Material Flows in Production Networks

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Joint work with:
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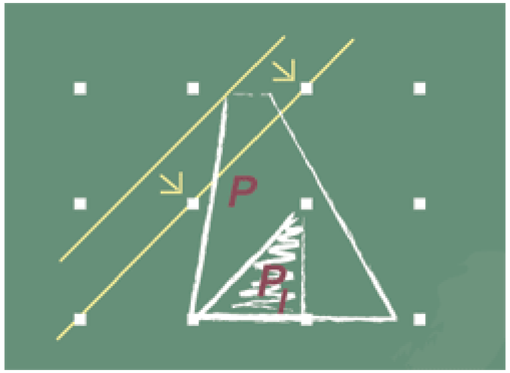
IPAM, Los Angeles, May 2009

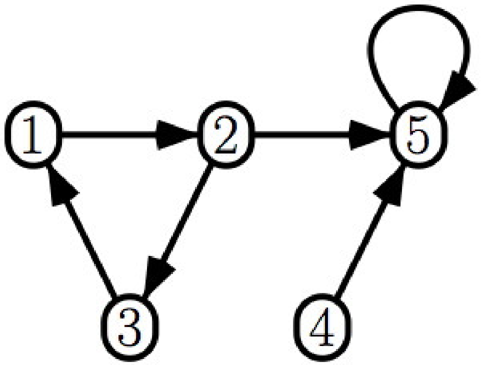
Continuous and Discrete models

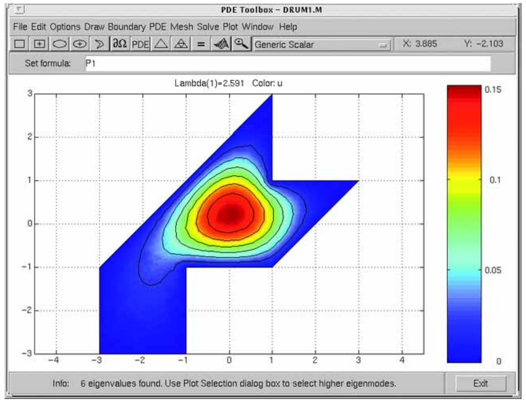
Motivation and Application:

- Water Networks
- Gas Networks
- Traffic Flow Networks
- Supply Chains
- …



MIP meets PDE





Traffic Flow Networks

See **Fügenschuh, Herty**, **Klar, Martin (2006):** *Combinatorial and Continuous Models for the Optimization of Traffic Flows on Networks*

- **Minimize driving time subject to**

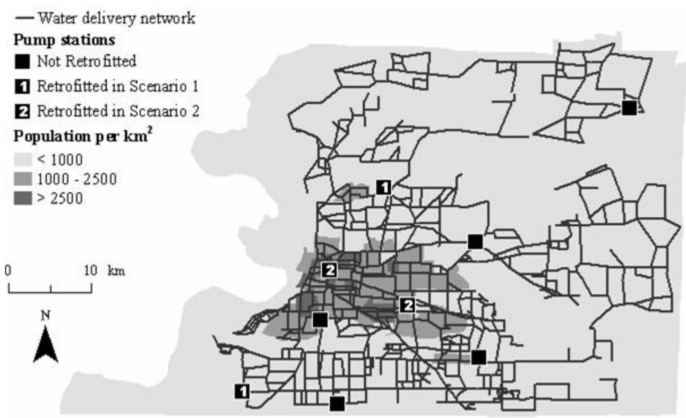
$$\partial_t \rho_j(x,t) + \partial_x f_j(\rho_j(x,t)) = 0$$

$$f_j(\rho) = \rho u^e(\rho)$$

$$\sum_{j=1}^{n} f(\rho_j(b_j-,t)) = \sum_{j=n+1}^{n+m} f(\rho_j(a_j+,t))$$

Protection of Drinking Water Systems

See **Fügenschuh, Göttlich, Herty (2007):** *Water Contamination Detection*



Objective

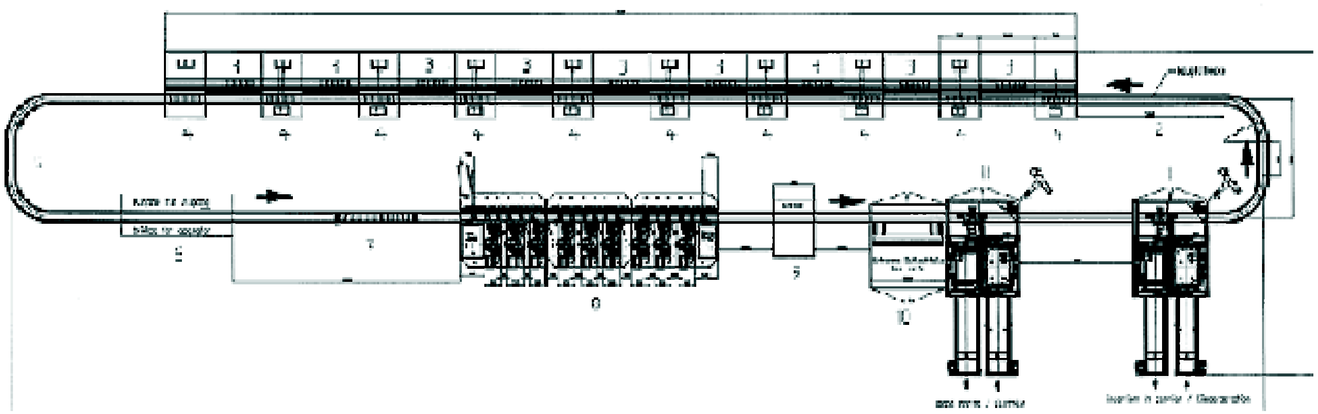
Try to identify sources such that the time evolved concentrations coincide with given measurements

$$\sum_{j\in\mathcal{A}_{meas}} \max_{t\in(0,T)} (c^j(\bar{x}^j,t) - \bar{c}^j(\bar{x}^j,t)) + \rho \sum_{v\in\mathcal{V}} \max_{t\in(0,T)} q^v(t).$$

Subject to

Water network model (advection, decay, coupling)

Production Systems

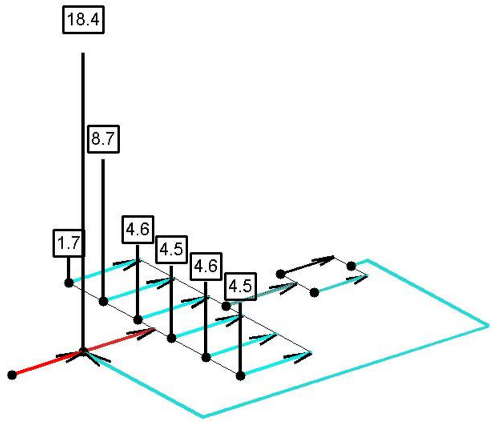


• **Maximize output subject to**

$$\partial_t \rho^e(x,t) + \partial_x f^e(\rho^e(x,t)) = 0$$

$$\partial_t q^e(t) = \alpha_v^e(t) \sum_{\bar{e} \in \delta_v^-} f^{\bar{e}}(\rho^{\bar{e}}(b^{\bar{e}},t)) - f^e(\rho^e(a^e,t))$$

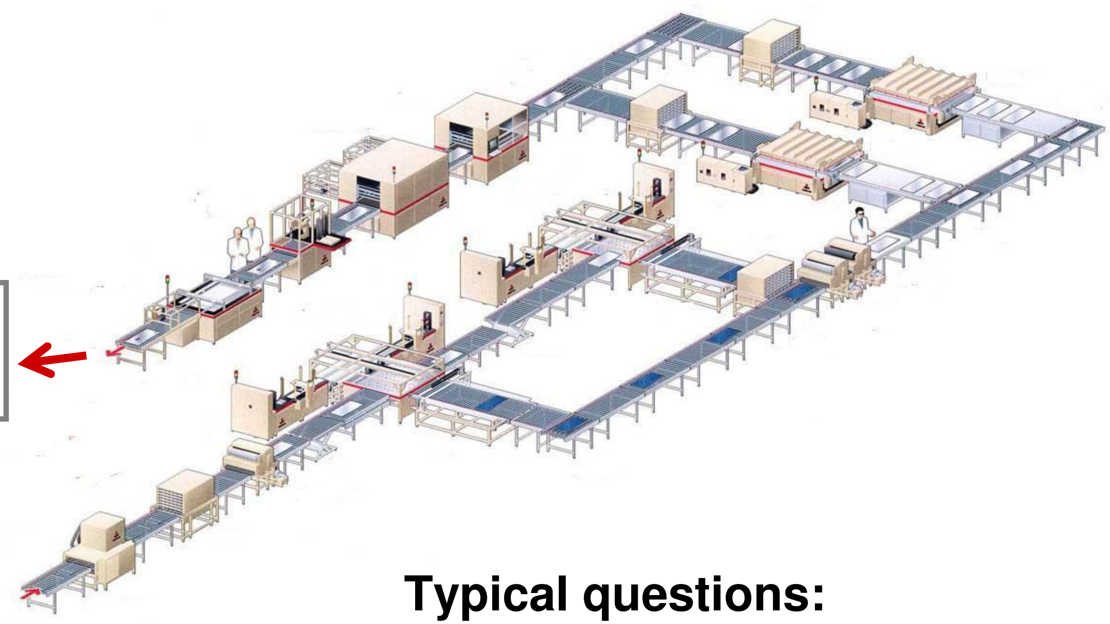
$$f^e(\rho^e(a^e,t)) = \min\{\mu^e, \frac{q^e(t)}{\epsilon}\}$$

