Traveling fronts on networks under reaction–diffusion equations

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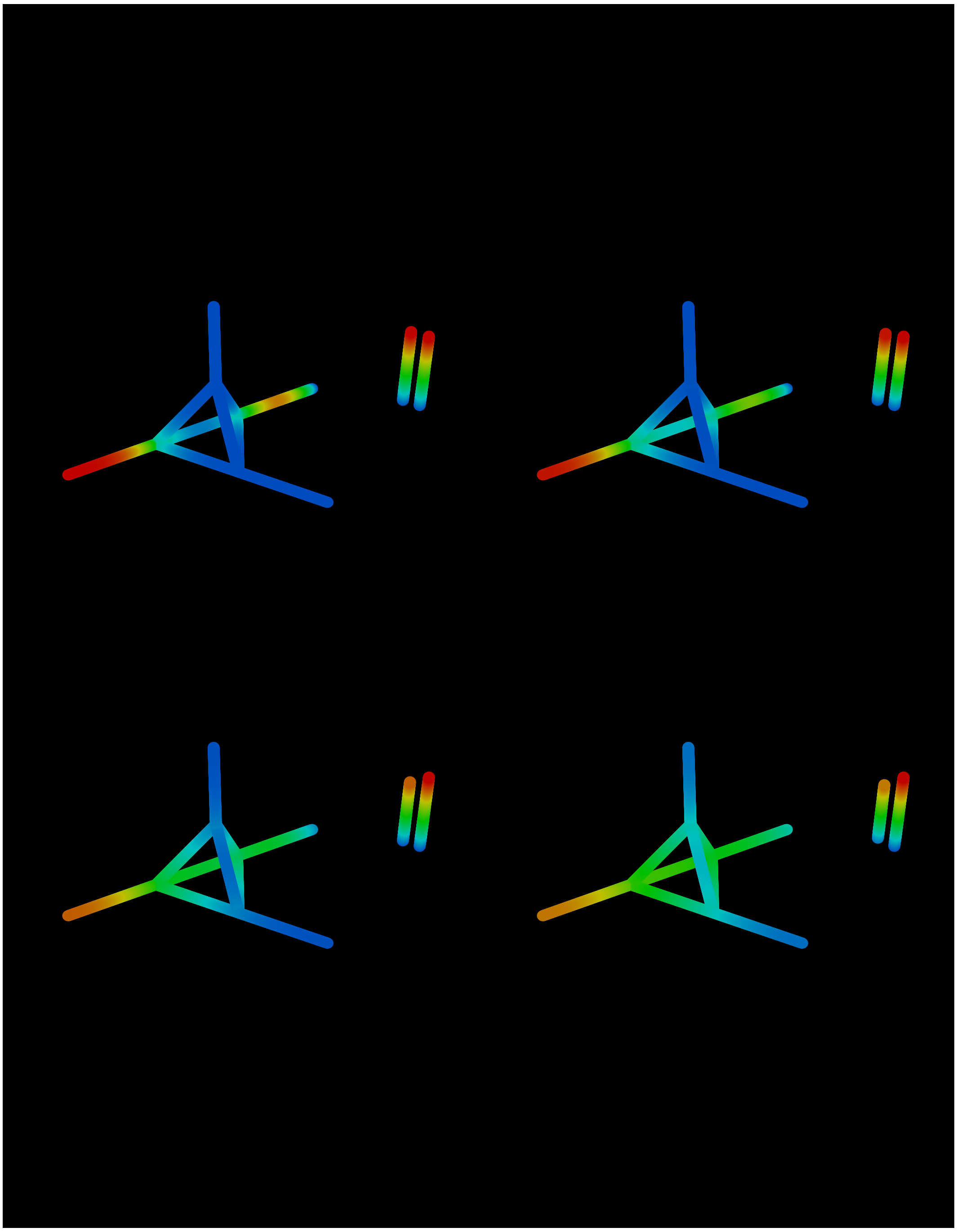
References: J. v. Below: Parabolic network equations, 2nd edition 1994; -: Front propagation in diffusion problems on trees, *Pitman Res. Notes Math. Ser.* **326** (1995); -: Similarity solutions for diffusion problems in ramified spaces, *in progress*.

On the edges of the network $\partial_t u_j = \partial^2_{x_j} u_j + 4u_j\,(1 - u_j)$



At the nodes continuity

$$\sum_j d_{ij}\partial_{x_j} u_j(v_i, t) + \sigma_i \partial_t u(v_i, t) = 0$$

Graphs and networks

$$G = \bigcup_{j \in \mathcal{N}} k_j \subset \mathbb{R}^m$$

$\Gamma = (V, K, \in)$ simple, connected, locally finite

vertices $V = \{v_i \big| i \in n\}, \quad n \subset \mathbb{N}$
edges $K = \{k_j \big| j \in \mathcal{N}\}, \quad \mathcal{N} \subset \mathbb{N}$
$\pi_j \in C^\nu([0, \ell_j]; \mathbb{R}^m)$, $\nu \geq 2$

x_j arc length variable of k_j

$$\partial_j = \frac{\partial}{\partial x_j}$$

valency $\gamma_i = \gamma(v_i) = \#\{k_j \in K \big| v_i \in k_j\} < \infty$
ramification nodes $V_r = \{v_i \in E \big| \gamma_i > 1\}$
boundary vertices $V_b = \{v_i \in V \big| \gamma_i = 1\}$
incidence matrix $\mathcal{D}(\Gamma) = (d_{ij})_{n \times \mathcal{N}}$

$$d_{ij} = \begin{cases} 1 & \text{if } \pi_j(\ell_j) = v_i, \\ -1 & \text{if } \pi_j(0) = v_i, \\ 0 & \text{otherwise.} \end{cases}$$

$T > 0 \quad u : G \times [0, T] \to \mathbb{R}$

$$u_j := u \circ (\pi_j, id) : [0, \ell_j] \times [0, T] \to \mathbb{R}$$

$$u_j(v_i, t) := u_j(\pi_j^{-1}(v_i), t)$$

$$\partial_j u_j(v_i, t) := \frac{\partial}{\partial x_j} u_j(x_j, t)\Big|_{\pi_j^{-1}(v_i)} \quad \text{etc.}$$

Transition conditions at the vertices

continuity condition (included in "$u \in \mathcal{C}(G)$")

$$\forall v_i \in V_r \;:\; k_j \cap k_s = \{v_i\} \implies u_j(v_i) = u_s(v_i),$$

classical weighted Kirchhoff flow condition at all $v_i \in V_K$

$$(K) \qquad \sum_{j \in \mathcal{N}} d_{ij} c_{ij} \partial_j u_j(v_i, t) = 0$$

dynamical Kirchhoff flow condition at all $v_i \in V_K$

$$(DK) \qquad \sum_{j \in \mathcal{N}} d_{ij} c_{ij} \partial_j u_j(v_i, t) + \sigma_i \partial_t u(v_i, t) = 0$$

General dynamical linear Kirchhoff condition (GK) in V_K

$$V_i(u, t) := \rho_i(t) u(v_i, t) - \sum_{j \in \mathcal{N}} d_{ij} c_{ij}(t) \partial_j u_j(v_i, t) - \sigma_i(t) \partial_t u(v_i, t) = 0$$

Dissipativity: all $c_{ij} > 0,\ \sigma_i \geq 0$

$\Omega = G \times [0, T]$. Throughout $V_r \subset V_K$, thus $V_{\mathrm{Dir}} \subset V_b$.

