Coupling conditions for transport problems on networks governed by conservation laws

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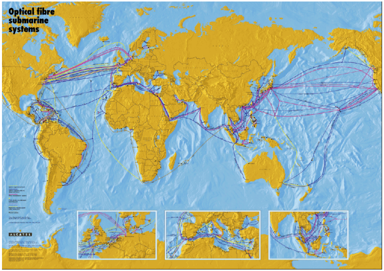
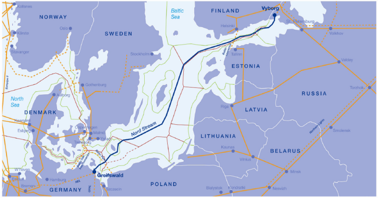
Outline of the Talk

Scope: Boundary Conditions For Hyperbolic Balance Laws on Networks

1. Applications and typical questions
2. Example based on Gas dynamics and Burger's equation
3. Further theoretical and numerical results
4. Questions of control of networked systems

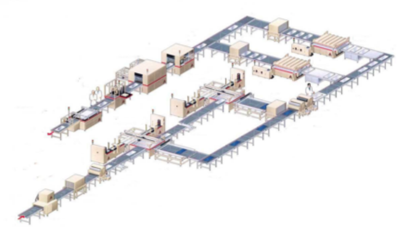


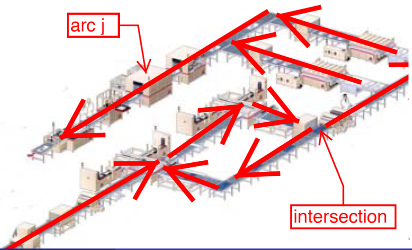


Mathematical Setting

Coupled systems of one–dimensional (systems) of nonlinear hyperbolic balance or conservation laws

- Dynamics of a physical system on arc j given by a hyperbolic pde
- Interest in the coupling of different dynamics at intersection ($x = 0$)
- A priori prescribed coupling introduce boundary conditions
- Questions on well–posed boundary conditions for nonlinear pdes
- Applications: Traffic flow, gas flow, supply chains, internet / communication, water flow in canals, irrigation channels, blood flow, ...





Applications: Traffic Flow On Road Networks

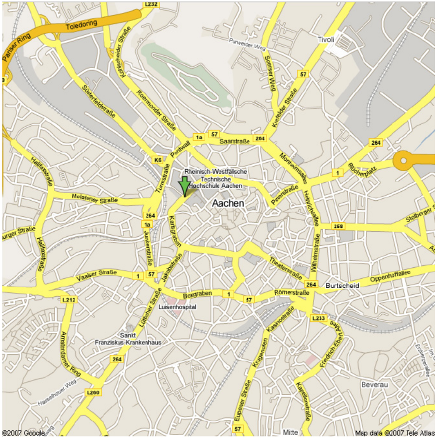
Macroscopic description of traffic flow on one–way road j by density $\rho^j(x,t)$ and average velocity $u^j(x,t)$

- Models based on scalar conservation laws (LWR)

$$\partial_t \rho^j + \partial_x \rho^j u(\rho^j) = 0$$

  or 1d systems (ARZ, Colombo)

- Coupling conditions at traffic intersections or on– and off–ramps to highways
- Many contributions since $\approx$ 1995 with results by Colombo, Holden, Lebacque Piccoli, Rascle, ...



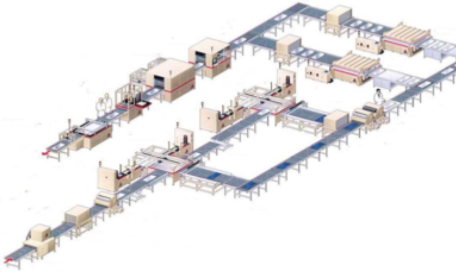
Applications: Supply Chain Management

Macroscopic description of large–volume production facilities by density of parts ρ^j

- Example of a one–phase model for a re–entrant factory

$$\partial_t \rho^j + \partial_x \frac{\rho^j}{1 + \int \rho^j dx} = 0$$

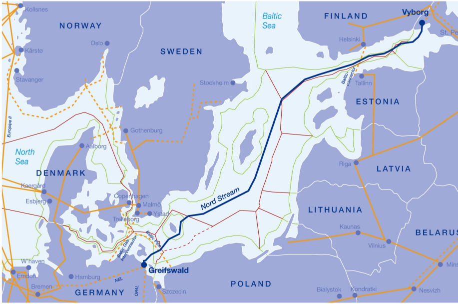
- Coupling conditions at machine–to–machine connections by buffers or storage tracks
- Contributions with results since $\approx$ 2000 by Armbruster, d'Apice, Degond, Göttlich, Klar, Ringhofer, . . .



Applications: gas networks

Gas flow in pipe networks described by the $p-$system or Euler's equation

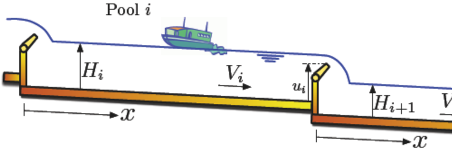
- 2×2 system of hyperbolic conservation laws with source term due to pipewall friction
- Coupling conditions trhough compressor stations, pipe–to–pipe fittings or valves
- Contributions with results since $\approx$ 2006 by Banda, Colombo, Klar, Garavello, Guerra, Schleper, ...



