Statistical Stability of Kinetic Models for High Frequency Waves

Guillaume Bal

Department of Applied Physics & Applied Mathematics

Columbia University

http://www.columbia.edu/~gb2030

gb2030@columbia.edu

Joint with **Ian Langmore, Olivier Pinaud, Kui Ren, Lenya Ryzhik**

Outline

1. **Formal derivation of kinetic models**

2. Numerical validity of kinetic models

3. Statistical stability

4. Stability/Instability in parabolic Anderson model

5. Optimal stability in simplified regimes

Scalar wave equation

The pressure $p(t, \mathbf{x})$ solves following closed form scalar equation

$$\frac{\partial^2 p}{\partial t^2} = c^2(\mathbf{x})\Delta p, \qquad c^2(\mathbf{x}) = \frac{1}{\rho_0 \kappa(\mathbf{x})}.$$

Moreover

$$\mathcal{E}_H(t) = \frac{1}{2}\int_{\mathbb{R}^d} \Big(\kappa(\mathbf{x})\Big(\frac{\partial p}{\partial t}\Big)^2(t,\mathbf{x}) + \frac{|\nabla p|^2(t,\mathbf{x})}{\rho_0}\Big) d\mathbf{x} = \mathcal{E}_H(0).$$

The role of a kinetic model is to **predict the spatial distribution** of the above conserved quantity.

We are interested in the **high frequency** regime, where the initial conditions have the form $p(0, \mathbf{x}) = p\Big(\dfrac{\mathbf{x}}{\varepsilon}\Big)$.

Wigner Transform

Let ε be a small parameter modeling typical wavelength.

The Wigner transform of two propagating vector fields is defined by:

$$W_\varepsilon[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}) = \int_{\mathbb{R}^d} e^{i\mathbf{y}\cdot\mathbf{k}}\mathbf{u}(\mathbf{x}-\varepsilon\frac{\mathbf{y}}{2})\mathbf{v}^*(\mathbf{x}+\varepsilon\frac{\mathbf{y}}{2})\frac{d\mathbf{y}}{(2\pi)^d}.$$

It is the inverse Fourier transform of the product:

$$W_\varepsilon[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k}) = \mathcal{F}^{-1}\Big(\mathbf{u}(\mathbf{x}+\varepsilon\frac{\mathbf{y}}{2})\mathbf{v}^*(\mathbf{x}-\varepsilon\frac{\mathbf{y}}{2})\Big).$$

We verify that

$$\int_{\mathbb{R}^d} W[\mathbf{u},\mathbf{v}](\mathbf{x},\mathbf{k})d\mathbf{k} \;=\; (\mathbf{u}\mathbf{v}^*)(\mathbf{x})$$

The Wigner transform captures field-field correlations.

Weak-Coupling Regime

In the weak coupling regime, the random fluctuations of the media are modeled by

$$\boxed{(c_\varepsilon^\varphi)^2(\mathbf{x}) = c_0^2 - \sqrt{\varepsilon} V^\varphi(\frac{\mathbf{x}}{\varepsilon}),} \quad \varphi = 1, 2,$$

$$c_0^2 = \frac{1}{\kappa_0 \rho_0}, \qquad V^\varphi(\mathbf{x}) = \frac{c_0^2}{\kappa_0} \kappa_1^\varphi(\mathbf{x}),$$

where c_0 is the average background speed and κ_1^φ and V^φ are random fluctuations in the compressibility and sound speed, respectively. We assume that $V^\varphi(\mathbf{x})$, $\varphi = 1, 2$, are **statistically homogeneous** mean-zero random fields with correlation functions and power spectra given by:

$$\boxed{\begin{aligned} c_0^4 R^{\varphi\psi}(\mathbf{x}) &= \langle V^\varphi(\mathbf{y}) V^\psi(\mathbf{y}+\mathbf{x})\rangle, \qquad 1 \le \varphi, \psi \le 2, \\ (2\pi)^d c_0^4 \hat{R}^{\varphi\psi}(\mathbf{p}) \delta(\mathbf{p}+\mathbf{q}) &= \langle \hat{V}^\varphi(\mathbf{p}) \hat{V}^\psi(\mathbf{q})\rangle. \end{aligned}}$$