Parabolic network interior $\Omega_p = (G \backslash V_{\mathrm{Dir}}) \times (0, T]$

Parabolic network boundary $\omega_p = (G \times \{0\}) \cup (V_{\mathrm{Dir}} \times (0, T])$

Parabolic qualitative and existence theory under dissipative Kirchhoff conditions on finite networks

$$\partial_t u_j = F_j\left(x_j, t, u_j, \partial_j u_j, \partial_j^2 u_j\right) =: F_j[u_j] \uparrow w.r.t.\, q = \partial_j^2 u_j$$

Comparison principle with respect to ω_p:
Suppose $F_j \in LIP_u^+$ *and* $\exists b > 0$ *s.th.* $\rho_i(t) \leq b\sigma_i(t)$ *in* $V_K \times (0,T]$.

Let $u, v \in C(\Omega) \cap C^{2,1}(\Omega_p)$ *s.th.*

$$V_i(v,t) \leq V_i(u,t) \quad in \quad V_K \times (0,T],$$

$$\partial_t u_j - F_j[u_j] \leq \partial_t v_j - F_j[v_j] \quad in \quad \Omega_{jp}^{\bullet} \text{ for all } j \in \{1, ..., N\}.$$

Then $u \leq v$ *on* ω_p *implies* $u \leq v$ *in* Ω_p.

Uniqueness for IBVP with respect to ω_p, positivity of the flow.

Strong and weak maximum - minimum principles with respect to the parabolic boundary ω_p.

Weak and classical $(C^{2+\alpha,1+\alpha/2}(\Omega))$ solvability of linear and semi-linear parabolic network equations under usual growth conditions.

In general, no flow under nondissipative Kirchhoff conditions, blow up and non - uniqueness phenomena.

G. Lumer (1979, 1980), J. von Below (1981, 1984, 1993 ..), S. Nicaise (1986, 1987, ..) J. P. Roth (1983, 1984), F. Ali Mehmeti (1987..) a.m.o.

Extension to ramified spaces, e.g. B. & Nicaise Comm.PDE 1996

Continuous traveling waves on networks

u traveling wave on G : $\Longleftrightarrow$

1. $u \in \mathcal{C}(G \times \mathbb{R})$
2. $u_j(x_j, t) = \varphi_j\left(x_j - \tau_j t\right)$ for $j \in \mathcal{N}$
3. $\varphi_j \in C^1(\mathbb{R})$ $\tau_j \partial_j u_j + \partial_t u_j = 0$ on k_j

W.l.o.g. all $\tau_j \geq 0$,

but we have to find a suitable incidence matrix $\mathcal{D}(\Gamma)$.

$$\varepsilon_{ij} := \ell_j \frac{1 + d_{ij}}{2} \qquad (d_{ij} \neq 0).$$

$$k_j \cap k_s = \{v_i\} \implies \varphi_j\left(\varepsilon_{ij} - \tau_j t\right) = \varphi_s\left(\varepsilon_{is} - \tau_s t\right).$$

$$k_j \cap k_s = \{v_i\} \implies \tau_j \varphi_j'\left(\varepsilon_{ij} - \tau_j t\right) = \tau_s \varphi_s'\left(\varepsilon_{is} - \tau_s t\right).$$

u stationary $\Longleftrightarrow$ all $\tau_j \varphi_j' \equiv 0$

u is nonstationary $\Longleftrightarrow$ $\exists z_0 \in \mathbb{R} : \tau_j \varphi_j'(z_0) \neq 0 \Longrightarrow$ all $\tau_j > 0$

$$k_j \cap k_s = \{v_i\} \implies \varphi_s(z) = \varphi_j\left(\varepsilon_{ij} - \frac{\tau_j}{\tau_s}\varepsilon_{is} + \frac{\tau_j}{\tau_s} z\right)$$

$$\varphi_h(z) = \varphi_j \left(C(\Pi, h, j) + \frac{\tau_j}{\tau_h} z \right), \qquad (j, h \in \mathcal{N})$$

$$\zeta\left(k_j, \Delta \leq \Gamma\right) = \begin{cases} +1 & \text{if } k_j \in K(\Delta) \,\&\, k_j(\Delta) \uparrow\uparrow k_j(\Gamma), \\ -1 & \text{if } k_j \in K(\Delta) \,\&\, k_j(\Delta) \uparrow\downarrow k_j(\Gamma), \\ 0 & \text{otherwise .} \end{cases}$$

If $k_1, \ldots, k_p$ denote the consecutive branches of a path Π from v_i via $v_{i+1}, \ldots$ to v_r, then

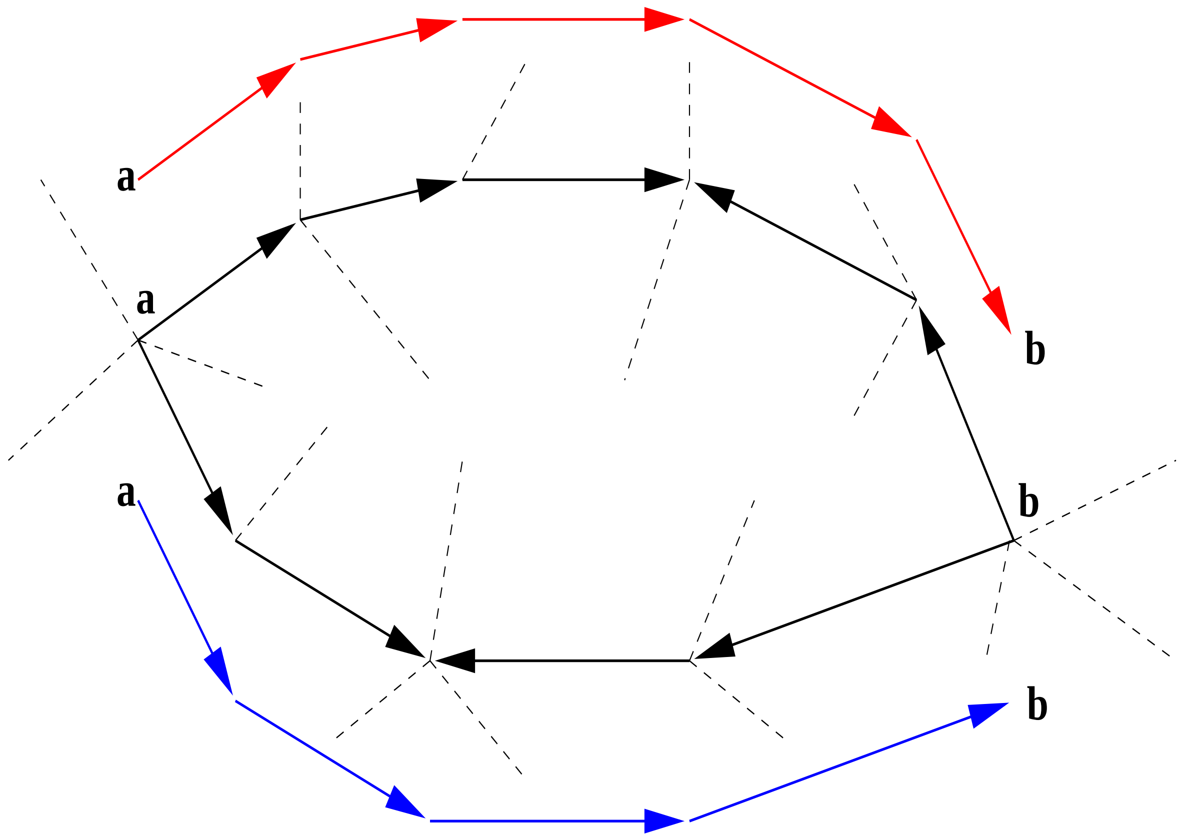
$$\varphi_p(z) = \varphi_1 \left(\varepsilon_{i1} - \frac{\tau_1}{\tau_2} \varepsilon_{i2} + \frac{\tau_1}{\tau_2} \varepsilon_{i+1,2} \mp \ldots - \frac{\tau_1}{\tau_p} \varepsilon_{r-1,p} + \frac{\tau_1}{\tau_p} z \right),$$