Applications: Water Networks

Control of a water level of river's by St. Venant equation

- 2×2 nonlinear hyperbolic equations with source terms due to slope of canal
- Coupling conditions through controllable gates
- Question of stabilization: maximal allowed deviation in height is 3 cm on 200 km
- Contributions with results since $\approx$ 2003 by Bastin, Coron, Gugat, Li Tatsien, Leugering, ...



Preliminary discussion

- Given a system of balance laws on a network define a weak solution as

$$\sum_i \int \int \partial_t \vec{\phi}_i \vec{u}_i + \partial_x \vec{\phi}_i \vec{f}_i(\vec{u}_i) + g(\vec{u}_i)\vec{\phi}_i dxdt = 0 \; \forall \vec{\phi}_i$$

- Using test functions with $\vec{\phi}_i(0-, t) = \vec{\phi}_j(0+, t)$ obtain Rankine–Hugenoit conditions at the node as

$$\sum_i \pm \vec{f}_i(\vec{u}_i(0\pm, t) = 0$$

- For system of m equations one obtains m conditions for a junction with n connected arcs $\implies$ further conditions need to be imposed
- System is non–linear, hyperbolic and the number of boundary conditions depend on the state of the system at most $n \times m$ conditions can be prescribed
- Regularity of the solutions as for 1d hyperbolic systems, i.e., BV in space ensures fulfillment of coupling conditions at $x = 0\pm$

Example from gas dynamics: Well–posedness?

- Gas dynamics in pipe j by $p-$system

$$\partial_t \begin{pmatrix} \rho_j \\ \rho_j u_j \end{pmatrix} + \partial_x \begin{pmatrix} \rho_j u_j \\ p(\rho_j) + \rho_j u_j^2 \end{pmatrix} = \begin{pmatrix} 0 \\ f(\rho_j, u_j) \end{pmatrix}$$

- coupled through the dynamics on other pipes by
  - conservation of mass

$$\sum_j \pm \rho_j u_j(0\pm, t) = 0$$

  - and additionally equal pressure (engineering community)

$$p(\rho_j(0+, t)) = p(\rho_i(0-, t)) \forall i, j$$

  - or additionally equal momentum

$$p(\rho_j(0+, t)) + (\rho_j u_j^2)(0+, t) = p(\rho_i(0-, t)) + (\rho_i u_i^2)(0-, t) \forall i, j$$

  - or ...

Discussion of derivation of well-posedness results

$$\partial_t \begin{pmatrix} \rho_j \\ \rho_j u_j \end{pmatrix} + \partial_x \begin{pmatrix} \rho_j u_j \\ p(\rho_j) + \rho_j u_j^2 \end{pmatrix} = \begin{pmatrix} 0 \\ f(\rho_j, u_j) \end{pmatrix},$$

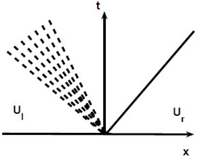
$$\sum_j \pm(\rho_j u_j)(x_j, t) = 0, \; p(\rho_j(x_j, t)) = p(\rho_i(x_i, t))$$

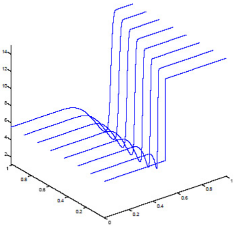
1. Notation for solutions: weak solutions $C^0(t, BV(x))$ and $C^1(t, L^1(x))$
2. Approximate solutions by piecewise constant initial data
3. Piecewise constant data generates a sequence of waves as solutions to Riemann problems (on each arc)
4. Need to construct solutions to Riemann problems at the junction
5. TV bounds on wave interactions
6. Compactness argument yields existence on weak solution (Helly's Theorem)

Recall: Riemann Problem

$$\partial_t U + \partial_x F(U) = 0,\ U(x,0) = \begin{pmatrix} U_l & x < 0 \\ U_r & x > 0 \end{pmatrix},\ U(x,t) : \mathbb{R}^2 \to \mathbb{R}^n$$

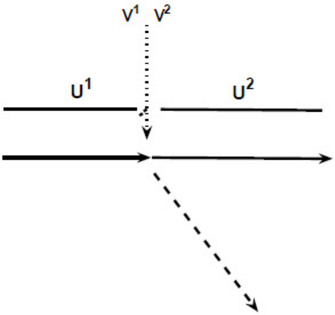
Theorem for strictly hyperbolic systems: Existence of a self–similar solution $U(x,t) = V(x/t)$. *Solution consists of at most* $n+1$ *constant states separated by entropy–shocks, rarefaction waves or contact discontinuities.*

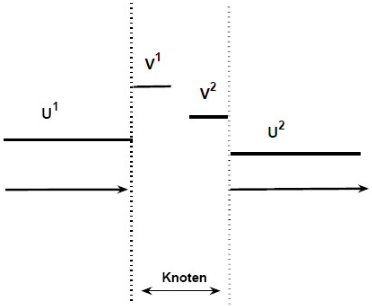




Riemann problems at the junction

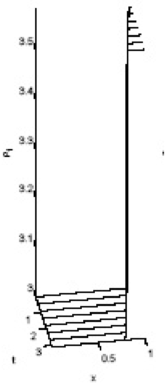
- Consider the situation of piecewise constant initial data in each arc $U_0^j = (\rho_j^0, \rho_j^0 u_j^0)$ – coupling conditions are not necessarily satisfied
- Introduce unknown, artifical states V^j for each arc
- Solve a Riemann problem on each arc with an artifical state V^j at the node





Constraints on V_j

- Compute $\Omega_j \in \mathbb{R}^2$, such that for all $V \in \Omega_j$, the self–similar solution $U_j(x,t)$ to a Riemann problem for U^j and V^j consists of waves of non–positive speed (incoming arcs)
- A wave tracking solution satisfies at the node $U_j(0-,t) = V_j \; \forall t > 0$
- Reduced problem: Find $V_j \in \Omega_j \subset \mathbb{R}^2$, such that the coupling conditions are fulfilled



Computation of the admissible sets Ω_j?