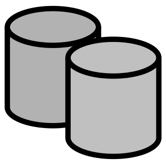


Motivation

Planning of a Production Network

Departing products





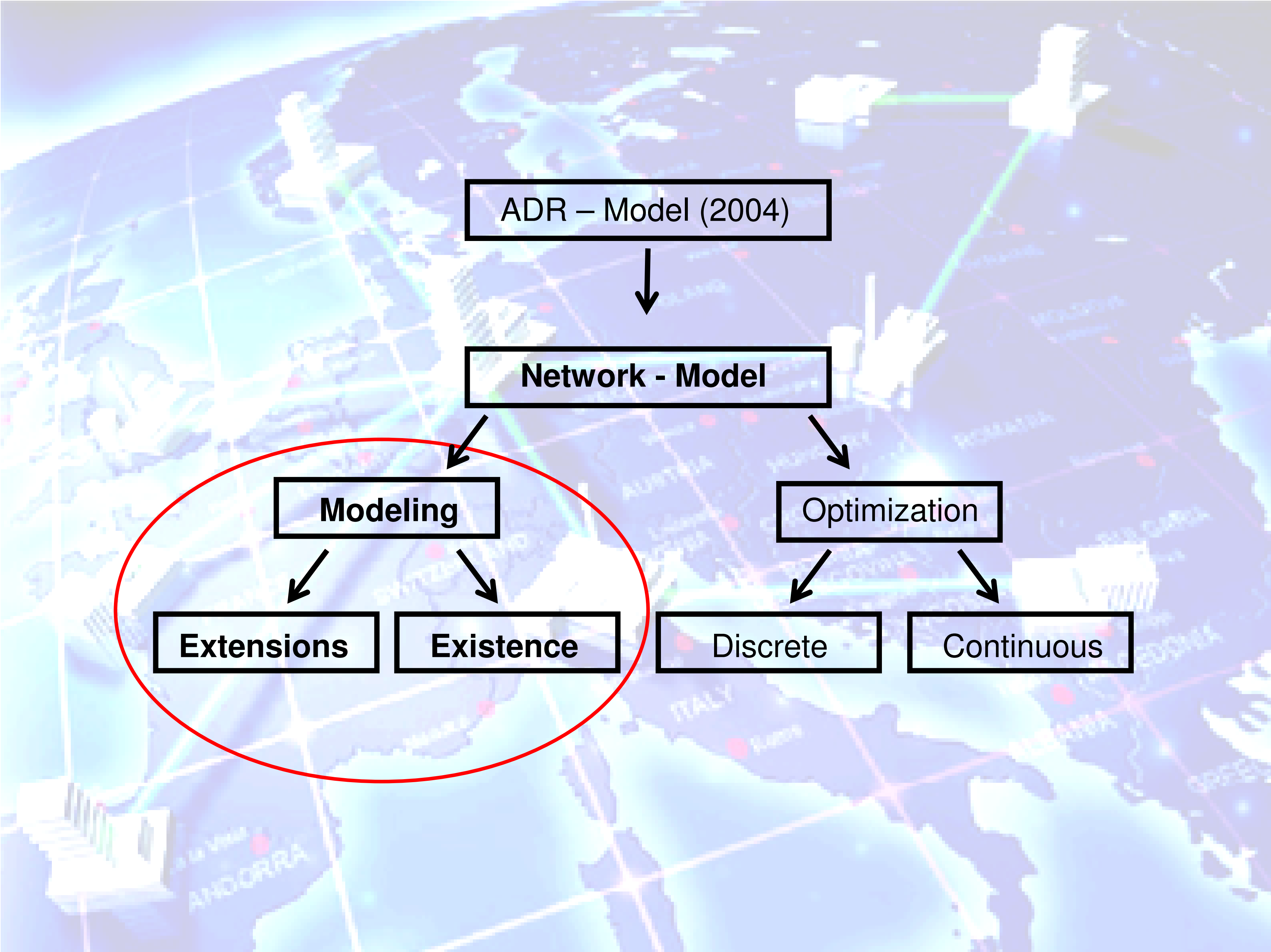
Arriving products

Typical questions:

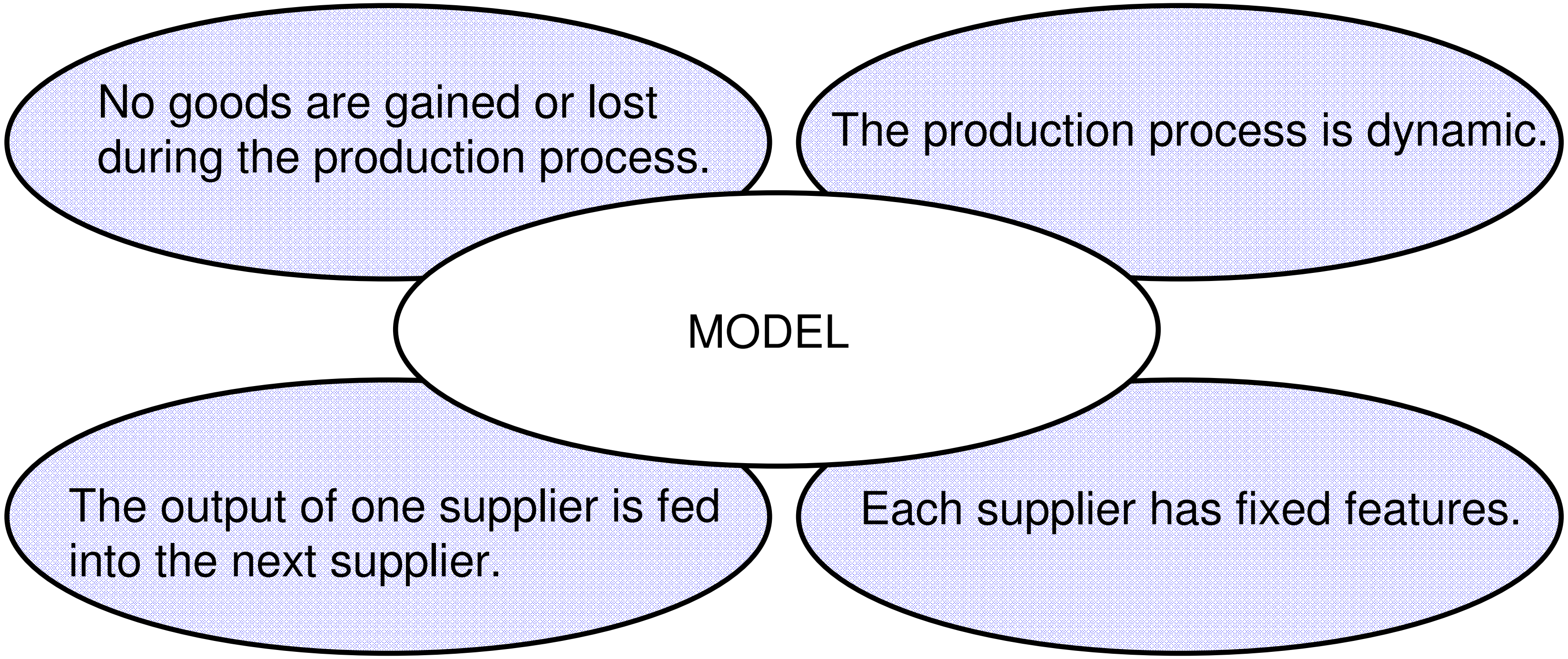
- Investment costs and profit
- Topology of the network
- Production mix and priority rules

Literature

System Dynamics	**Forrester (1961):** *Industrial Dynamics* **Baumol (1970):** *Economic Dynamics*
Discrete Event Simulation	**Banks et al. (1996):** *Discrete-Event System Simulation* **Fishman (2001):** *Discrete-Event Simulation*
Discrete Optimization	**Voß, Woodruff (2003):** *Introduction to Computational Optimization Models for Production Planning in a SC* **Wolsey, Pochet (2006):** *Production Planning by Mixed Integer Programming*
Queueing Theory	**Bolch et al. (1998):** *Queueing Networks and Markov Chains* **Chen, Yao (2001):** *Fundamentals of Queueing Networks*


ADR – Model (2004)
Network - Model
Modeling
Optimization
Extensions
Existence
Discrete
Continuous

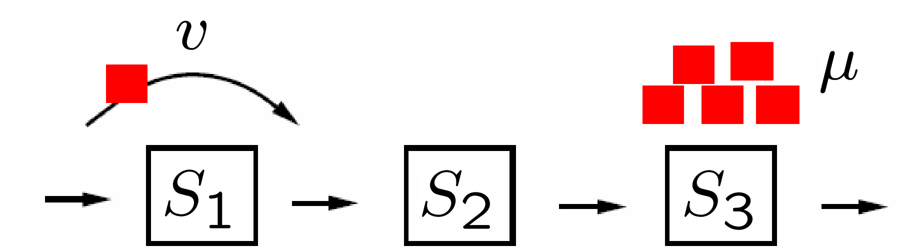
Assumptions



Continuous models are valid for large quantity production!