Kinetic theory in weak coupling regime

Let $W_\varepsilon(t,\mathbf{x},\mathbf{k})$ be the Wigner transform of two fields $\mathbf{u}_1$ and $\mathbf{u}_2$ propagating in the two media $\varphi = 1,2$.
The limit Wigner distribution is decomposed over propagating modes as: $W(t,\mathbf{x},\mathbf{k}) = a_+(t,\mathbf{x},\mathbf{k})\mathbf{b}_+\mathbf{b}_+^* + a_-(t,\mathbf{x},\mathbf{k})\mathbf{b}_-\mathbf{b}_-^*$. Furthermore, the **radiative transfer equation** for a_+ is (with $\omega_\pm = \pm c_0|\mathbf{k}|$):

$$\begin{aligned}\frac{\partial a_+}{\partial t} + c_0\hat{\mathbf{k}}\cdot\nabla a_+ + (\Sigma(\mathbf{k}) + i\Pi(\mathbf{k}))a_+ \\ = \frac{\pi\omega_+^2(\mathbf{k})}{2(2\pi)^d}\int_{\mathbb{R}^d}\hat{R}^{12}(\mathbf{k}-\mathbf{q})a_+(\mathbf{q})\delta\Big(\omega_+(\mathbf{q})-\omega_+(\mathbf{k})\Big)d\mathbf{q},\end{aligned}$$

$$\Sigma(\mathbf{k}) = \frac{\pi\omega_+^2(\mathbf{k})}{2(2\pi)^d}\int_{\mathbb{R}^d}\frac{\hat{R}^{11}+\hat{R}^{22}}{2}(\mathbf{k}-\mathbf{q})\delta\Big(\omega_+(\mathbf{q})-\omega_+(\mathbf{k})\Big)d\mathbf{q}$$

$$i\Pi(\mathbf{k}) = \frac{i\pi\sum_{j=\pm}}{4(2\pi)^d}\ \text{p.v.}\int_{\mathbb{R}^d}\Big(\hat{R}^{11}-\hat{R}^{22}\Big)(\mathbf{k}-\mathbf{q})\frac{\omega_j(\mathbf{k})\omega_+(\mathbf{q})}{\omega_j(\mathbf{q})-\omega_+(\mathbf{k})}d\mathbf{q}.$$

Equation for spatial Wigner transform

All models start with an equation for the Wigner transform, which requires the field equation to be first-order in the time variable:

$$\varepsilon \frac{\partial \mathbf{u}_\varepsilon^\varphi}{\partial t} + A_\varepsilon^\varphi \mathbf{u}_\varepsilon^\varphi = 0, \qquad \varphi = 1, 2,$$

The Wigner transform of the two fields defined as

$$W_\varepsilon(t, \mathbf{x}, \mathbf{k}) = W[\mathbf{u}_\varepsilon^1(t, \cdot), \mathbf{u}_\varepsilon^2(t, \cdot)](\mathbf{x}, \mathbf{k}),$$

solves the equation

$$\varepsilon \frac{\partial W_\varepsilon}{\partial t} + W[A_\varepsilon^1 \mathbf{u}_\varepsilon^1, \mathbf{u}_\varepsilon^2] + W[\mathbf{u}_\varepsilon^1, A_\varepsilon^2 \mathbf{u}_\varepsilon^2] = 0.$$

Some pseudo-differential calculus allows us to write $W[A_\varepsilon^1 \mathbf{u}_\varepsilon^1, \mathbf{u}_\varepsilon^2]$ in terms of $W_\varepsilon(t, \mathbf{x}, \mathbf{k})$ and thus get a closed form equation for $W_\varepsilon(t, \mathbf{x}, \mathbf{k})$ amenable to (non-rigorous) asymptotic expansions.