Admissible sets Ω_j for Burger's equation

$$\partial_t u_j + \partial_x \frac{1}{2} u_j^2 = 0, u_j(x,0) = \begin{pmatrix} u_l^j & x < 0 \\ v^j & x > 0 \end{pmatrix}$$

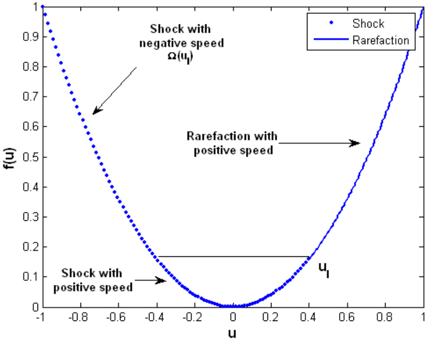
- Setting: Incoming arc j, u_l^j is the initial value, $v^j \in \Omega_j$ such that coupling condition is satisfied
- If $v^j < u_l^j$ then the solution $u(x,t)$ is a shock wave of velocity

$$s = \frac{f(u_l) - f(v)}{u_l - v}$$

- If $v^j > u_l^j$ then the solution is a rarefaction wave with velocity $f'(u_j)$

Admissible sets Ω_j for Burger's equation

Burger's equation $\partial_t u_j + \partial_x \frac{1}{2} u_j^2 = 0,\ u_j(x,0) = \begin{pmatrix} u_l^j & x < 0 \\ v^j & x > 0 \end{pmatrix}$



- $\Omega_j(u_l) = \{v : -\infty < v \leq \min\{-u_l, 0\}\}$

Riemann solver at the junction for Burger's equation

$\partial_t u_j + \partial_x \frac{1}{2} u_j^2 = 0, \; \sum_j \pm u_j^2(0\pm, t) = 0$

- Given constant data u_0^j close to the junction
- Compute the admissible set $\Omega_j := \Omega_j(u_j^0)$
- Obtain states $v_j \in \Omega_j \subset \mathbb{R}$ such that

$$\sum_j \pm v_j^2 = 0, \; v_j \in \Omega_j$$

  (v_j not necessarily unique $\implies$ additional conditions necessary!)

- Solve on each arc a Riemann problem with data u_j^0 and v_j (yields wave with signed speed, careful estimates on TV–bounds necessary!)

Gas & Water Networks: p-system

$$\partial_t \begin{pmatrix} \rho_j \\ \rho_j u_j \end{pmatrix} + \partial_x \begin{pmatrix} \rho_j u_j \\ p(\rho_j) + \rho_j u_j^2 \end{pmatrix} = \begin{pmatrix} 0 \\ f(\rho_j, u_j) \end{pmatrix}$$

- Need: conservation of mass and either equal pressure or equal momentum assumption for uniqueness of Riemann solver
- Two characteristic families ρ, q
- Each solution might be a combination of shock and rarefaction waves
- Results so far: subsonic data only, single junction

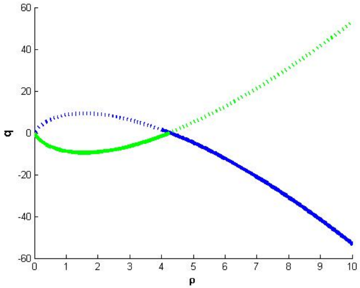
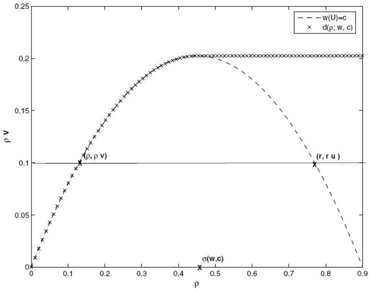


Figure: Phase diagram. dashed=rarefaction waves

Traffic Networks: LWR based models

$$\partial_t \rho_j + \partial_x \rho_j u_j(\rho_j) = 0$$

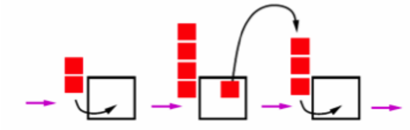
- Coupling condition $\vec{f}_j = A\vec{f}_j$ for A enjoying certain properties (distribution matrix with assumptions on the kernel) and drivers maximize the flux through the intersection
- or simple distribution matrix A and a right–of–way matrix B and maximization of the flux through the intersection
- Interpretation as demand and supply functions
- Existence of weak solutions for a



Supply Chains: piecewise linear flux and buffer

$$\partial_t \rho_j + \partial_x \min\{\mu_j, v_j \rho_j\} = 0, \ \partial_t q_j = v_i \rho_i - v_j \rho_j$$

- Need: Prescribe a rule how the machine empties its buffer
- No backwards moving information
- Results so far: Existence of solutions on a circle–free network and BV data