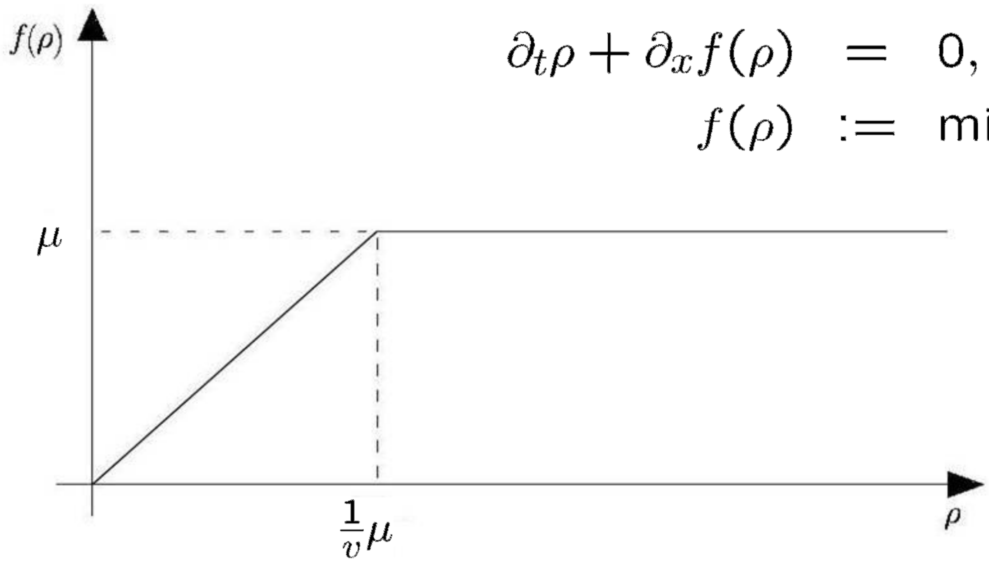
ADR - Model

See **Armbruster, Degond** and **Ringhofer (2004):** *A model for the dynamics of large queuing networks and supply chains*



$$\partial_t \rho + \partial_x f(\rho) = 0, \quad \forall x \in [a,b],\ t \geq 0$$

$$f(\rho) := \min\left\{v\rho, \mu\right\}$$



ρ : density of parts

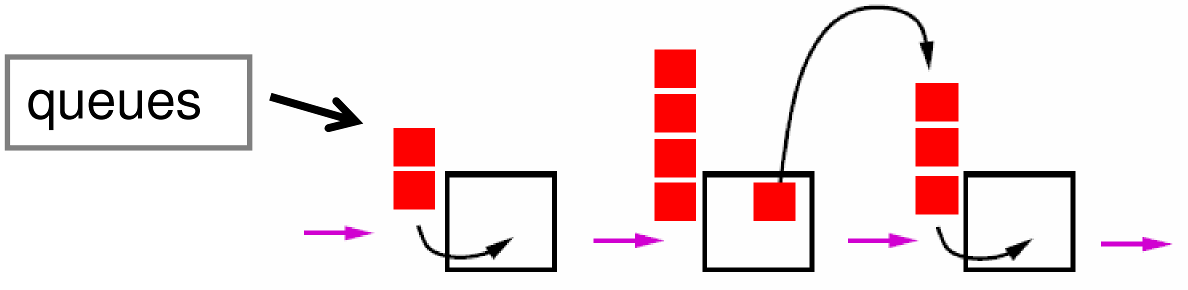
μ : maximum capacity

v : processing velocity

Network Model

- **Idea:**
  - Each processor is described by one arc
  - Use ADR Model for dynamics inside the processor
  - Add equations for queues in front of the processor

- **Advantage:**
  - Standard treatment of equations
  - Straightforward definitions for complex networks



Network Model

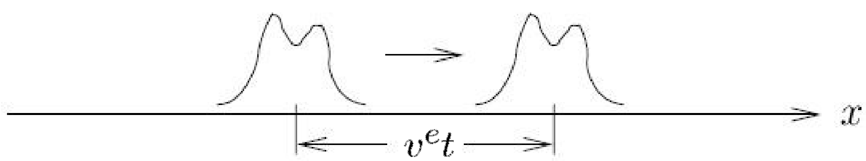
See **Göttlich, Herty**, **Klar (2005):** *Network models for supply chains*

Definition

A production network is a finite directed graph $(\mathcal{A}, \mathcal{V})$ where each arc $e \in \mathcal{A}$ corresponds to a processor on the intervall $[a^e, b^e]$. Each processor e has an associated queue q^e in front.

PDE: Linear transport equation

Processor: $\partial_t \rho^e(x,t) + v^e \partial_x \rho^e(x,t) = 0, \quad \forall x \in [a^e, b^e]$



Network Model

See **Göttlich, Herty**, **Klar (2006):** *Modeling and optimization of supply chains on complex networks*

Nonlinear ODE

Inflow

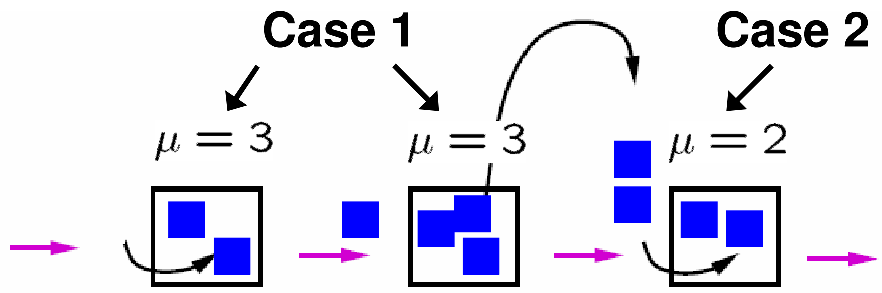
Outflow

Queue: $\partial_t q^e = h^e(\rho^e, A^{v,e}) - \psi^e(q^e)$

$$\psi^e(q^e) = \begin{cases} \min\{h^e(\rho^e, A^{v,e}), \mu^e\}; & q^e(t) = 0 \\ \mu^e; & q^e(t) > 0 \end{cases}$$

Case 1

Case 2

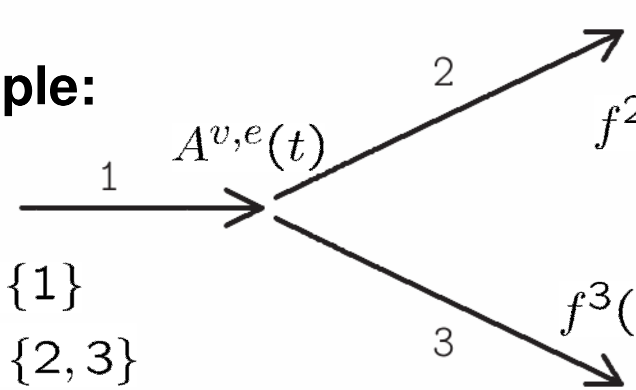


Network Coupling

Definition

We define time-dependent distribution rates $A^{v,e}(t)$ for each vertex with multiple outgoing arcs. The functions $A^{v,e}(t)$ are required to satisfy $\sum_{e \in \delta_v^+} A^{v,e}(t) = 1$ and $A^{v,e}(t) \in [0,1]$.

Example:



$\delta_v^- = \{1\}$

$\delta_v^+ = \{2,3\}$

Inflow into queue of arc 2

$$f^2(\rho^2(x_v,t)) = A^{v,2}(t) \cdot f^1(\rho^1(x_v,t))$$

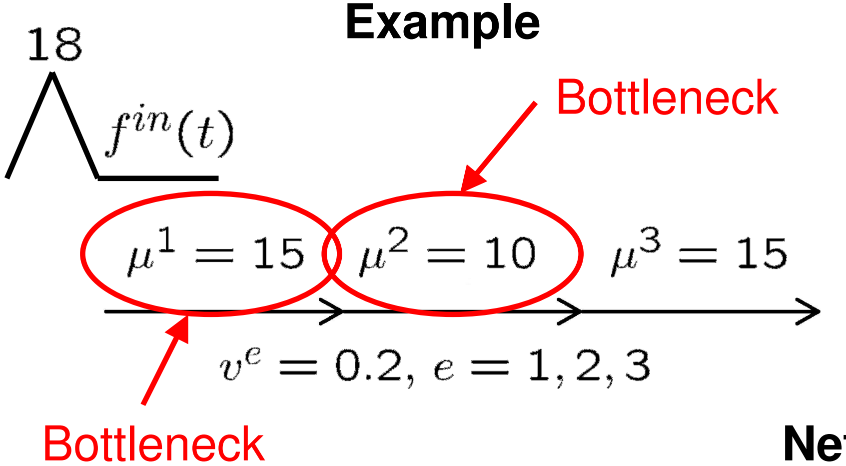
Inflow into queue of arc 3

$$f^3(\rho^3(x_v,t)) = (1 - A^{v,2}(t)) \cdot f^1(\rho^1(x_v,t))$$

$A^{v,e}(t)$ will be obtained as solutions of the optimization problem

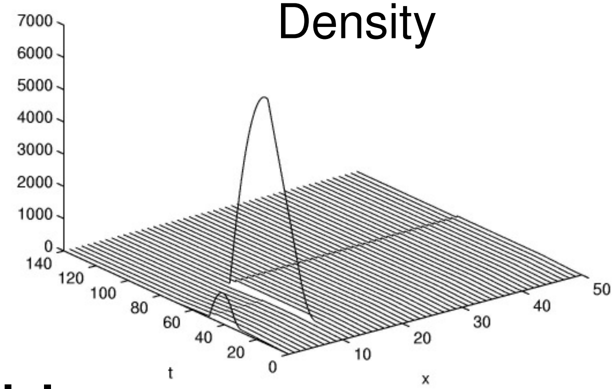
Numerical Results

Example



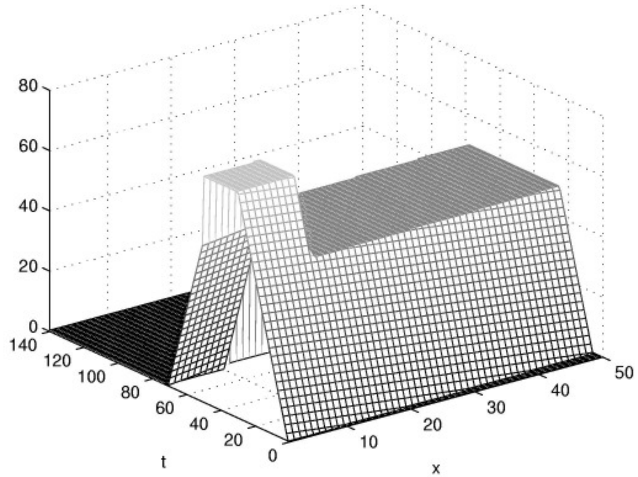
ADR-Model

Density

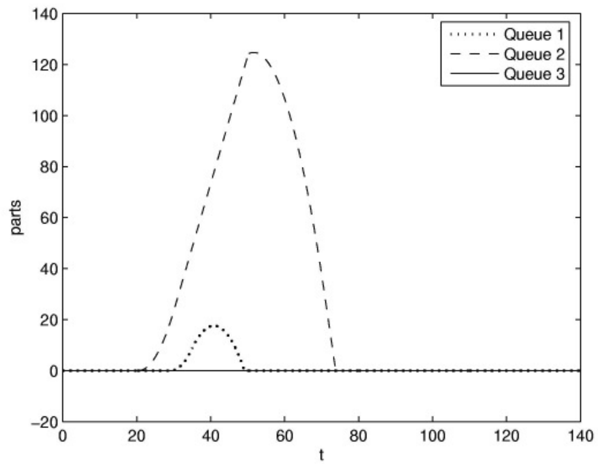


Network-Model

Density

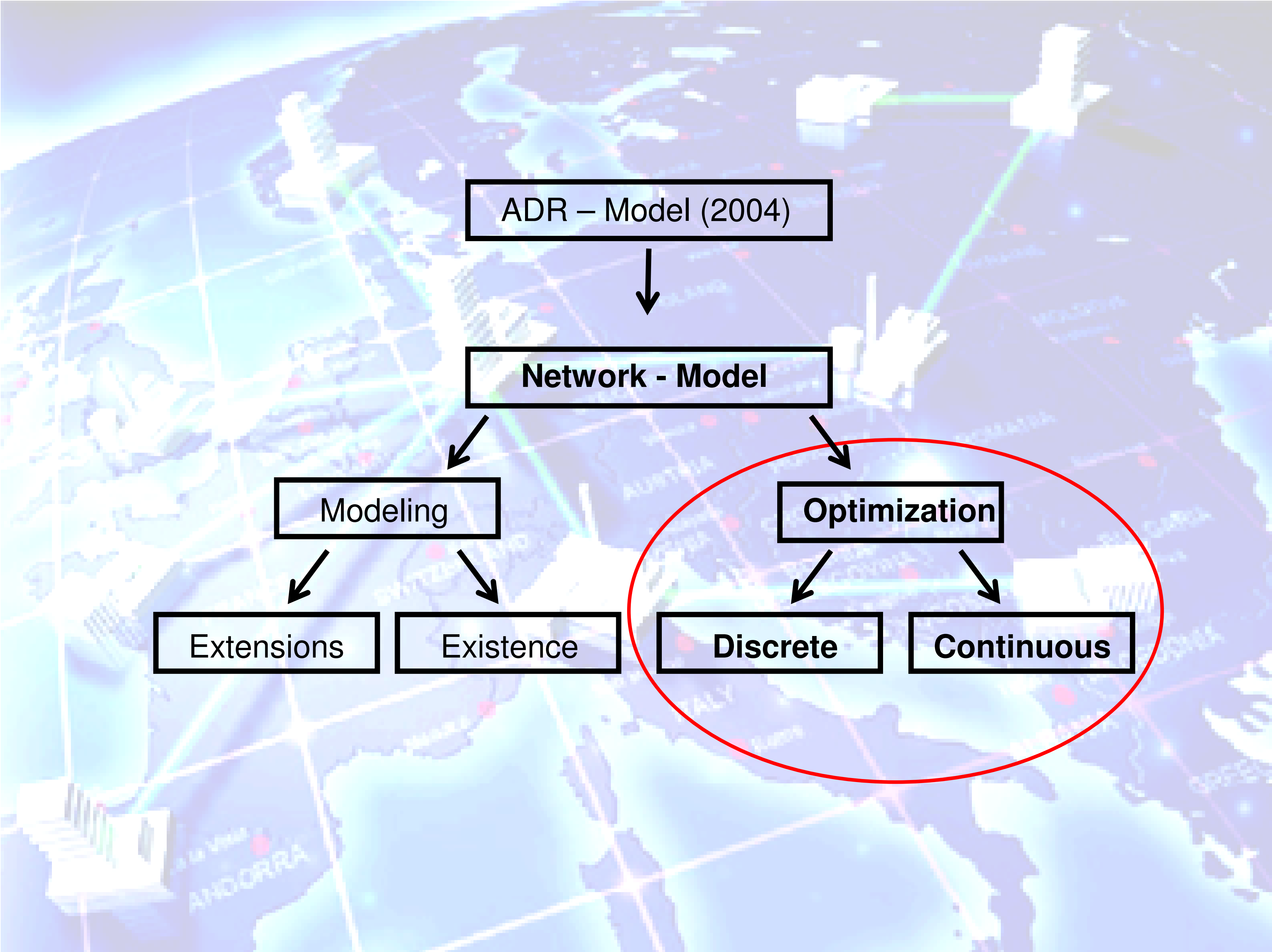


Queues



Continuous Production Models

Pros	Cons
Accurate description of supply chain behavior	Difficult to include discrete decisions
Fast computing times	PDE-constrained optimization problems
Opportunity to introduce non-linearities	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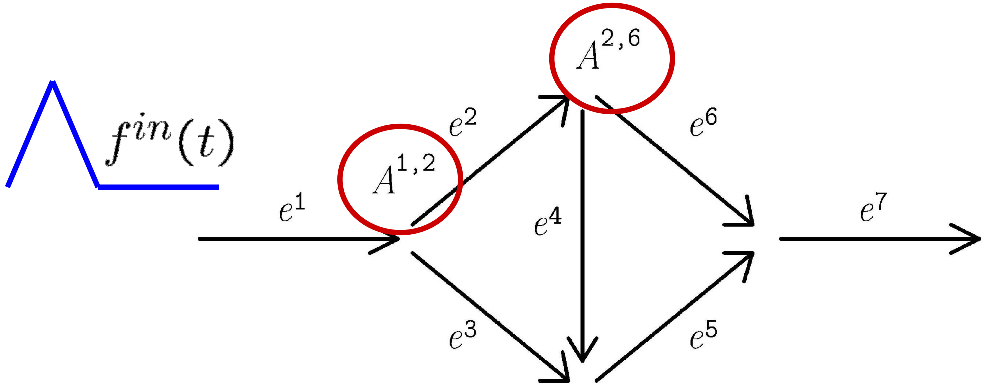
Optimization Problem

- Consider network with entry/exit suppliers



- Given are time-dependent inflow profiles
- Controls are the distribution rates at vertices

What is the optimal distribution of parts among the network?

Optimization Problem

Cost functional

Controls → Positive weights

$$\min_{A^{v,e}(t), v \in \mathcal{V}_d} \sum_{e \in \mathcal{A}} \int_0^T -b^e v^e \rho^e \, dt$$

Constraints

subject to $e \in \mathcal{A},\ v \in \mathcal{V},\ t \in (0,T),\ x \in [a^e, b^e]$

Processor → $\partial_t \rho^e + v^e \partial_x \rho^e = 0,$

Queue → $\partial_t q^e = h^e(\rho^e, A^{v,e}) - \psi^e(q^e)$

$$\psi^e(q^e) = \begin{cases} \min\{h^e(\rho^e, A^{v,e}), \mu^e\}; & q^e(t) = 0 \\ \mu^e; & q^e(t) > 0 \end{cases}$$

Initial conditions → $\rho^e(x,0) = 0,\ q^e(0) = 0$

Solution Techniques

Optimal control problem with PDE/ODE as constraints!

How to solve ?

↙ ↘

Mixed Integer Programming (MIP)

Adjoint Calculus

↓

MIP ⇔ LP

Mixed Integer Program (MIP)

See **Fügenschuh, Göttlich, Herty, Klar, Martin (2006)**: *A Discrete Optimization Approach to Large Scale Networks based on PDEs*

A mixed-integer program is the minimization/maximization of a **linear** function subject to **linear** constraints.

Problem: Modeling of the queue-outflow in a discete framework

$$\psi^e(q^e) = \begin{cases} \min\{h^e(\rho^e, A^{v,e}), \mu^e\}; & q^e(t) = 0 \\ \mu^e; & q^e(t) > 0 \end{cases}$$

Solution: Reduce complexity (as less as possible binary variables)

Relaxed queue-outflow[1]

$$\psi^e(q^e) = \min\left\{\frac{q^e(t)}{\epsilon}, \mu^e\right\}$$

[1]See **Armbruster et al. (2006)**: *Autonomous Control of Production Networks using a Pheromone Approach*

Derivation MIP

Problem: Suitable discretization of the min-nonlinearity

$$\psi(q^e(t)) = \min\left\{\frac{q^e(t)}{\epsilon}, \mu^e\right\}$$

Idea: Introduce binary variables (decision variables) ξ_t^e

$$\psi(q_t^e) = \min\left\{\frac{q_t^e}{\epsilon}, \mu^e\right\} \quad \Longleftrightarrow \quad \begin{aligned} \xi_t^e \in \{0,1\}, M >> 1 \\ \mu^e \xi_t^e \le \psi(q_t^e) \le \mu^e \\ \frac{q_t^e}{\epsilon} - M\xi_t^e \le \psi(q_t^e) \le \frac{q_t^e}{\epsilon} \end{aligned}$$

Outflow of the queue
= Inflow to a processor

Example: $\mu^e \le \frac{q_t^e}{\epsilon}$ implies $\xi_t^e = 1$ and $\psi(q_t^e) = \mu^e$