especially at the endpoint v_r

$$\begin{aligned} &\varphi_p\left(\varepsilon_{rp} - \tau_p t\right) \\ &= \varphi_1 \left(\varepsilon_{i1} - \frac{\tau_1}{\tau_2} \varepsilon_{i2} + \frac{\tau_1}{\tau_2} \varepsilon_{i+1,2} \mp \ldots - \frac{\tau_1}{\tau_p} \varepsilon_{r-1,p} + \frac{\tau_1}{\tau_p} \varepsilon_{r,p} - \tau_1 t \right) \end{aligned}$$

$$\varphi_p\left(\varepsilon_{rp} - \tau_p t\right) = \varphi_1 \left(\varepsilon_{i1} + \tau_1 \sum_{j=2}^{p} \zeta\left(k_j, \Pi\right) \ell_j \tau_j^{-1} - \tau_1 t \right)$$

$\varepsilon_{ij} := \ell_j \frac{1+d_{ij}}{2}$



MESH RULE: Let u be a nonstationary wave on G, Π_1 and Π_2 directed paths in Γ joining v_i and v_r, $k_h = k_{11}$. Then

$$\varphi_h\left(\ell_h\frac{1+d_{ih}}{2}+\tau_h\sum_{j\in\mathcal{N}}\zeta\left(k_j,\Pi_1\right)\frac{\ell_j}{\tau_j}-\tau_h t\right)$$
$$=\varphi_h\left(\ell_h\frac{1+d_{ih}}{2}+\tau_h\sum_{j\in\mathcal{N}}\zeta\left(k_j,\Pi_2\right)\frac{\ell_j}{\tau_j}-\tau_h t\right).$$

If in addition some φ_h is monotone, then $(\Gamma,\mathcal{D})$ is acyclic. More precisely: Along any circuit Z in Γ

$$\sum_{j\in\mathcal{N}}\zeta\left(k_j,Z\right)\frac{\ell_j}{\tau_j}=0.$$

Pf $\varphi_{p_m}^{(m)}\left(\varepsilon_{rp_m}-\tau_{p_m}t\right)=$ $m=1,2$

$$=\varphi_1^{(m)}\left(\varepsilon_{i1m}+\tau_{1m}\sum_{j=2}^{p_m}\zeta\left(k_{jm},\Pi_m\right)\ell_{jm}\tau_{jm}^{-1}-\tau_{1m}t\right)$$

$$\varphi_1^{(2)}(z)=\varphi_1^{(1)}\left(\varepsilon_{i11}-\frac{\tau_{11}}{\tau_{21}}\varepsilon_{i21}+\frac{\tau_{11}}{\tau_{21}}z\right) \qquad t=\frac{\varepsilon_{i+1,11}\ell_{11}}{\tau_{11}}+\tilde{t}$$

∎

Ex 1 If $\ell_j\equiv$ const., and if $\exists$ nonst. wave of const. speed, then Γ is bipartite:

$$\sum_{j\in\mathcal{N}}\zeta\left(k_j,Z\right)=0.$$

indegree $\gamma_i^+=\#\{k_j\in K \,|\, d_{ij}=1\}$
outdegree $\gamma_i^-=\#\{k_j\in K \,|\, d_{ij}=-1\}$
v_i sink $:\Longleftrightarrow\ \gamma_i^-=0$ $\qquad$ v_i source $:\Longleftrightarrow\ \gamma_i^+=0$
$V_b^+=\{v_i\in V_b \,|\, \gamma_i^+=1\}$ $\qquad$ $V_b^-=\{v_i\in V_b \,|\, \gamma_i^-=1\}$

Continuous nonstationary waves on networks under Kirchhoff conditions

Clearly $V_b\cap V_K=\emptyset$. $\qquad$ $k_j\cap k_s=\{v_i\}\implies \tau_j\varphi_j'\left(\varepsilon_{ij}-\tau_jt\right)=\tau_s\varphi_s'\left(\varepsilon_{is}-\tau_st\right)$

$$k_j\cap k_s=\{v_i\}\implies \partial_ju_j(v_i,t)=\frac{\tau_s}{\tau_j}\varphi_s'\left(\varepsilon_{is}-\tau_st\right)$$

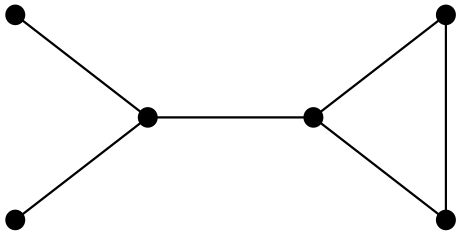
$$\sum_{j\in\mathcal{N}} d_{ij}c_{ij}\partial_j u_j(v_i,t) = 0 \quad \text{in } V_r \iff \sum_{j\in\mathcal{N}} d_{ij}c_{ij}\frac{1}{\tau_j} = 0 \quad \text{in } V_r$$

(K) dissipative $\Longrightarrow$ there are neither sinks nor sources in V_r $(\star)$

(K) diss. and $(\Gamma,\mathcal{D})$ acyclic & finite $\Longrightarrow$ $(\star)$ & $V_b^+ \neq \emptyset \neq V_b^-$

$$(\star) \implies \sum_{d_{ij}=1} \frac{\tau_j}{\gamma_i^+}\partial_j u_j(v_i,t) = \sum_{d_{ij}=-1} \frac{\tau_j}{\gamma_i^-}\partial_j u_j(v_i,t) \quad \text{in } V_r$$

No nonstationary traveling wave with dissipative (K):



$$(DK) \qquad \sum_{j\in\mathcal{N}} d_{ij}c_{ij}\partial_j u_j(v_i,t) + \sigma_i\partial_t u(v_i,t) = 0 \quad \text{in } V_K$$

$$\iff \sigma_i = \sum_{j\in\mathcal{N}} d_{ij}c_{ij}\frac{1}{\tau_j} \quad \text{in } V_K$$