Notion of a solution

Definition

Fix $\hat{u} \in \Omega^n$ and $T \in]0, +\infty]$. A weak Ψ-solution to

$$\begin{cases} \partial_t u_l + \partial_x f(u_l) = 0 & t \in \mathbb{R}^+ & l \in \{1, \ldots, n\} \\ u(0,x) = u_o(x) & x \in \mathbb{R}^+ & u_o \in \hat{u} + L1(\mathbb{R}^+; \Omega^n) . \end{cases} \qquad (1)$$

on $[0, T]$ is a map $u \in C^0\left([0, T]; \hat{u} + L1(\mathbb{R}^+; \Omega^n)\right)$ such that

(W) For all $\phi \in C^\infty\left(]-\infty, T[\times \mathbb{R}^+; \mathbb{R}\right)$ and for $l = 1, \ldots, n$

$$\int_0^T \int_{\mathbb{R}^+} \left(u_l\, \partial_t \phi + f(u_l)\, \partial_x \phi\right)\, dx\, dt + \int_{\mathbb{R}^+} u_{o,l}(x)\, \phi(0,x)\, dx = 0 .$$

(Ψ) The condition at the junction is met: for a.e. $t \in \mathbb{R}^+$, $\Psi\left(u(t, 0+)\right) = 0$.

Result on 2x2 conservation laws on networks

Theorem

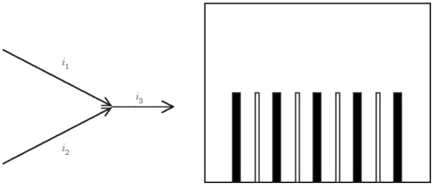
Let $n \in N$, $n \geq 2$. Fix the pairwise distinct vectors $\nu_1, \ldots, \nu_n$ in $\mathbb{R}^3 \setminus \{0\}$. Fix an n-tuple of states $\bar{u} \in \Omega^n$ such that f satisfies **(F)** *at $\bar{u}$ and the Riemann Problem with initial datum $\bar{u}$ admits the stationary solution. Let $\Psi \in C^1(\Omega^n; \mathbb{R}^n)$ satisfy a condition on its determinant and let the data be subsonic. Then, there exist positive δ, L and a map $S\colon [0, +\infty[\times \mathcal{D} \to \mathcal{D}$ such that:*

1. $\mathcal{D} \supseteq \{u \in \bar{u} + L1(\mathbb{R}^+; \Omega^n)\colon TV(u) \leq \delta\}$;
2. *for $u \in \mathcal{D}$, $S_0 u = u$ and for $s, t \geq 0$, $S_s S_t u = S_{s+t} u$;*
3. *for $u, w \in \mathcal{D}$ and $s, t \geq 0$, $\|S_t u - S_s w\|_{L1} \leq L \cdot (\|u - w\|_{L1} + \|t - s\|)$.*
4. *If $u \in \mathcal{D}$ is piecewise constant, then for $t > 0$ sufficiently small, $S_t u$ coincides with the juxtaposition of the solutions to Riemann Problems centered at the points of jumps or at the junction.*

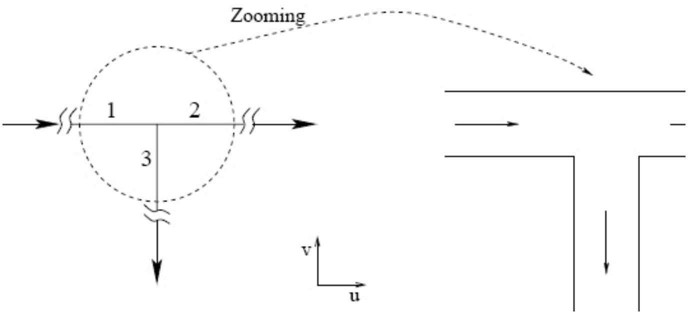
Moreover, for every $u \in \mathcal{D}$, the map $t \mapsto S_t u$ is a Ψ-solution.
For any $\tilde{\Psi} \in C^1(\Omega^n; \mathbb{R}^n)$ with $\|\tilde{\Psi} - \Psi\|_{C^1} < \delta$, $\tilde{\Psi}$ generates a semigroup of solutions on $\mathcal{D}$ and for $u \in \mathcal{D}$

Other Approaches (Theoretical)

- Second–order traffic flow model due to Aw-Rascle-Zhang
- LWR + information traveling with car and influencing it's speed (e.g., truck/car property)
- Junction introduces a mixture of cars on the outgoing road
- Instead of solving a Riemann problem solve an initial-value problem with oscillating initial data.

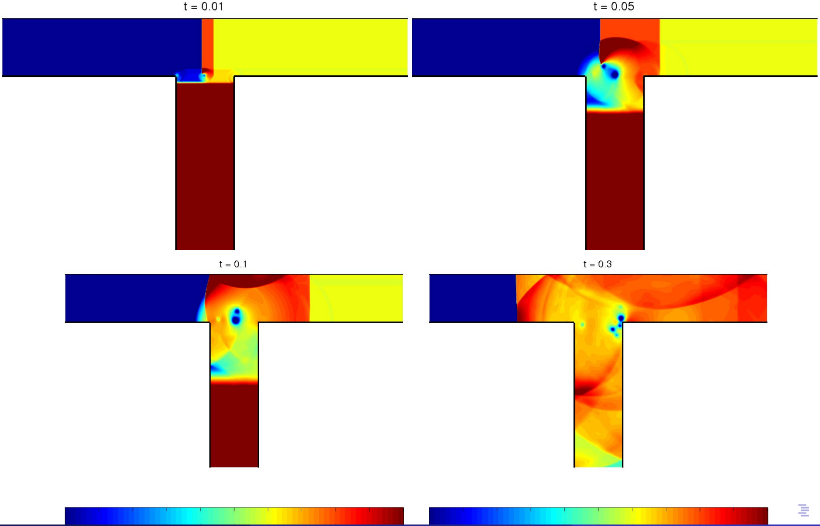


Numerical approaches: Validation of coupling conditions by 2d simulations



- Node is locally a 2d domain
- Prescribe constant initial data
- Simulation until nearly a steady–state is obtained
- Average to obtain similar values compared with 1–d model