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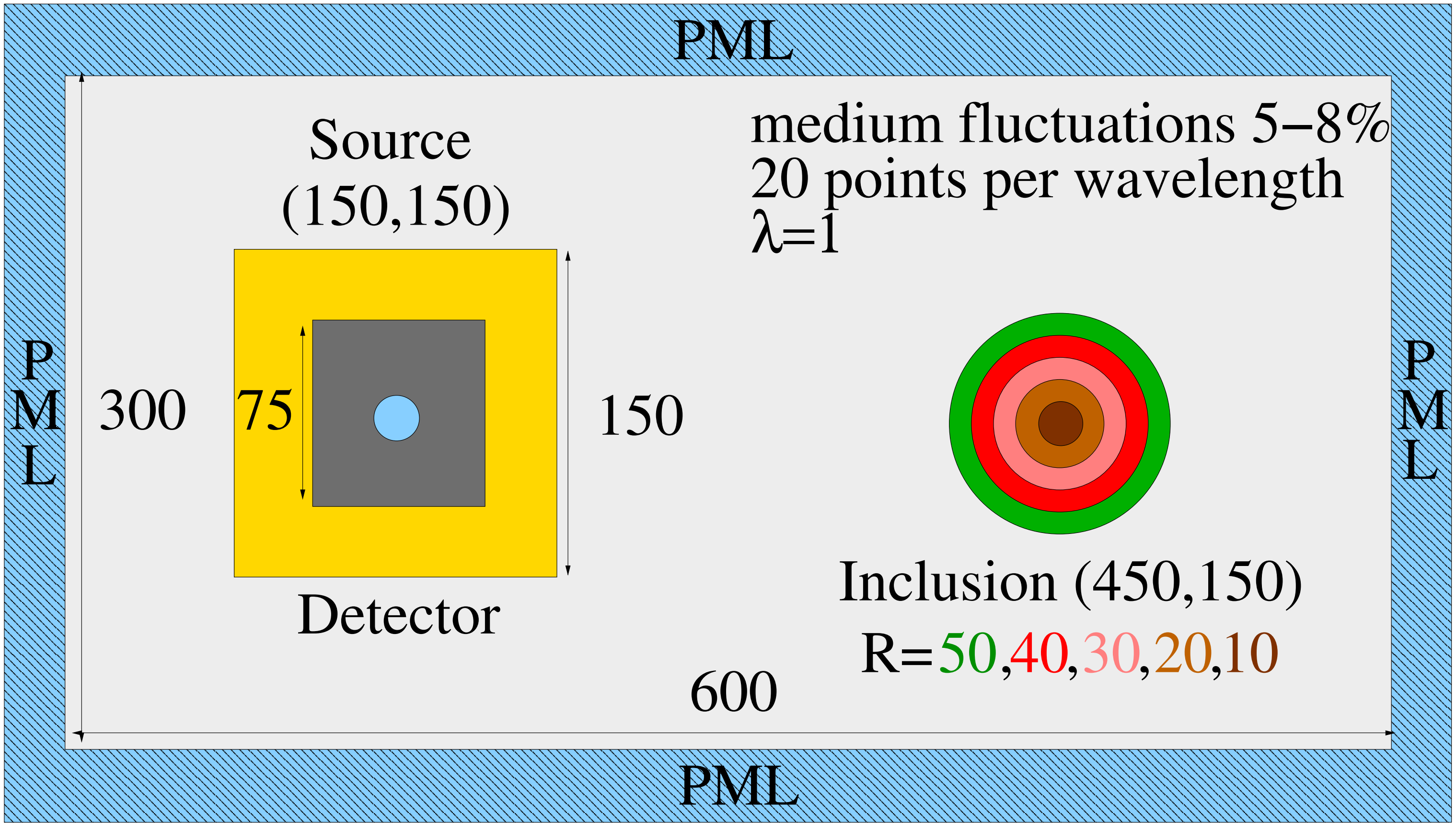
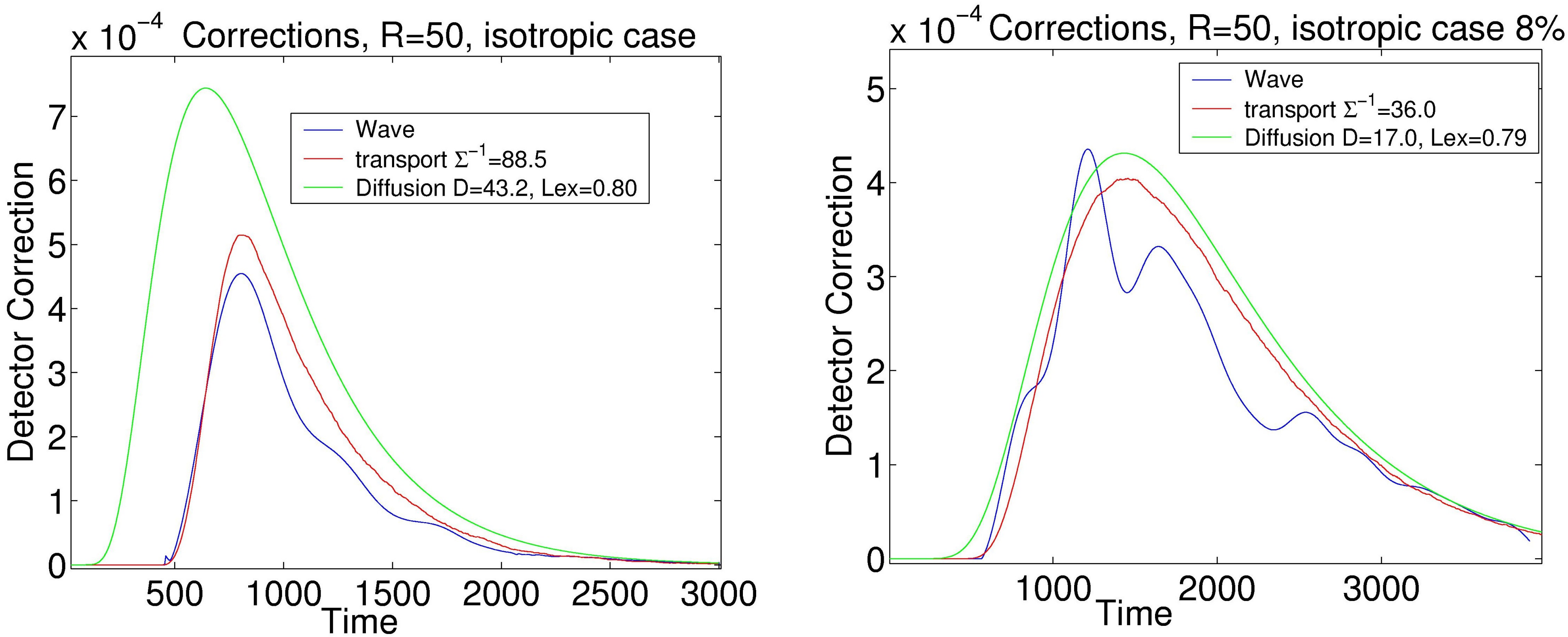
Numerical validation