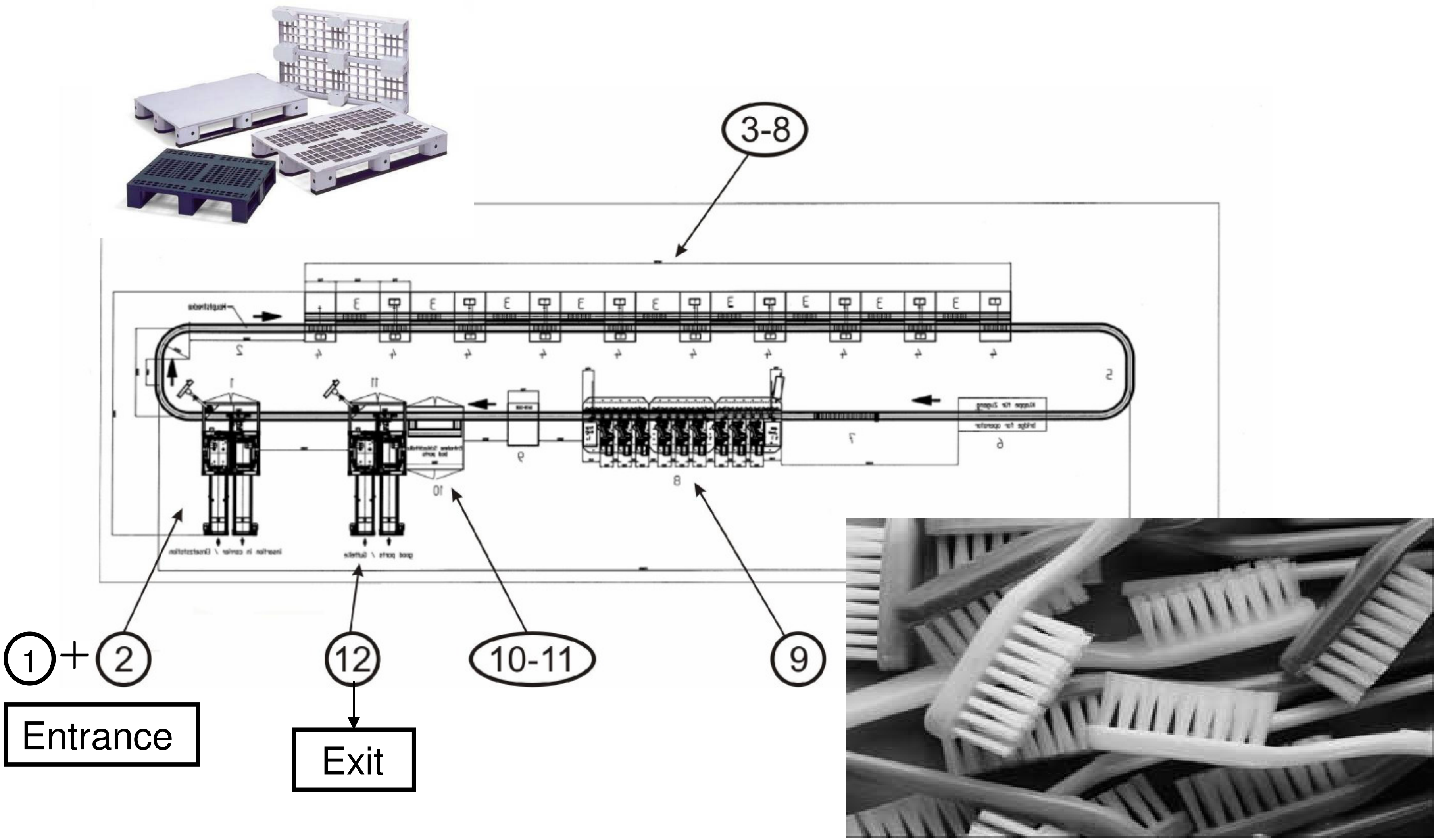
! The problem has $|\#Arcs| \cdot |\#Timesteps|$ binary variables!

Mixed Integer Program

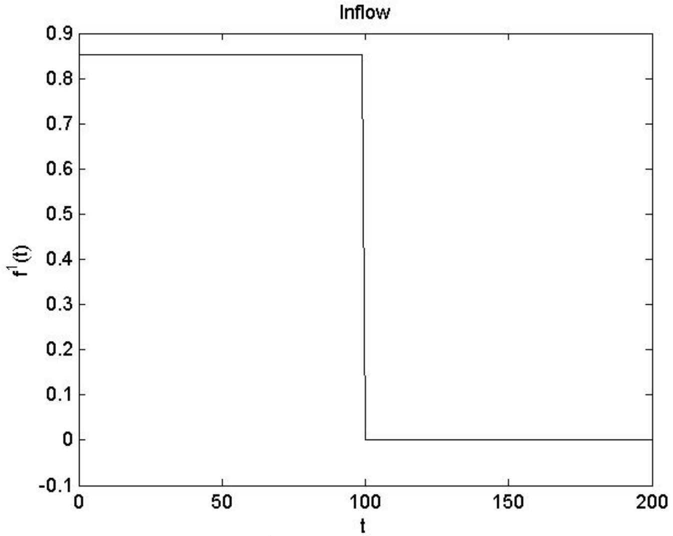
Two-point Upwind discretization (PDE) and explicit Euler discretization (ODE) leads to

maximize outflux	$\longrightarrow$	$\min_{A_t^{v,e}, v \in \mathcal{V}_d} \sum_e \sum_t -\Delta t \, b_t \, v^e \rho_t^{e,b}$ subject to
processor	$\longrightarrow$	$\rho_{t+1}^{e,b} = \rho_t^{e,b} + \frac{\Delta t}{\Delta x} \left(\psi(q_t^e) - v^e \rho_t^{e,b} \right)$
queue	$\longrightarrow$	$q_{t+1}^e = q_t^e + \Delta t \, (h_t^e - \psi(q_t^e))$
queue outflow	$\longrightarrow$	$\mu^e \xi_t^e \leq \psi(q_t^e) \leq \mu^e$ $\frac{q_t^e}{\epsilon} - M \xi_t^e \leq \psi(q_t^e) \leq \frac{q_t^e}{\epsilon}, \qquad \xi_t^e \{0, 1\}$
initial conditions	$\longrightarrow$	$q_0^e = 0, \; \rho_0^{e,b} = 0, \; \psi(q_0^e) = 0$

Toothbrush Manufacturing



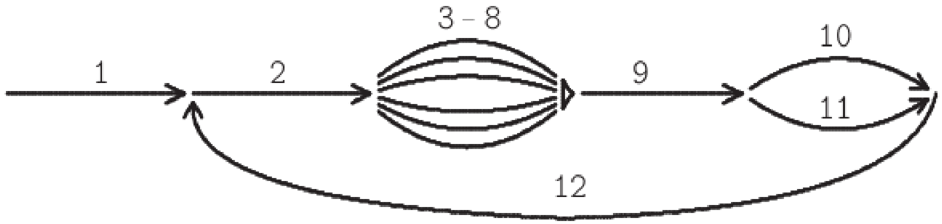
Example



Constant inflow until $t \leq 100$

Solved by ILOG CPLEX 10.0

- Solution time = 11.16 sec
- # Variables = 9600

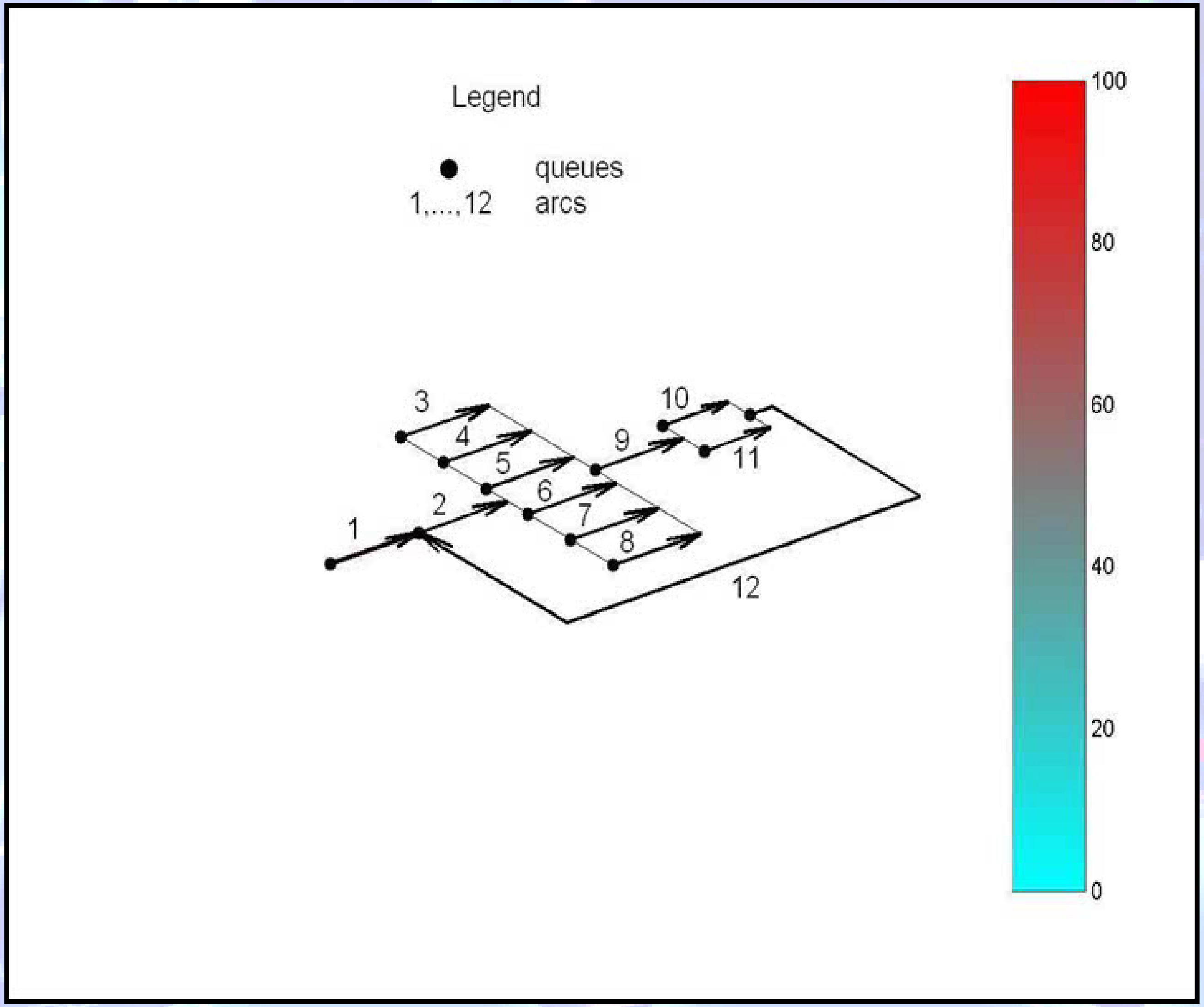


e	μ^e	v^e
1	100	0.01
2	0.71	0.35
3 - 8	0.07	0.01
9	0.71	0.05
10 - 11	0.24	0.12
12	0.71	0.35