Example: time evolution of the density $\rho(x, y, t)$ for p-system



Example: Comparison of predicted (1–d) values at intersection and results of numerical simulation

Pressure in three pipes

	$u_1 = 4$	$u_1 = 5$	$u_1 = 6$	$u_1 = 6.5$	$u_1 = 7.5$	$u_1 = 8.5$	$u_1 = 9$
Pipe 1	115.2418	118.3104	120.8469	122.1585	124.8776	127.3623	128.5093
	(124.492)	(131.586)	(139.071)	(142.965)	(151.065)	(159.597)	(164.028)
Pipe 2	115.0216	118.2603	120.8546	122.0521	124.4525	126.8142	127.9266
	(124.492)	(131.586)	(139.071)	(142.965)	(151.065)	(159.597)	(164.028)
Pipe 3	117.0552	119.1032	121.9611	123.4902	125.6128	128.5010	129.8158
	(124.492)	(131.586)	(139.071)	(142.965)	(151.065)	(159.597)	(164.028)

- Equal pressure at node is a reasonable assumption for $1 \longrightarrow 2$ situation
- Absolute values differ up to 30%
- Picture different in the $2 \longrightarrow 1$ situation

Control problems at the node is common to many applications

- Control P acts through a modified coupling conditions, e.g., compressor in gas networks

$$P = c\, q_i \left(\left(\frac{p(\rho_i)}{p(\rho_j)} \right)^{\kappa} - 1 \right), \quad q_i = q_j$$

- use the result on continuous dependence on the coupling condition itself to obtain results on optimal nodal control (weak solutions)
- use linearization and Lyapunov stability criteria to obtain controllability (strong solutions)

Mathematical formulation of the control problem

$$\min \int_{x_a}^{x_b} |p\left(y_1^n(T,x)\right) - \bar{p}| dx \text{ subject to}$$

$$\partial_t y^e + \partial_x f\left(y^e\right) = g\left(x, y^e\right),\ \Psi\left(y^1, \dots, y^n\right) = P\left(t\right), y \in \mathbb{R}^2$$

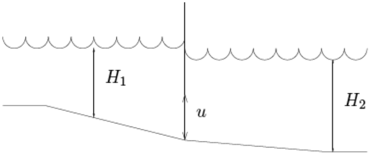
- Theorem on weak solutions: there is a continuous dependence on the coupling condition

$$\|\mathcal{E}\left(t, t_0, y_0, P\right) - \mathcal{E}\left(t, t_0, \tilde{y}_0, \tilde{P}\right)\| \leq L \cdot \left(\|y_0 - \tilde{y}_0\| + \int_{t_0}^{t_0+t} \|P(\tau) - \tilde{P}(\tau)\| d\tau\right)$$

- Used to state existence results for optimal control problems on finite time horizons

Interest of the industry: Optimal control and controllability or stabilization of instationary flow patterns

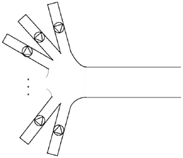
Applications: Existence and control results – Water I



- Cost functional $\mathcal{J} = \int_0^T d|\partial_x H_j|$
- Equation in each pipe
  $$\partial_t \begin{pmatrix} H_j \\ Q_j \end{pmatrix} + \partial_x \begin{pmatrix} Q_j \\ \frac{g}{2} H_j^2 + Q_j^2/H_j^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -\chi_{0,L} Q_j|Q_j|/H_j - gH_j \sin \alpha_j(x) \end{pmatrix}$$
- Coupling condition $\Psi = \begin{pmatrix} b_1 Q_1 - b_2 Q_2 \\ Q_1/(H_1 - H_2) - u(t) \end{pmatrix}$
- H_j is the water level, bQ the total water flow, similar to Coron, Bastin et. al., Automatica, 2003.

Problem is well–posed and existence of an optimal control with $TV(u)$ small is proven.

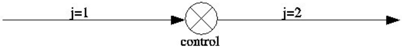
Applications: Existence and control results – Water II



- Cost functional $\mathcal{J} = \int_0^T \int_0^L (H_n - \bar{h})^+ dx\tau$
- Equation in each pipe
$$\partial_t \begin{pmatrix} H_j \\ Q_j \end{pmatrix} + \partial_x \begin{pmatrix} Q_j \\ \frac{g}{2} H_j^2 + Q_j^2 / H_j^2 \end{pmatrix} = \begin{pmatrix} 0 \\ -\chi_{0,L} Q_j |Q_j| / H_j - g H_j \sin \alpha_j(x) \end{pmatrix}$$
- Coupling condition $\Psi = \begin{pmatrix} b_n Q_n - \sum_{i=1}^n b_i Q_i \\ Q_2 - u_1(t) \\ \dots \\ b_{n-1} Q_{n-1} - u_{n-1} \end{pmatrix}$

Problem is well–posed and existence of an optimal controls $\vec{u}$ with small $TV-$norm is proven.