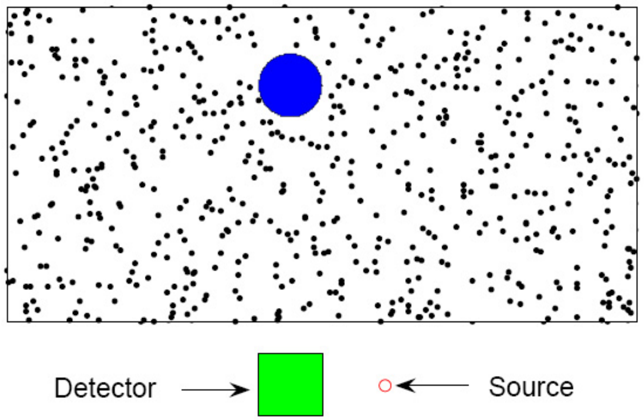
The domain size is roughly $20,000 \times 10,000 = 200M$ nodes

Effect of void inclusions



Correction generated by an inclusion of radius $R = 50$ where the random fluctuations are suppressed. Left: 5% RMS. Right: 8% RMS. Transport and diffusion generated by best energy fit. The diffusion fit is now much more accurate.

Discrete Scatterers and frequency domain



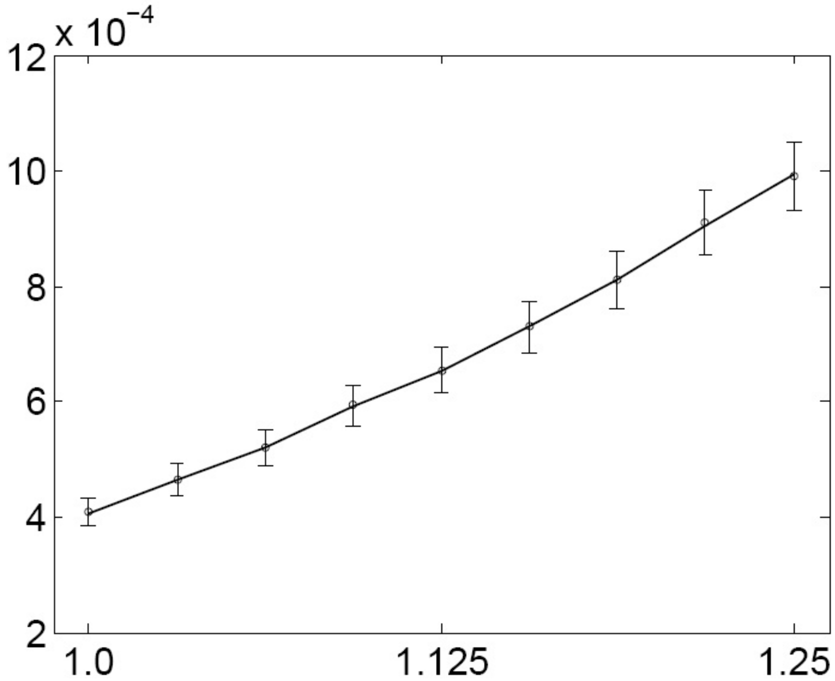
The sound speed fluctuations take the form $\sqrt{\varepsilon}V_\varepsilon(\frac{\mathbf{x}}{\varepsilon})$, where