Specific dynamical Kirchhoff condition (SDK):

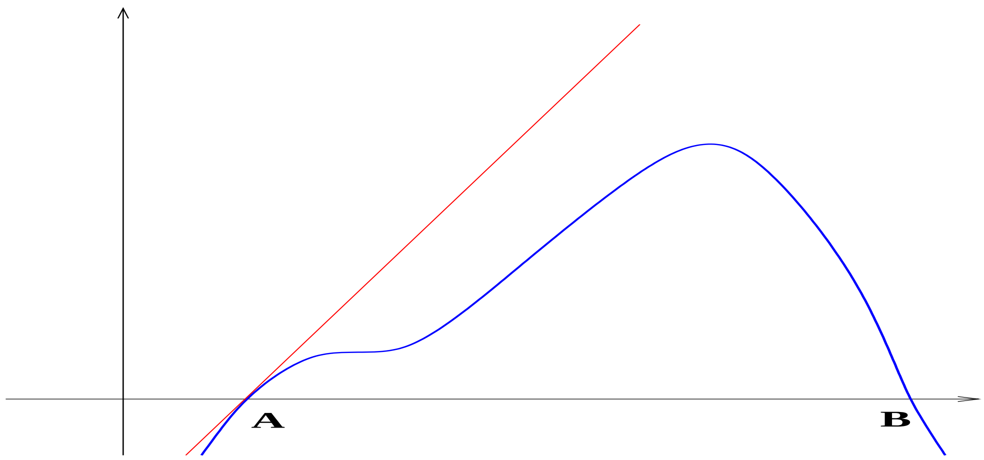
$$\partial_t u(v_i,t) + \sum_{j\in\mathcal{N}} d_{ij}\frac{d_{ij}\tau_j}{\gamma_i}\partial_j u_j(v_i,t) = 0 \qquad (v_i \in V)$$

Traveling fronts for RDBE

$$(RDBE) \qquad \partial_t u_j = a_j \partial_j^2 u_j + f_j\left(u_j\right) \qquad \text{for } \ j \in \mathcal{N}$$

$a_j > 0$, $A < B$, $f_j \in C^1([A,B])$, $f_j(A) = f_j(B) = 0$, $f_j > 0$ in (A,B), $f_j'(A) > 0$, $f_j'(B) < 0$,

often $f_j(u) \leq f_j'(A)(u-A)$ in $[A,B]$.

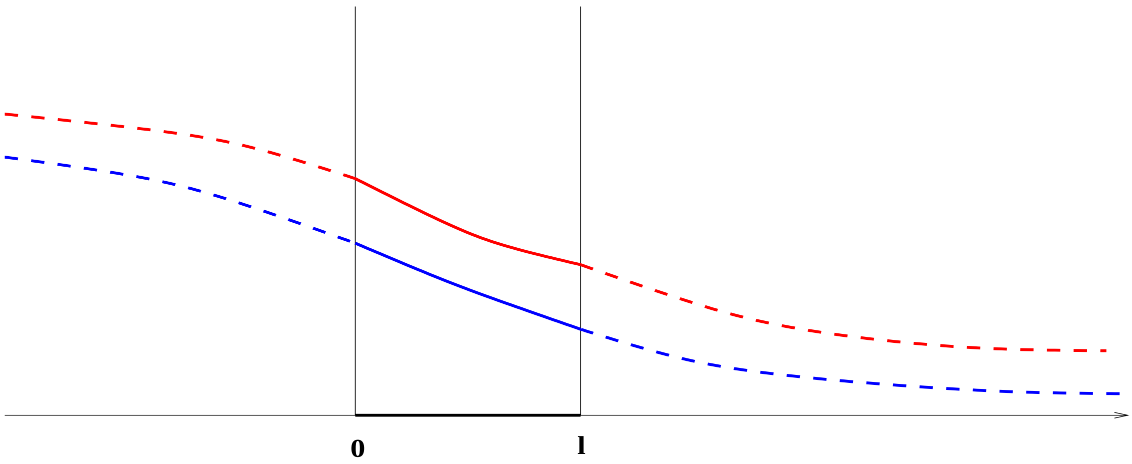


$u \in C(G \times \mathbb{R})$ traveling front solution on $G :\Longleftrightarrow$

1. on each k_j, $u_j(x_j,t) = \varphi_j\left(x_j - \tau_j t\right)$ & $u \in (RDBE)$

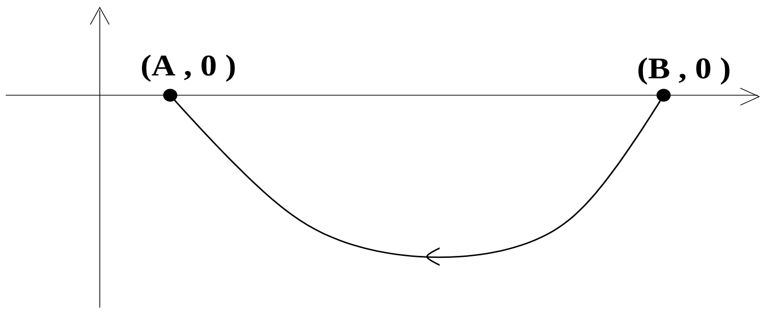
2. $\tau_j > 0$, & $\varphi_j \in C^2(\mathbb{R})$

3. $A \leq \varphi_j(z) \leq B$, $\lim\limits_{z\to\infty} \varphi_j(z) = A$ & $\lim\limits_{z\to-\infty} \varphi_j(z) = B$



On a single edge $$-\tau_j \varphi_j' = a_j \varphi_j'' + f_j(\varphi_j)$$

$$\begin{cases} \varphi_j' = \psi_j \\ \psi_j' = -\dfrac{\tau_j}{a_j}\psi_j - \dfrac{1}{a_j} f_j(\varphi_j) \end{cases}$$



$\exists$ single branch front $\iff$ $\exists$ heteroclinic orbit between $(B, 0)$ and $(A, 0)$ with $A \leq \varphi \leq B$

(slight extension of a result by Hadeler & Rothe, Aronson & Weinberger 1975:)

Necessarily: $(B, 0)$ saddle $\iff$ $f_j'(B) < 0$

Necessarily: $(A, 0)$ stable and attractive node and no vortex $\iff$

$f_j'(A) > 0$, $-\frac{\tau_j}{a_j} < 0$ and the discriminant $\frac{\tau_j^2}{a_j^2} - 4\frac{f_j'(A)}{a_j} \geq 0$

Conclusion: $\exists$ single branch fronts for $\tau_j \in [\tau_j^*, \infty)$,

$$\tau_j^* \overset{\geq}{=} 2\sqrt{a_j f_j'(A)}$$

NECESSARY CONDITIONS ON THE NETWORK

(a) $(\Gamma, \mathcal{D})$ acyclic

(b) mesh rule $\displaystyle \sum_j \zeta(k_j, Z)\frac{\ell_j}{\tau_j} = 0$

(c)

$$\frac{\tau_h}{\tau_j} = \sqrt{\frac{a_h}{a_j}\frac{\tau_j\varphi_j' + f_j(\varphi_j)}{\tau_j\varphi_j' + f_h(\varphi_j)}} \qquad (h, j \in \mathcal{N})$$

$$k_j \cap k_h \neq \emptyset \Longrightarrow \begin{cases} -\tau_j\varphi_j' = a_j\varphi_j'' + f_j(\varphi_j) \\ -\tau_j\varphi_j' = a_h\frac{\tau_j^2}{\tau_h^2}\varphi_j'' + f_h(\varphi_j) \end{cases}$$

since

$$k_j \cap k_h = \{v_i\} \implies \varphi_h(z) = \varphi_j\left(\varepsilon_{ij} - \frac{\tau_j}{\tau_h}\varepsilon_{ih} + \frac{\tau_j}{\tau_h}z\right)$$