Application probem: Controllability for a system with compressors



Problem: Two connected pipes connected with a compressor at $x = 0$. The customer requires certain pressure and flow $y_B(t)$ for times $t \geq t^{**}$ and we need operator $u(\rho, q)$ to fulfiling the demand.
Assumptions: $\lambda_1(y_i) < 0 < \lambda_2(y_i)$, smooth solutions

$$\partial_t \begin{pmatrix} \rho_i \\ q_i \end{pmatrix} + A(\rho_i, q_i)\partial_x \begin{pmatrix} \rho_i \\ q_i \end{pmatrix} = G(t, x, \rho_i, q_i) \text{ on } \mathcal{D}_i$$

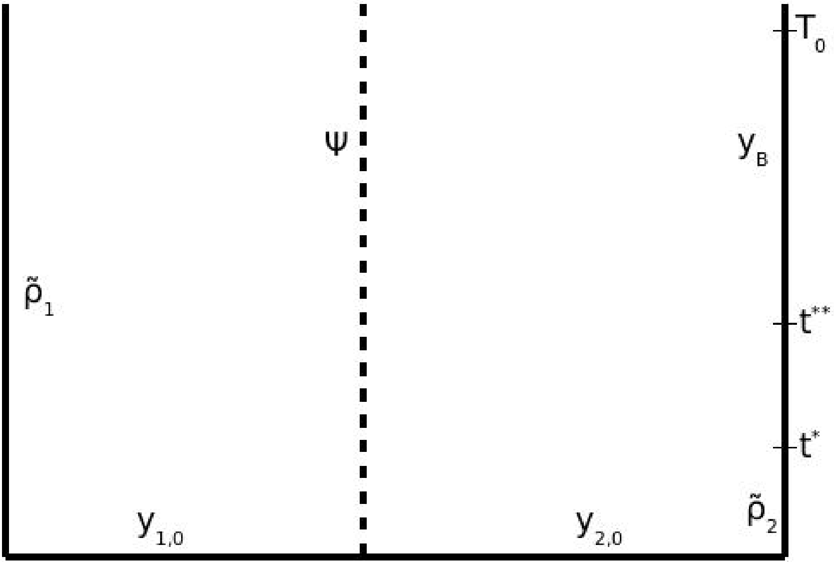
$$\begin{aligned} \mathcal{D}_1 &= \{(t,x) : t \geq 0, -L \leq x \leq 0\} \\ \mathcal{D}_2 &= \{(t,x) : t \geq 0,\ 0 \leq x \leq L\} \end{aligned}$$

Existing results mainly due Li Ta-Tsien et. al., also Coron et. al.

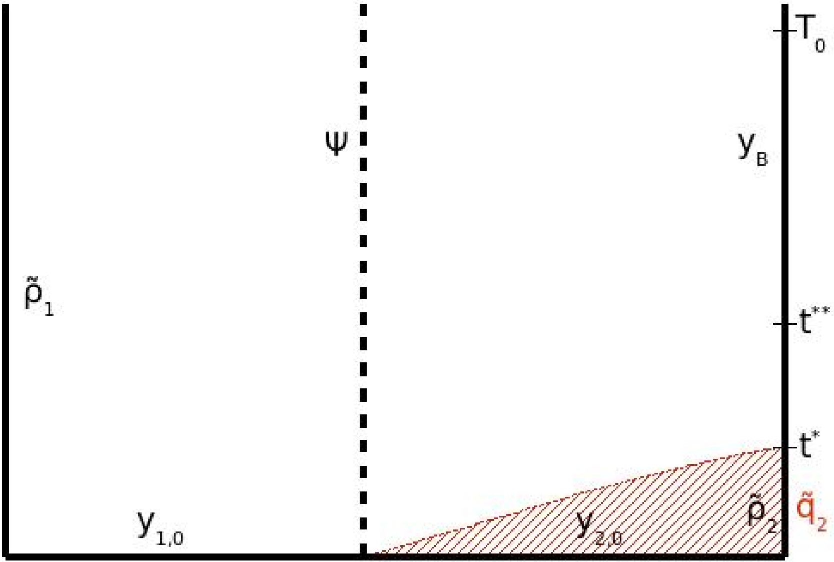
Wang (2006)
We assume that all given functions are $\mathbf{C^1}$ with respect to their arguments and that $G(t,x,0)=0$ as well as $\det(L(y))\neq 0$. Furthermore, the conditions of C^1 compatibility at the boundary points $(0,a)$ and $(0,b)$ are fulfilled.
Then, for **any** given time $T_0>0$ and suitably small C^1 norm of the initial and boundary conditions, the initial boundary value problem has a C^1 solution $y(t,x)$.