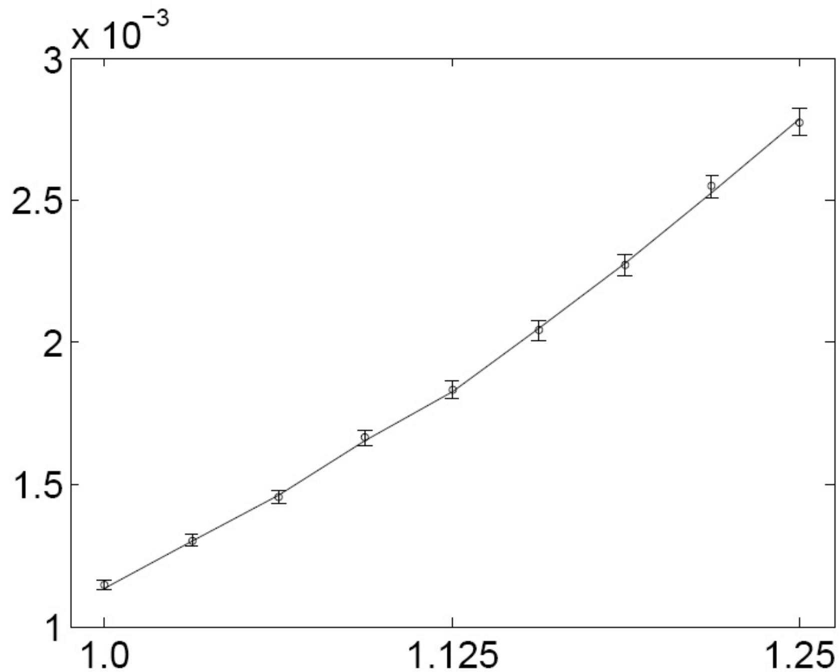
$$V_\varepsilon(\mathbf{x}) = \varepsilon^{-\frac{(\gamma+2\beta)d}{2}} \sum_j \tau_j V\Big(\frac{\mathbf{x}-\mathbf{x}_j^\varepsilon}{\varepsilon^\beta}\Big)$$

with $\beta > 0$ and $\gamma < 1$. The scatterers are modeled as a Poisson point process: the points $\mathbf{x}_j^\varepsilon(\omega)$ form a Poisson point process of density $\nu_\varepsilon = \varepsilon^{\gamma d} n_0$.

At the wave level, the scatterers are sufficiently localized so that we can use a Foldy-Lax model. At the transport level, we observe that the power spectrum $\hat{R}_\varepsilon$ converges to $\hat{R}_0 = L^{2d}\mathbb{E}\{\tau^2\}n_0$.