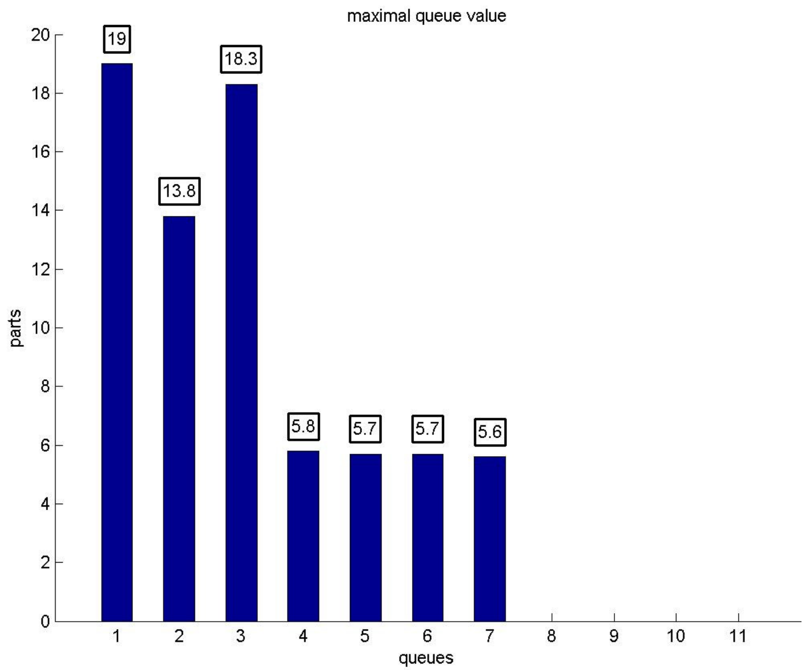
Maximization of outflow, i.e. optimizing the amount of parts passing processor 12

$$\min -\sum_t \frac{1}{t+1} g_t^{12}$$

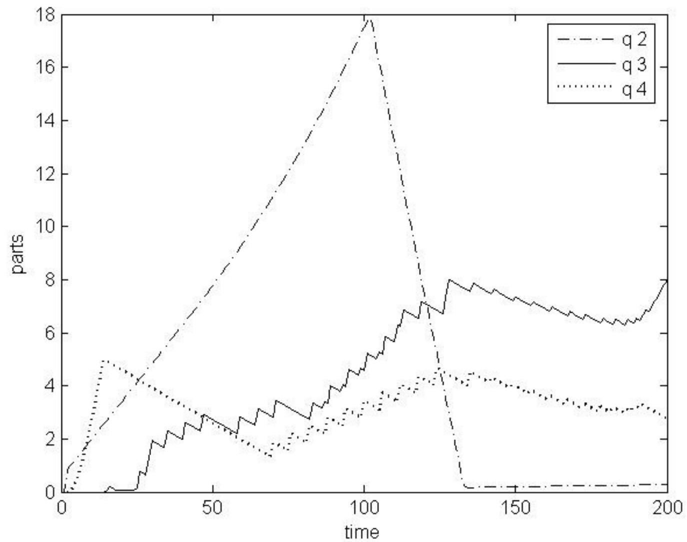
Optimization



Results



Maximal load of queues



Evolution of queues

Advantage MIP

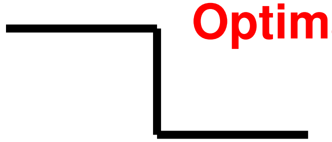
Constraints can be easily added to the MIP model:

- Bounded queues:

$$\mathsf{q}_t^e \leq \mathsf{const} \quad \forall e, t$$

- Optimal inflow profile:

$$\mathsf{max} \sum_{e=1,t} \mathsf{f}_t^e$$



Optimal ??

In other words: Find a maximum possible inflow to the network such that the queue-limits are not exceeded.

Advantage MIP

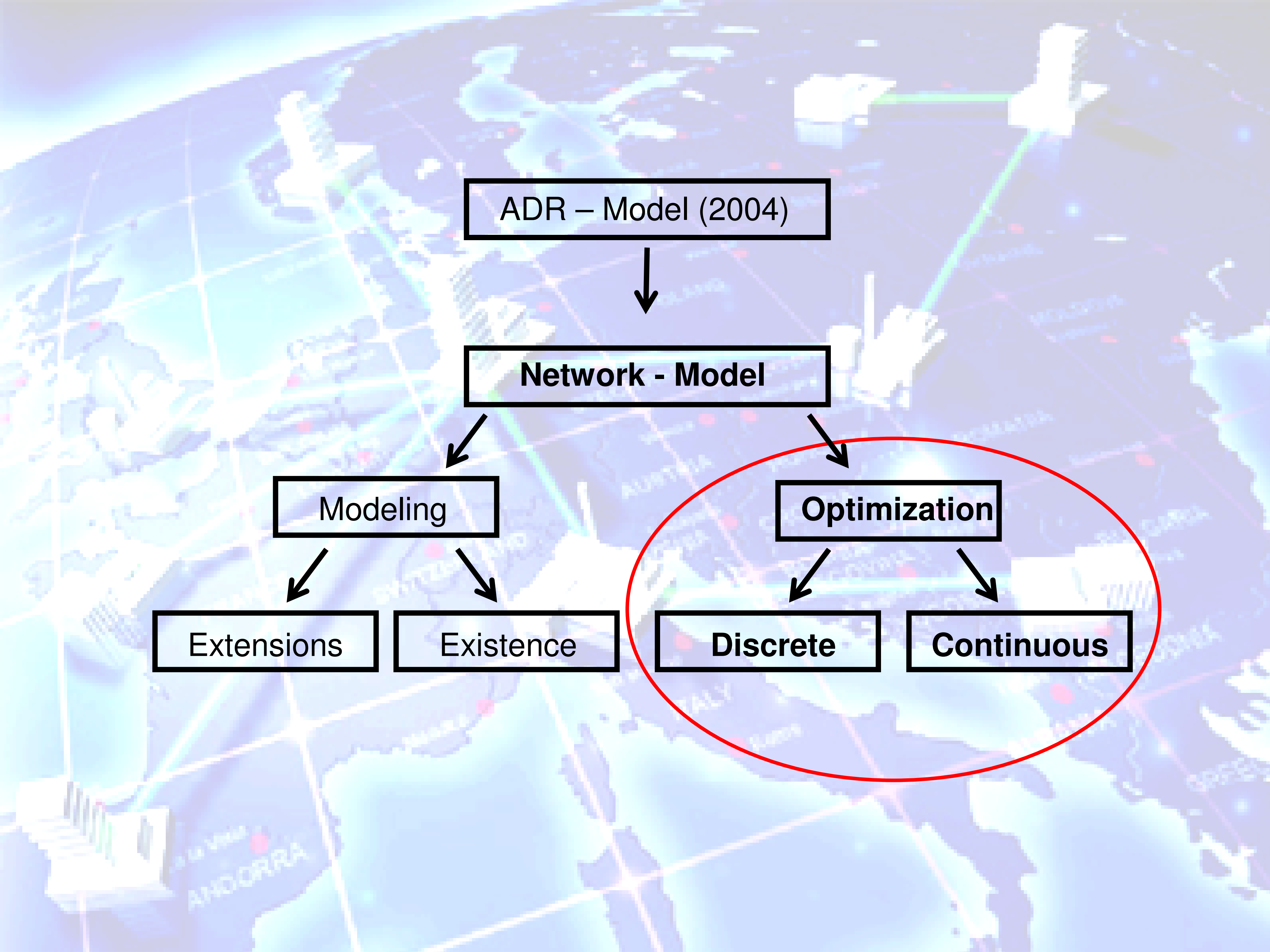
- Maintenance shut-down:

$$\phi_t^{\tilde{e}} \in \{0,1\},\ \forall t,\ \forall l = 0,\dots,N-1$$

$$\mathsf{h}_{t+l}^{\tilde{e}} \leq \max\{\mu^e : e \in E\}|E| \cdot (1-\phi_t^{\tilde{e}})$$

$$\sum_{t=1}^{N_T} \phi_t^{\tilde{e}} = N$$

- Processor $\tilde{e}$ has to be switched off for N consecutive time intervals during the total run time N_T .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Adjoint Calculus

See **Göttlich, Herty, Kirchner, Klar (2006)**: *Optimal Control for Continuous Supply Network Models*

Adjoint calculus is used to solve PDE and ODE **constrained** optimization problems. Following steps have to be performed:

1. Define the **Lagrange** – functional:

$$
\begin{aligned}
L(\rho^e, A^v, q^e, \Lambda^e, P^e) = & \sum_{e \in \mathcal{A}} \int_0^T \int_{a^e}^{b^e} \boxed{-b^e v^e \rho^e} \, dx \, dt \quad \text{Cost functional} \\
& - \sum_{e \in \mathcal{A}} \int_0^T \int_{a^e}^{b^e} \Lambda^e \boxed{\left(\partial_t \rho^e + v^e \partial_x \rho^e \right)} dx \, dt \quad \text{PDE processor} \\
& - \sum_{e \in \mathcal{A}} \int_0^T P^e \boxed{\left(\partial_t q^e - h^e(\rho^e, A^v) + \psi^e(q^e) \right)} dt \quad \text{ODE queue}
\end{aligned}
$$

with **Lagrange multipliers** Λ^e and P^e.