Used in an explicit construction of the desired control u

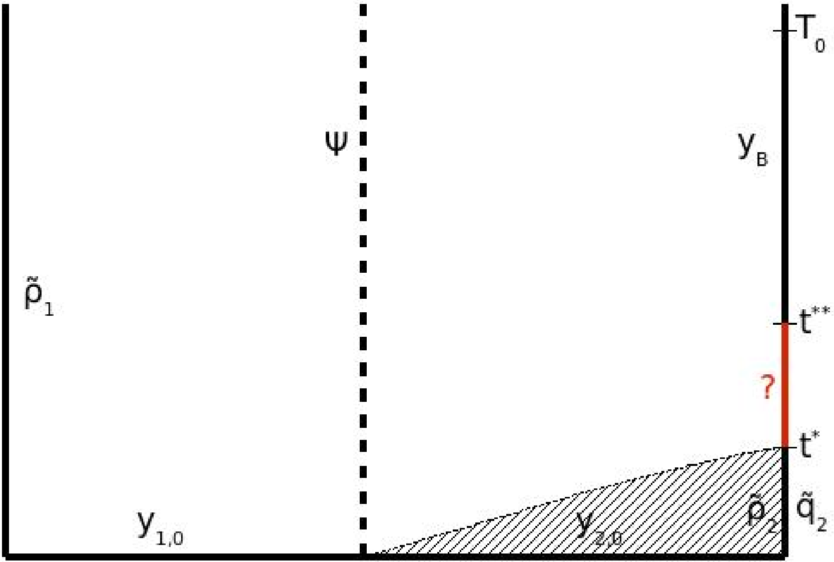
Construction of an exact control



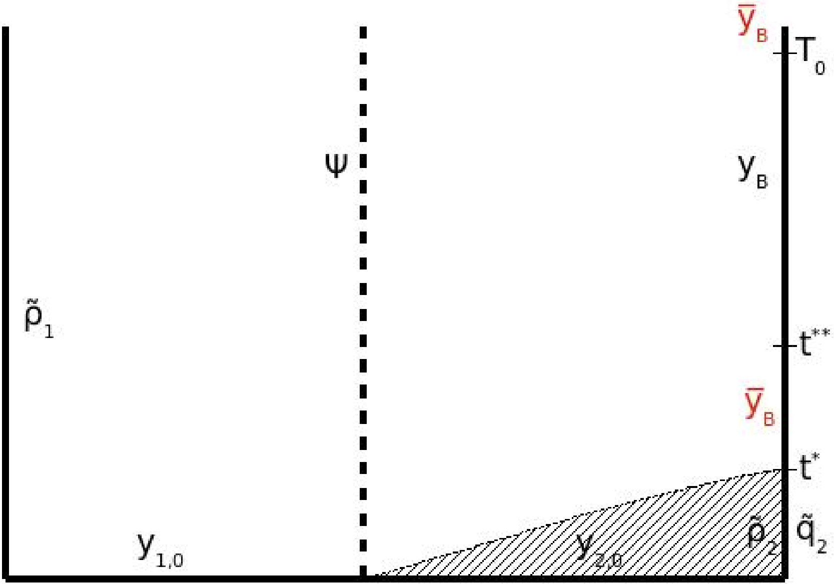
Construction of an exact control



Construction of an exact control



Construction of an exact control



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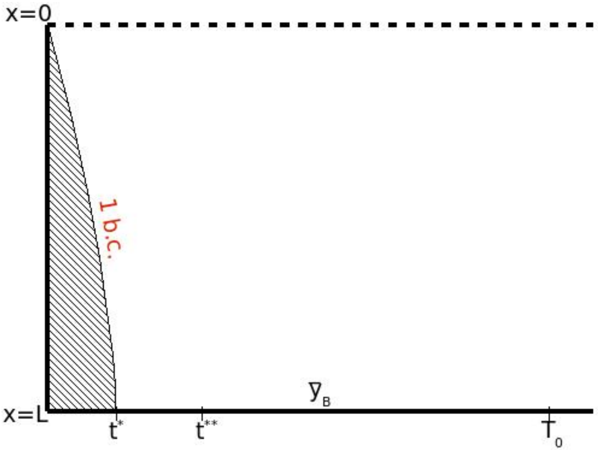
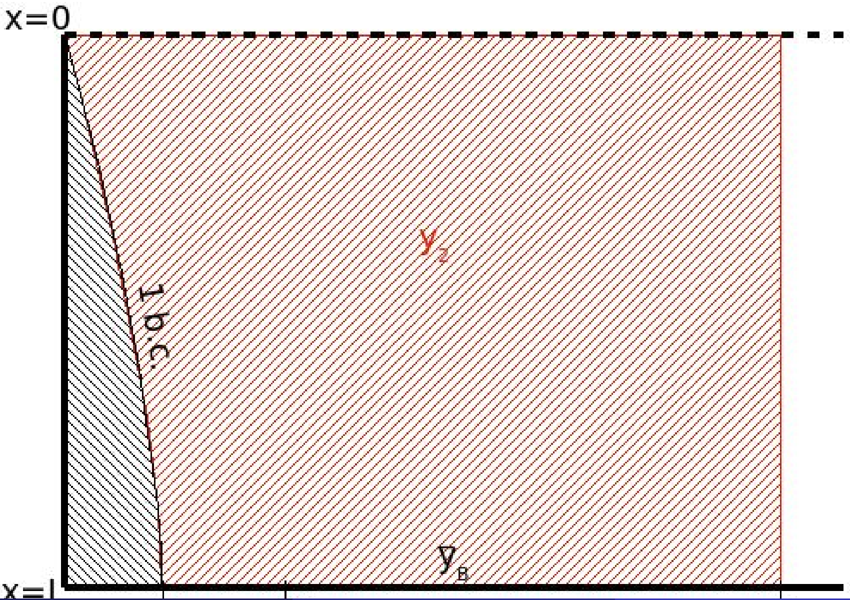


