{R}^6} |V_{1,p,2}|^2 f_{1,p,2}\delta(p_1-p-p_2)\delta(\omega_1-\omega_p-\omega_2)dp_1dp_2 \\
f_{p,1,2} &= n_1n_2 - n_p(n_1+n_2)
\end{aligned}$$

- Shallow water capillary waves: $\omega_k = C_1\,|p|^2$ and $V_{p,1,2} = C_2|p|^2$.
- Deep water capillary waves: $\omega_k = C_3\,|p|^{3/2}$ and

$$\begin{aligned}
V_{p,1,2} &= C_4\left[(p_1p_2+|p_1||p_2|)\left(\frac{|p_1||p_2|}{|p|}\right)^{1/4} + (p_1p+|p_1||p|)\left(\frac{|p_1||p|}{|p_2|}\right)^{1/4}\right. \\
&\quad \left. +(pp_2+|p||p_2|)\left(\frac{|p||p_2|}{|p_1|}\right)^{1/4}\right]
\end{aligned}$$