EXISTENCE CRITERION: Assume

(1) the natural case: $f_j =: f$ for $j \in \mathcal{N}$,

(2) $$\sum_{j\in\mathcal{N}} \zeta(k_j, Z)\frac{\ell_j}{\sqrt{a_j}} = 0 \quad \text{along all circuits } Z \leq \Gamma,$$

and $f(u) \leq f'(A)(u-A)$ in $[A,B]$. Choose any k_j and set $\tau_j^* = 2\sqrt{a_j f'(A)}$. Then for any $\tau_j \geq \tau_j^*$ there is a traveling front solution u of (RDBE), where

$$\tau_h = \tau_j\sqrt{\frac{a_h}{a_j}} \quad \text{for} \quad h \in \mathcal{N}.$$

In the finite case the minimal branch speed is given by

$$\tau^* = \tau^*(G, f, a_1, \ldots, a_N) := 2\sqrt{f'(A)} \min_{1\leq j\leq N} \sqrt{a_j},$$

while in the infinite one $\inf_{j\in\mathcal{N}} \tau_j = 2\sqrt{f'(A)} \inf_{j\in\mathcal{N}} \sqrt{a_j}$.

Proof: $\exists$ heteroclinic orbit (φ_j, ψ_j) joining $(B,0)$ and $(A,0)$. Set

$$\tau_h = \tau_j\sqrt{a_h a_j^{-1}}$$

and

$$\varphi_h(z) := \varphi_j\left(\varepsilon_{ij} - \frac{\tau_j}{\tau_h}\varepsilon_{ih} + \frac{\tau_j}{\tau_h}z\right) \qquad \text{for } k_j \cap k_h = \{v_i\}.$$

Then (φ_h, ψ_h) is a heteroclinic orbit joining $(B,0)$ and $(A,0)$ for

$$-\tau_h\varphi' = a_h\varphi'' + f(\varphi).$$

mesh rule & connectedness of $G \implies u$ well - defined by

$$u_j(x_j, t) := \varphi_j\,(x_j - \tau_j t)$$

■

network front continuum $\sim \tau \left(1, \sqrt{\frac{a_2}{a_1}}, \ldots, \sqrt{\frac{a_j}{a_1}}, \ldots\right),\ \tau \geq \tau_1^*$

Traveling fronts under the Kirchhoff conditions

$V_r = V_K!$ $\quad k_j \cap k_s = \{v_i\} \implies \partial_j u_j(v_i, t) = \frac{\tau_s}{\tau_j}\varphi_s'\,(\varepsilon_{is} - \tau_s t)$

$$(K) \qquad \sum_{j\in\mathcal{N}} d_{ij}c_{ij}\partial_j u_j(v_i, t) = 0 \quad \text{in } V_r$$

$$\left[\frac{\tau_h}{\tau_j} = \sqrt{\frac{a_h}{a_j}}\right] \qquad \iff \sum_{j\in\mathcal{N}} d_{ij}c_{ij}\frac{1}{\sqrt{a_j}} = 0 \quad \text{in } V_r$$

$$(DK) \qquad \sum_{j\in\mathcal{N}} d_{ij}c_{ij}\partial_j u_j(v_i, t) + \sigma_i \partial_t u(v_i, t) = 0 \quad \text{in } V_r$$

$$\iff \sigma_i = \frac{\sqrt{a_m}}{\tau_m}\sum_{j\in\mathcal{N}} d_{ij}c_{ij}\frac{1}{\sqrt{a_j}} \quad \text{in } V_r$$

$V_r^0 = V_r \setminus V_r^d$ with $V_r^d := \{v_i \in V \,\big|\, \sigma_i \neq 0\}$

Lem 2 *Ass.* (1), (2).

1. *If the c_{ij} are given, then any traveling front solution satisfies a (DK) with σ_i given by*

$$\sigma_i = \frac{\sqrt{a_m}}{\tau_m} \sum_{j \in \mathcal{N}} d_{ij} c_{ij} \frac{1}{\sqrt{a_j}}.$$

2. *If (DK) is given with $V_r^d \neq \emptyset$, then at most one branch speed vector is admissible.*

Thm 3 *Supp. (DK) dissipative, $(\Gamma, \mathcal{D})$ acyclic, and $f(u) \leq f'(A)(u - A)$ in $[A, B]$. Then the (RDBE) admit a traveling front solution $u \in C^{2,1}_{DK}(G \times \mathbb{R})$ iff the mesh rule*

$$\sum_{j \in \mathcal{N}} \zeta(k_j, Z) \frac{\ell_j}{\sqrt{a_j}} = 0 \quad \textit{holds along all circuits } Z \leq \Gamma$$

and with $\delta_i := \sum_{j \in \mathcal{N}} d_{ij} c_{ij} \frac{1}{\sqrt{a_j}}$

$$\delta_i \geq 2\sigma_i \sqrt{f'(A)} \quad \textit{in } V_r, \quad \delta_i = 0 \quad \textit{in } V_r^0, \quad \frac{\sigma_i}{\sigma_h} = \frac{\delta_i}{\delta_h}, \quad \textit{in } V_r^d.$$

In this case the branch speeds satisfy for $h, j \in \mathcal{N}$

$$\frac{\tau_h}{\tau_j} = \sqrt{\frac{a_h}{a_j}} \quad \textit{and} \quad \sigma_i = \frac{\sqrt{a_m}}{\tau_m} \sum_{j \in \mathcal{N}} d_{ij} c_{ij} \frac{1}{\sqrt{a_j}} \ \textit{ in } V_r^d.$$

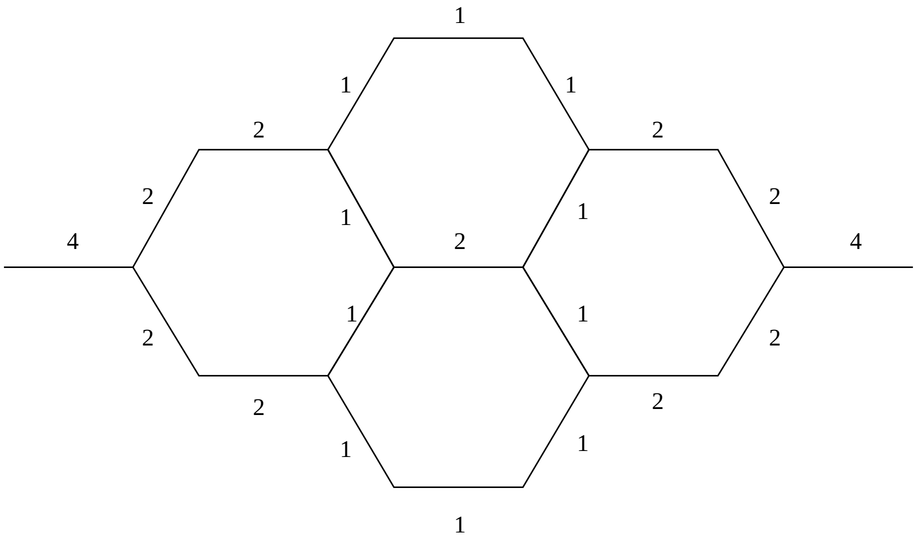
In addition,

- $V_r^d \neq \emptyset \quad \Longrightarrow \quad (\tau_j)_{j \in \mathcal{N}}$ *unique,*
- $V_r = V_r^0$: $\exists$ *front continuum* $\iff$ *all* $\delta_i = 0$ *in* V_r.