transposed problem:

$$\partial_x y_2 + (A(y_2))^{-1}\,\partial_t y_2 = (A(y_2))^{-1}\,G(t,x,y_2)$$

Construction of an exact control



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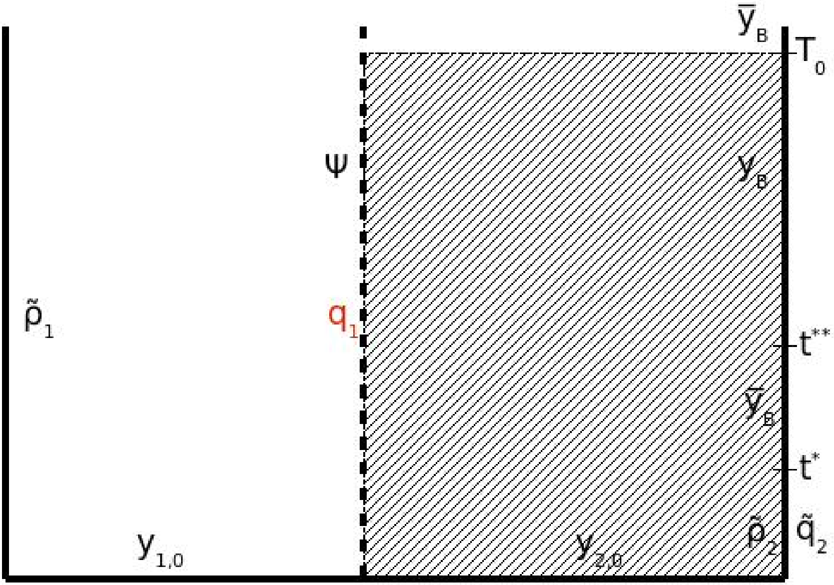

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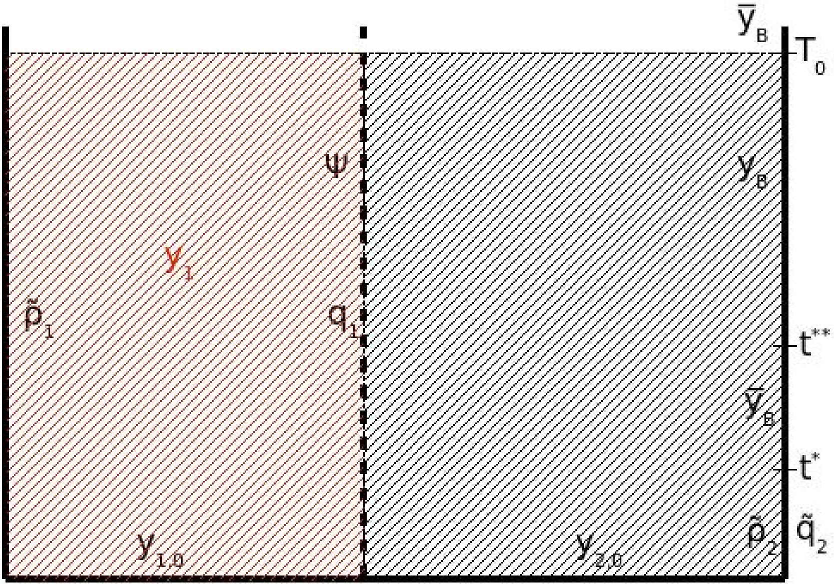

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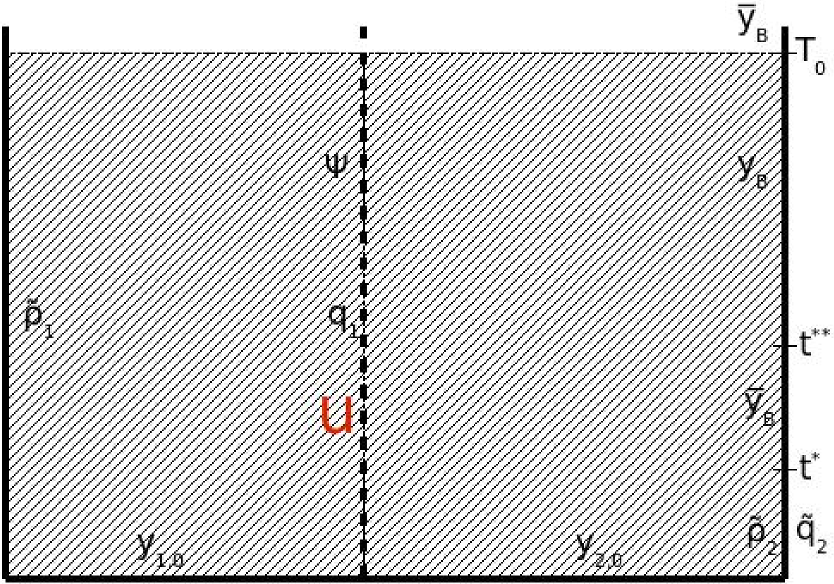

Summary

- Well–posedness for 2×2 systems on the arc and results in the scalar case for traffic, supply chain and communication networks by wave front tracking
- Restrictions on initial data for coupling conditions in the 2×2 case, trans–sonic states in the traffic flow model possible
- Existence results for optimal controls and common coupling conditions including shock waves
- Construction of feedback control laws based on classical solutions for controllability and stabilization