Adjoint Calculus

2. Derive the first order optimality system (**KKT-system**):

Forward (state) equations:

$$\partial_t \rho^e + v^e \partial_x \rho^e = 0, \; \rho^e(x,0) = 0, \; v^e \rho^e(a,t) = \psi^e(q^e),$$
$$\partial_t q^e = h^e(\rho^e, A^v) - \psi^e(q^e), \; q^e(0) = 0,$$

Backward (adjoint) equations:

$$-\partial_t \Lambda^e - v^e \partial_x \Lambda^e = v^e, \; \Lambda^e(x,T) = 0,$$
$$v^e \Lambda^e(b,t) = \sum_{\bar{e} \in \delta_v^+ \text{ s.t. } e \in \delta_v^-} P^{\bar{e}}(t) \frac{\partial}{\partial \rho^{\bar{e}}} h^{\bar{e}}(\rho^e, A^v),$$
$$-\partial_t P^e = 1 - (P^e - \Lambda^e(a,t)) (\psi^e)'(q^e), \; P^e(T) = 0,$$

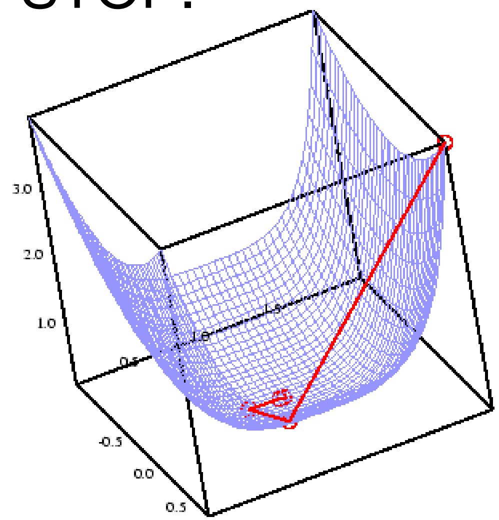
Gradient equation:

$$\sum_{e \in \delta_v^+} P^e \frac{\partial}{\partial A^{v,\bar{e}}} h^e(\rho^e, A^v) = 0.$$

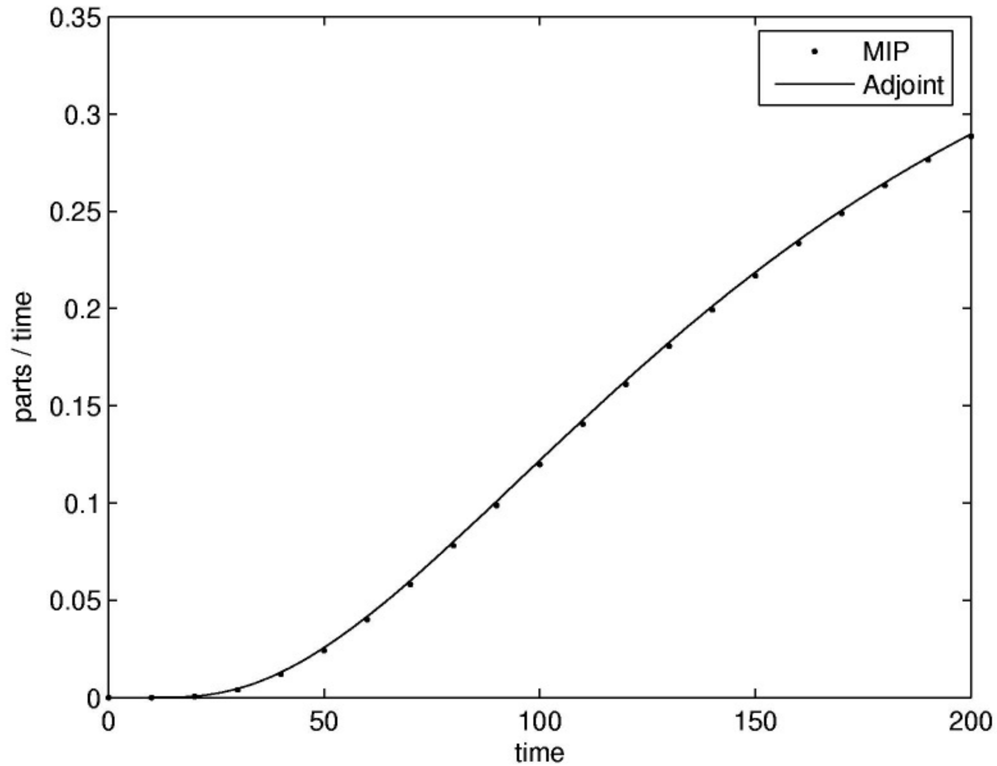
Optimization Algorithm

Projected steepest descent method:

1. Choose initial control vector A_0
2. Compute for A_0 the solution of state and adjoint equations
3. Compute the gradient. If it is zero, then STOP.
4. Update the control vector.
5. Go to 2.



Optimal outflow profile



Optimal functional value: $J^*(\vec{A}^v) = -0.19$

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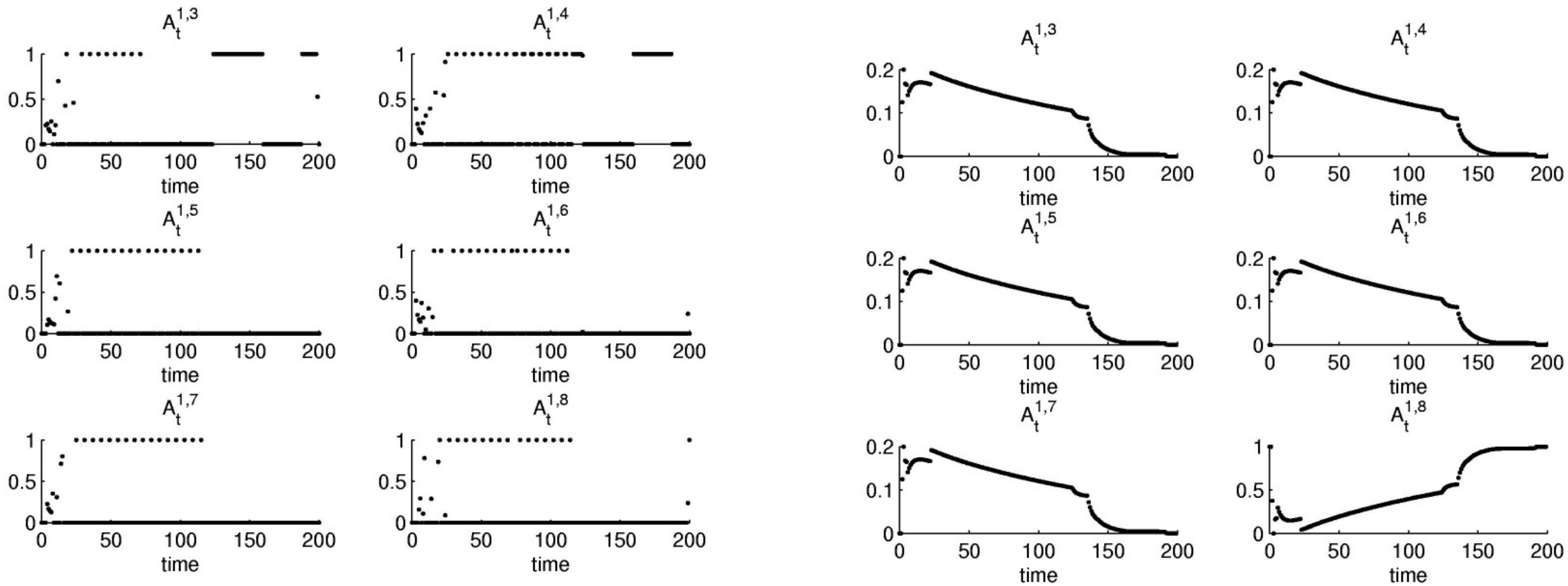
Optimal control

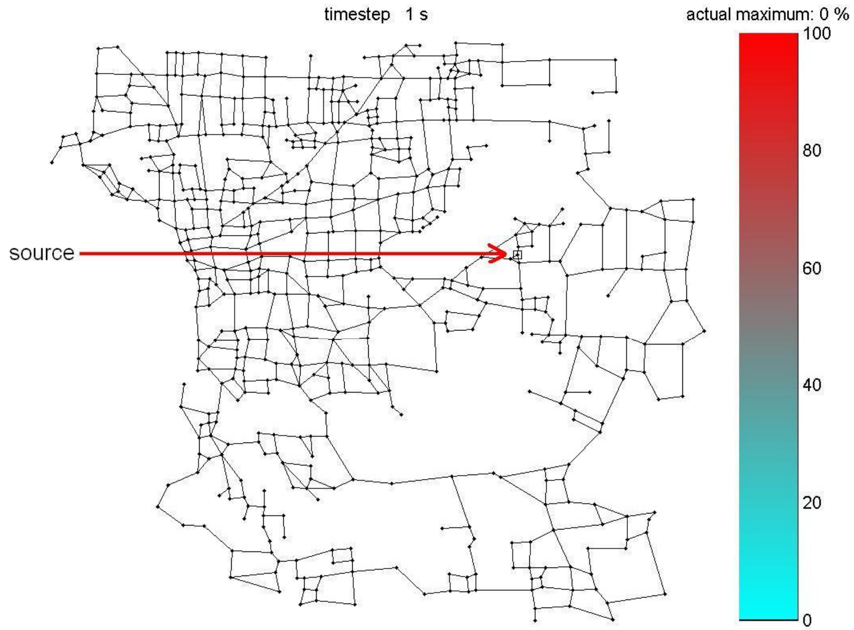


Distribution rates A_t^e, $e = 3, \dots, 8$ computed by the MIP (left) and the adjoint approach (right)

$\longrightarrow$ **No uniqueness!**

Motivation

Optimizing large-scale networks with hundreds of arcs and vertices



Solving the adjoint system or the MIP is computationally expensive!