Consistent Kirchhoff condition

$$(CK) \qquad \sum_{j\in\mathcal{N}} d_{ij}a_j\partial_j u_j(v_i,t) = 0 \qquad \text{in } V_r$$

$$\iff \sum_{j\in\mathcal{N}} d_{ij}\tau_j = 0 \quad \text{in } V_r \iff \sum_{j\in\mathcal{N}} d_{ij}\sqrt{a_j} = 0 \quad \text{in } V_r$$



$$TNSC \qquad \tau_{\text{eff}}(G,a_i,f) := \sum_{v_i\in V_b} d_{ij}\tau_j = 0$$

Consistent dynamical Kirchhoff condition:

$$(CDK) \qquad \sigma_i\partial_t u(v_i,t) + \sum_{j\in\mathcal{N}} d_{ij}a_j\partial_j u_j(v_i,t) = 0 \qquad \text{in } V_K$$

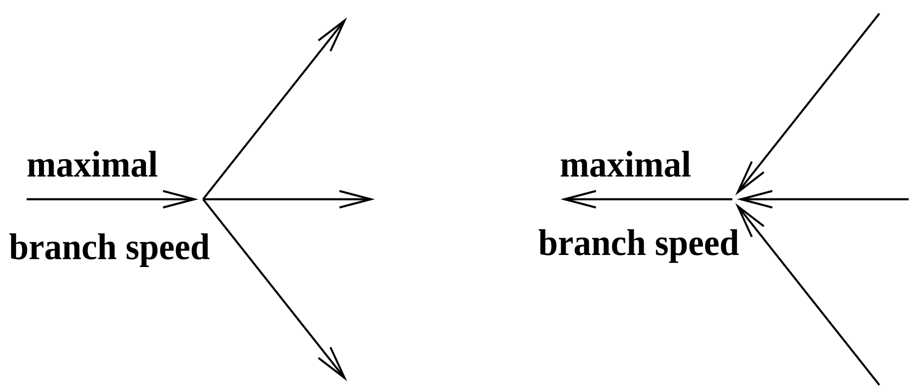
$$\iff \sigma_i = \frac{\sqrt{a_m}}{\tau_m}\sum_{j\in\mathcal{N}} d_{ij}\sqrt{a_j} = \frac{a_m}{\tau_m^2}\sum_{j\in\mathcal{N}} d_{ij}\tau_j \quad \text{in } V_K$$

$$(CDK) \text{ dissipative at } v_i \iff \sum_{j \in \mathcal{N}} d_{ij}\tau_j \geq 0$$

$$(CDK) \text{ nondissipative at } v_i \iff \sum_{j \in \mathcal{N}} d_{ij}\tau_j < 0$$

Cor 4 *Suppose* Γ *is a tree.*

1. *If* $\sum_{j\in\mathcal{N}} d_{ij}\sqrt{a_j} \geq 0$ *in* V_r, *then* $\exists$ *traveling front in* G *satisfying a diss.* (CDK). $\max_j \tau_j$ *occurs only at* V_b^-, *while* $\min_j \tau_j$ *occurs only at* V_b^+. $\tau_{\text{eff}} \leq 0$.
2. *If* $\sum_{j\in\mathcal{N}} d_{ij}\sqrt{a_j} < 0$ *in* V_r, *then* $\exists$ *traveling front in* G *satisfying a nondiss.* (CDK). $\max_j \tau_j$ *occurs only at* V_b^+, *while* $\min_j \tau_j$ *occurs only at* V_b^-. $\tau_{\text{eff}} > 0$.



Speed dissipation under diss. (CDK) vs. speed concentration under nondiss. (CDK)

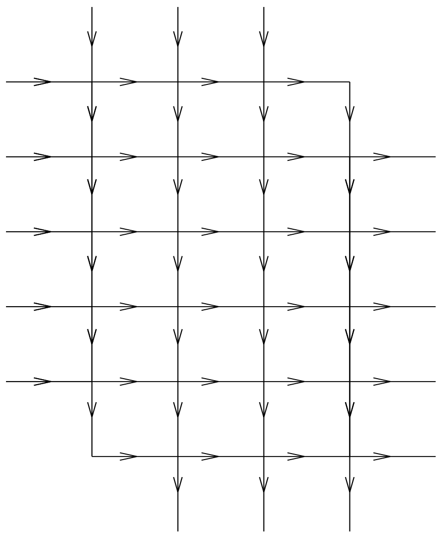
Isotachic fronts under (CK) and (CDK):

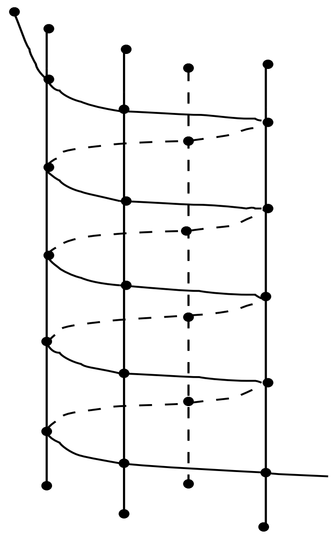
$$\text{all } \tau_j \text{ are equal} \iff \text{all } a_j \text{ are equal}$$

isotachic front & $(CK) \Longrightarrow \quad \gamma_i^+ = \gamma_i^- \quad$ in V_r and

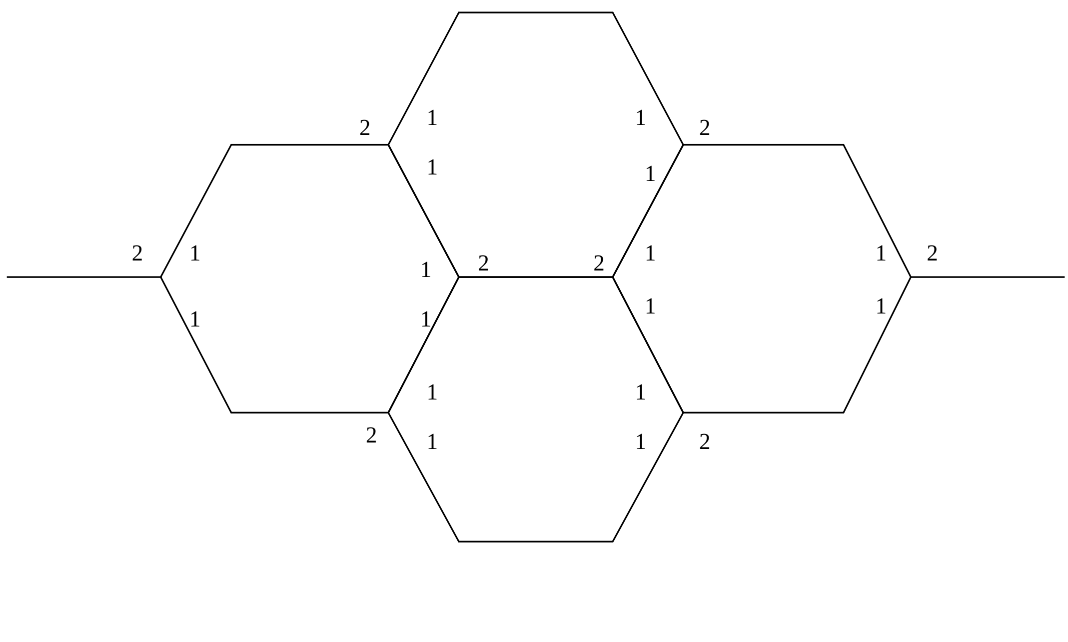
$$\sum_{j \in \mathcal{N}} \zeta\left(k_j, Z\right) \ell_j = 0 \qquad \text{along any circuit } Z \text{ in } \Gamma.$$

Examples: trees with even degrees in V_r or:





Isotachic fronts only under inconsistent (K):



isotachic front & $(CDK) \Longrightarrow \quad \sigma_i = a\tau^{-1}\left(\gamma_i^+ - \gamma_i^-\right) \quad \text{in } V_r$

Application to neuron models with isotachic propagation on the dendritic tree, nonlinear Camerer model

Attractivity and stability of equilibria on finite networks

$f_j \in C^1([A,\infty))$, $f_j(A) = f_j(B) = 0$, $f_j > 0$ in (A,B), $f_j'(A) > 0$, $f_j'(B) < 0$, $f_j < 0$ in (B,∞).

$$(**) \qquad \begin{cases} (a)\ u \in C(G\times[0,\infty)) \cap C^{2,1}_{DK}(G\times(0,\infty)), \\ (b)\ \partial_t u_j = a_j \partial_j^2 u_j + f_j(u_j) \quad \text{on } k_j,\ 1 \le j \le N, \\ (c)\ u\big|_{G\times\{0\}} \geq\not\equiv A. \end{cases}$$

Thm 5 *Impose diss. (K) in V:*

$$\sum_{j\in\mathcal{N}} d_{ij} c_{ij} \partial_j u_j(v_i,t) = 0. \qquad (v_i \in V)$$

If $u \in (**)$, then

$$\lim_{t\to\infty} |u(\cdot,t) - B|_{\infty,G} = 0.$$

Pf Choose $A_1 > A, t_1 > 0$ s.th. $u \geq A_1$ for $t \geq t_1$. $B_1 := \max_G u(\cdot,0)$. Let $\underline{u}, \overline{u} \in C^1([t_1,\infty))$ solve

$$\begin{cases} \dot{\underline{u}} = \min_j f_j(\underline{u}), \\ \underline{u}(t_1) = A_1, \end{cases} \qquad \begin{cases} \dot{\overline{u}} = \max_j f_j(\overline{u}), \\ \overline{u}(0) = B_1. \end{cases}$$

Comparison Principle: $\underline{u} \leq u \leq \overline{u}$ in $G \times [t_1,\infty)$. ■

Advantage: arg. applies also to the infinite case.

Thm 6 *Impose consistent (CDK) in V with all $\sigma_i \geq 0$:*

$$\sigma_i \partial_t u(v_i, t) + \sum_{j \in \mathcal{N}} d_{ij} a_j \partial_j u_j(v_i, t) = 0. \qquad (v_i \in V)$$

If $u \in (**)$, then

$$\lim_{t \to \infty} | u(\cdot, t) - B |_{\infty, G} = 0.$$

Pf Set

$$F_j(z) = \int_A^z f_j(s) ds$$

and

$$\mathcal{E}(u) = \sum_{j \in \mathcal{N}} \frac{1}{2} \int_0^{\ell_j} \Big(a_j \left(\partial_j u_j \right)^2 - F_j(u_j) \Big) dx_j \geq \text{cst.}$$

Then

$$\frac{d}{dt} \mathcal{E}(u(\cdot, t)) = - \sum_{j \in \mathcal{N}} \int_0^{\ell_j} \left(\partial_t u_j(x_j, t) \right)^2 dx_j + \sum_{i \in n} \partial_t u(v_i, t) \underbrace{\sum_{j \in \mathcal{N}} d_{ij} a_j \partial_j u_j(v_i, t)}_{= -\sigma_i \partial_t u(v_i, t)}$$

$$= - \sum_{j \in \mathcal{N}} \int_0^{\ell_j} \left(\partial_t u_j(x_j, t) \right)^2 dx_j - \sum_{v_i \in V} \sigma_i \left(\partial_t u(v_i, t) \right)^2 \leq 0.$$

Apply Lasalle's invariance principle: $\omega \left(u(\cdot, 0) \right) = \{ B \}$. ∎

Thm 7 *Suppose diss.* (DK) *and* $f := f_1 = \ldots = f_N$ *and*

$$\delta_i := \sum_{j \in \mathcal{N}} d_{ij} c_{ij} \sqrt{a_j^{-1}} \geq \sigma_i 2\sqrt{f'(A)}$$

for all $v_i \in V$ *and*

$$\sum_{j \in \mathcal{N}} \zeta(k_j, Z) \frac{\ell_j}{\sqrt{a_j}} = 0 \qquad (Z \text{ circuit in } \Gamma).$$

If $u \in C(G \times [0, \infty); [A, B])$, $u \in (**)$, *then*

$$\lim_{t \to \infty} |u(\cdot, t) - B|_{\infty, G} = 0.$$

Pf $\exists$ front $w \in C^{2,2}(G \times \mathbb{R})$ s.th. $w \in (RDBE)$,

$$\tau_j \geq \tau_j^* \geq 2\sqrt{a_j f'(A)}$$

and for $d_{is} \neq 0$,

$$\sum_{j \in \mathcal{N}} d_{ij} c_{ij} \partial_j w_j(v_i, t) + \sigma_i(t) \partial_t w(v_i, t)$$

$$= -\varphi_s'(\varepsilon_{is} - \tau_s t) \{\delta_i \sqrt{a_s} - \tau_s \sigma_i\} \geq 0.$$

Find $A_1 > A, t_1 > 0$ s.th. $u \geq A_1$ for $t \geq t_1$ by the StMP and $w \leq u$ for $t = t_1$.