Solution Techniques

Optimal control problem with PDE/ODE as constraints!

How to solve ?

Mixed Integer Programming (MIP)

Adjoint Calculus

↓

MIP ⟷ LP

Linear Program (LP)

Idea: Use adjoint equations to prove the reformulation of the MIP as a LP

$$\psi(q_t^e) = \min\left\{\frac{q_t^e}{\epsilon}, \mu^e\right\}$$

↑ Outflow of the queue

MIP ⟷

$$\xi_t^e \in \{0,1\}, M >> 1$$
$$\mu^e \xi_t^e \le \psi(q_t^e) \le \mu^e$$
$$\frac{q_t^e}{\epsilon} - M\xi_t^e \le \psi(q_t^e) \le \frac{q_t^e}{\epsilon}$$

LP ⟷

$$u_t^{e\pm} \ge 0$$
$$\psi(q_t^e) = \frac{1}{2}(\mu^e + \frac{q_t^e}{\epsilon}) - \frac{1}{2}(u_t^{e+} + u_t^{e-})$$
$$\mu^e - \frac{q_t^e}{\epsilon} = u_t^{e+} - u_t^{e-}$$
$$u_t^{e+} \cdot u_t^{e-} = 0$$

Remove the complementarity condition! →

Remark: The remaining equations remain unchanged!

From MIP to LP

Theorem

Assume the inflow at a fixed vertex is non-zero and $b^e_{N_T} < 0$.
Let either the costs b^e_t be monotone increasing in time or $\epsilon > \Delta t$.
Then, every optimal solution to the LP automatically satifies the complementarity condition $u^{e-}_t \cdot u^{e+}_t = 0$.

In other words:

We can solve the LP model and obtain the same solution as for the complementary problem!

Idea of the Proof: Part I

Constraints can be reformulated as follows:

$$\min_{A^{v,e}} \sum_e \sum_t -\Delta t b_t^e v^e \rho_t^{e,b} = \sum_e \sum_t -\Delta t b_t^e x_t^e + \dots$$

$$q_{t+1}^e = \dots,\ \rho_{t+1}^{e,b} = \dots$$

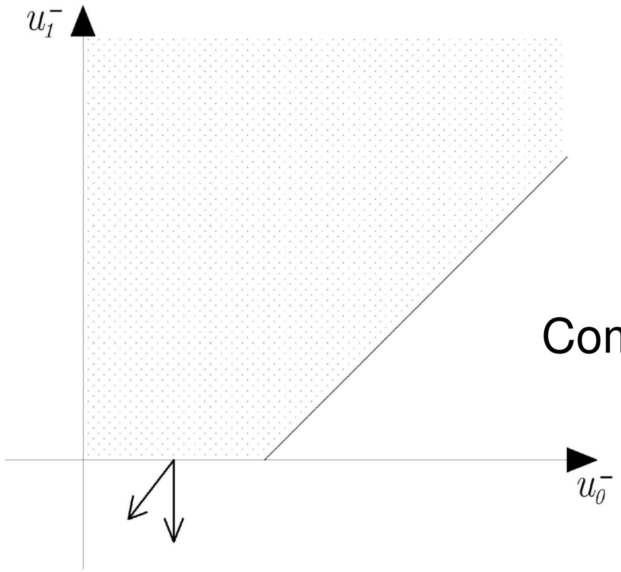
$$x_t^e + u_t^{e-} = \frac{q_t^e}{\epsilon}$$

$$x_t^e + u_t^{e+} = \mu^e$$

$$u_t^{\pm} \geq 0$$

- Complementary condition is satisfied whenever the inflow to the arc is maximized
- Monotone increasing costs b_t^e imply maximizing x_t^e
- Maximizing x_t^e is minimizing either u_t^{e-} or u_t^{e+}
  $\longrightarrow$ complementary condition is satisfied

Idea of the Proof: Part II



Complementary solution is at $u_0- = u_1^- = 0$

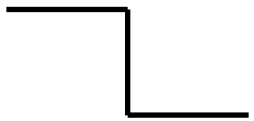
- Consider a single supplier with constant costs $b_t^e = -1$ and time horizon *T=2*
- Complementary solution is still an optimal solution
- But we obtain another solution if we store all incoming parts in the queue for one time-step and extract later $\longrightarrow$ complementary condition is not satisfied
- For $\epsilon = \Delta t$ the cost vector is orthogonal to the boundary

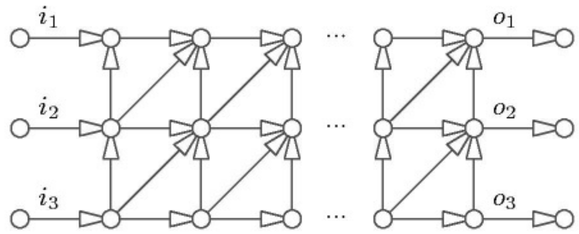
Numerical Results I

Example

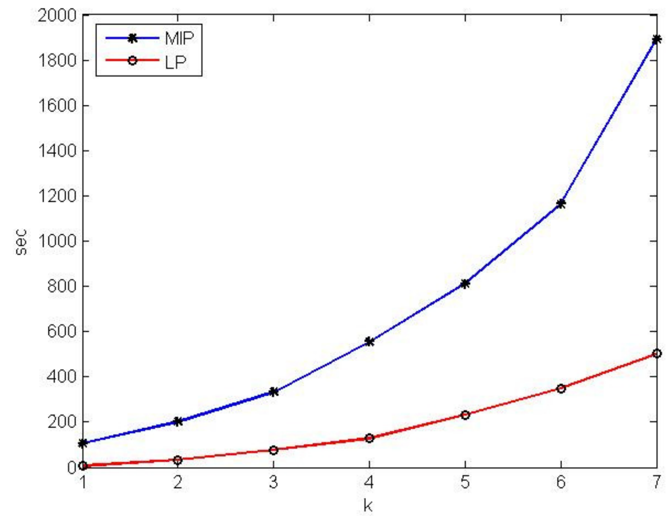
Solved by ILOG CPLEX 10.0

$f^{in}_{\{i_1,i_2,i_3\}}(t)$





$k \times 3$ interior network points



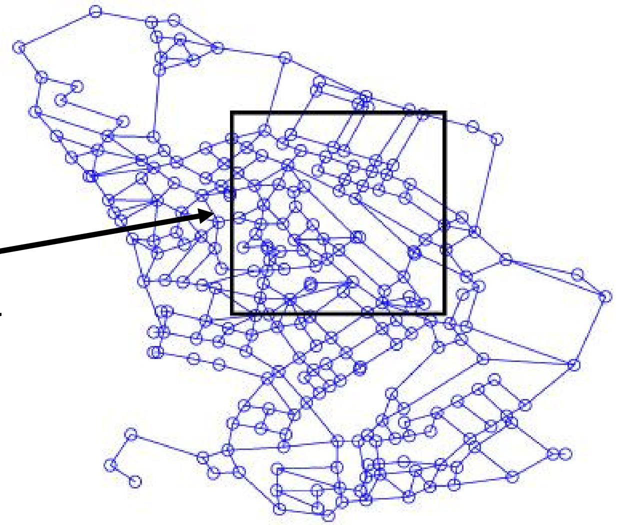
k	Presolve MIP	Solution MIP	Presolve LP	Solution LP
1	78.87	105.33	0.11	6.26
2	159.09	201.64	0.32	33.54
3	236.59	332.78	0.54	76.91
4	352.08	555.11	0.75	127.24
5	477.50	810.44	0.99	232.68
6	590.09	1163.25	1.22	348.53
7	983.55	1891.90	1.42	502.72
8	907.01	infeasible	1.73	768.47

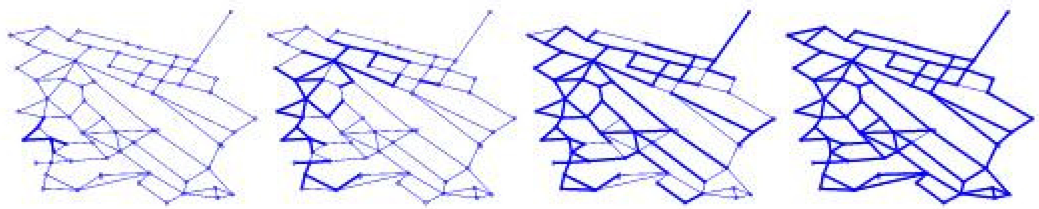
Numerical Results II

A production network consisting of 418 arcs and 233 vertices.

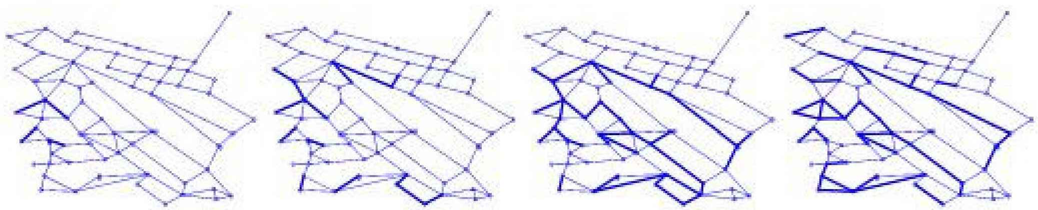
~15 Mio. variables!

Clipping area





Simulation results for t=10,23,36,50



Optimization results for t=10,23,36,50

MIP: no result after one day!

LP: ca. 5 hours

Thank you!