Comparison Principle: $w \leq u$ in $G \times [0, \infty)$. ∎

Thm 8 *Supp. diss. (DK). Let $[B_1, B_2]$ be the component of $\bigcap_j \{z \,\big|\, f_j'(z) \le 0\}$ containing B, $u \in (**)$ with*

$$u \in C(G \times [t_1, \infty); [B_1, B_2])$$

for some $t_1 \ge 0$. Then

$$\lim_{t \to \infty} |u(\cdot, t) - B|_{\infty, G} = 0.$$

Pf W.l.o.g. $t_1 = 0$. $\tilde{u}^h \in C^{2,1}_{DK}(G \times [0, \infty))$

$$\tilde{u}^h_j(x_j, t) := \frac{1}{h} \{u_j(x_j, t+h) - u_j(x_j, t)\}$$

$$\frac{\partial}{\partial t} \tilde{u}^h_j = a_j \tilde{u}^h_{j x_j x_j} + \tilde{u}^h_j \int_0^1 \underbrace{f_j'\left(\tau u_j(x_j, t+h) + (1-\tau) u_j(x_j, t)\right)}_{\le 0} d\tau.$$

$$|\tilde{u}^h|^{(0)}_{G \times 0, \infty)} \le \max_G |\tilde{u}^h(\cdot, 0)|,$$

$$|u_t|^{(0)}_{G \times [0, \infty)} \le \max_G |u_t(\cdot, 0)|.$$

$U = \{u(\cdot, t) \,\big|\, t \ge 0\}$ is $|\cdot|^{(2)}_G$-bounded, each $|\cdot|^{(1+\beta)}_G$ - limit u_∞ satisfies (K) and is a bounded weak solution of

$$0 = a_j u_{j x_j x_j} + f_j(u_j) \qquad \text{on } k_j,\ 1 \le j \le N,$$

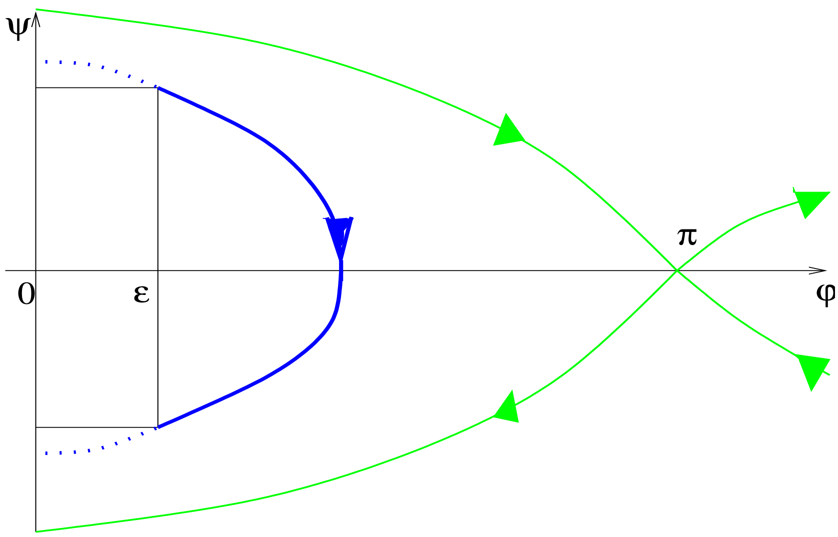
$$u_\infty \in C^2(G), \qquad u_\infty \equiv B.$$

∎

The equilibrium B is not attractive without dissipativity of (DK):

$a_j \equiv 1,\ f(u) = f_j(u) = \sin u,$ $\{(\varphi(z), \psi(z)) \big| z \in [0, l]\}$

$$\begin{cases} \varphi' = \psi, \\ \psi' = -f(\varphi), \end{cases} \qquad \begin{array}{l} 0 < \varepsilon \le \varphi < \pi \text{ in } [0, l] \\ \varphi(0) = \varphi(l) = \varepsilon \quad \& \quad \psi(l) = -\psi(0) \end{array}$$



Choose Γ with $V_b = \emptyset$, $V = V_K$, $\ell_j \equiv l$.

$u \in C^{2,1}(G \times \mathbb{R})$, $u_j(x_j, t) := \varphi(x_j)$ satisfies $u \in (**)$

$$0 < \psi(0) = d_{ij} u_{jx_j}(v_i, t) \quad \text{for} \quad d_{ij} \neq 0,$$

$$\sum_{j=1}^{N} d_{ij} c_{ij} \partial_j u_j(v_i, t) + \sigma_i \partial_t u(v_i, t) = 0, \qquad (v_i \in V_K)$$

with arbitrary σ_i and

$$c_{ij} = \begin{cases} 1 & \text{if } j = \min\{s \big| d_{is} \neq 0\}, \\ \dfrac{1}{1-\gamma_i} & \text{otherwise,} \end{cases}$$

$$\operatorname{dist}\left((\varphi(\ell/2), 0), (\pi, 0)\right) = \min_G |\pi - u(\cdot, t)| \equiv \text{const.} > 0$$

Conclusion: B is not locally attractive.

Rem 9 Extension to reaction terms with a sign change

$$f(u) = u\,(1-u)\,(u-\mu)$$

a.m.o.

Rem 10 Extension to given incidence ratios $g_{ijs} = g_{isj}^{-1} > 0$:

$$\forall v_i \in V_r \ : \ k_j \cap k_s = \{v_i\} \implies u_j(v_i) = g_{ijs} u_s(v_i).$$

Rem 11 Impossibility of traveling pulses with $A = B$ by the negative Bendixson Criterion: no homoclinic orbit for

$$\begin{cases} \varphi_j' = \psi_j \\ \psi_j' = -\dfrac{\tau_j}{a_j}\psi_j - \dfrac{1}{a_j} f_j\,(\varphi_j) \end{cases}$$

Rem 12 A general condition (GK)

$$V_i(u,t) := \rho_i u(v_i,t) - \sum_{j\in\mathcal{N}} d_{ij} c_{ij} \partial_j u_j(v_i,t) - \sigma_i \partial_t u(v_i,t) = 0$$

is much too restrictive. Suppose at some v_i, $\rho_i \neq 0$ and (GK) is imposed. For a nonstationary wave u this leads to a linear ODE for each profile φ_h with $d_{ih} \neq 0$, namely

$$\varphi_h' = \eta_h \varphi_h, \qquad \eta_h := \frac{\rho_i}{\tau_h}\left(\sum_{j\in\mathcal{N}} d_{ij} c_{ij} \frac{1}{\tau_j} - \sigma_i\right)^{-1}.$$

Thus, $\varphi_h(z) = z_0 \exp(\eta_h z)$ with $z_0 \neq 0$ and all profiles are completely determined by the branch speeds τ_j. Clearly, these profiles can never belong to a traveling front.

DAMPING EFFECT OF THE DYNAMICAL DISSIPATIVE KIRCHHOFF CONDITION

Tetrahedron K_4 $\partial_t u_j = \partial_j^2 u_j + 4u_j\,(1 - u_j)$

(CK) left $$\sum_{j \in \mathcal{N}} d_{ij} \partial_j u_j(v_i, t) = 0$$

$$\sigma_i \partial_t u(v_i, t) + \sum_{j \in \mathcal{N}} d_{ij} \partial_j u_j(v_i, t) = 0$$ right (CDK)

$\sigma_1 = 10,\ \sigma_3 = 1000,\ \sigma_2 = \sigma_4 = 0$