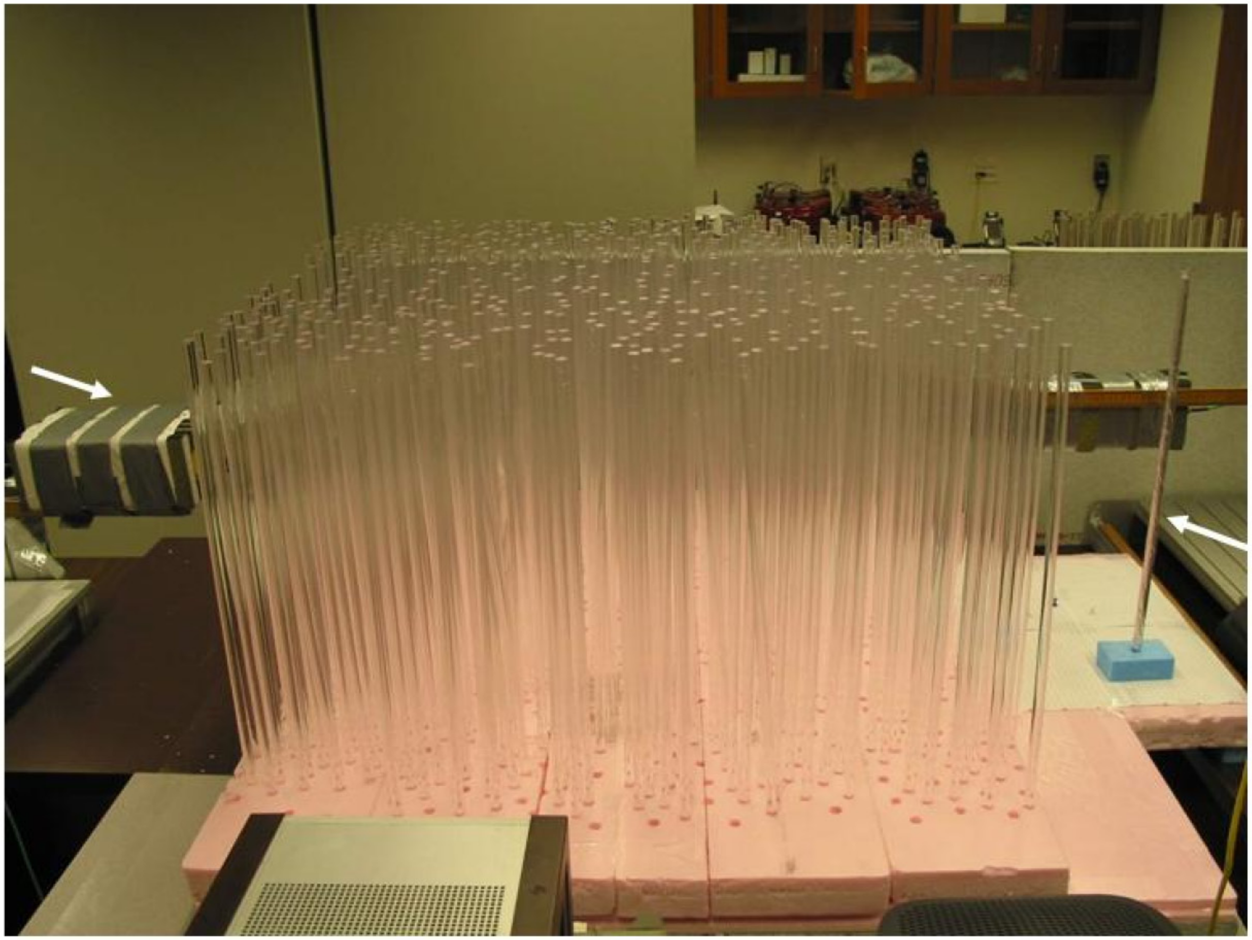
Comparison Wave Energy - Transport



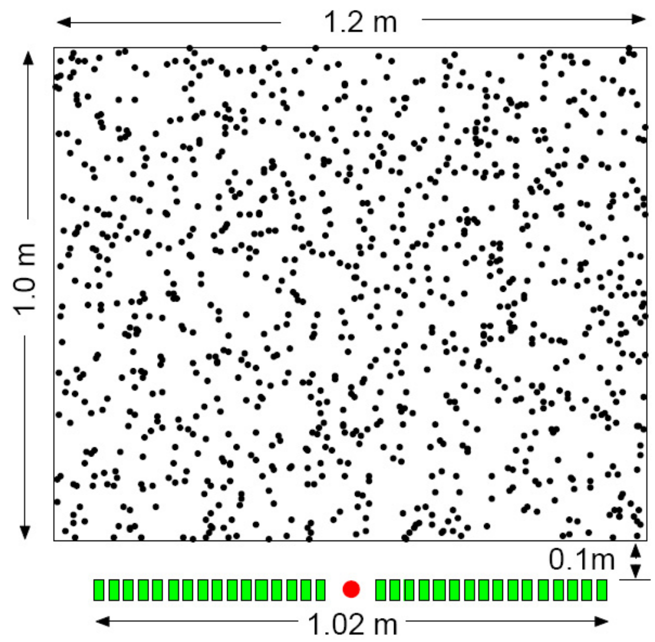


Small detector (left) and Large detector (right). Error bars = 1 standard deviation (50 realizations)

Duke experimental set-up

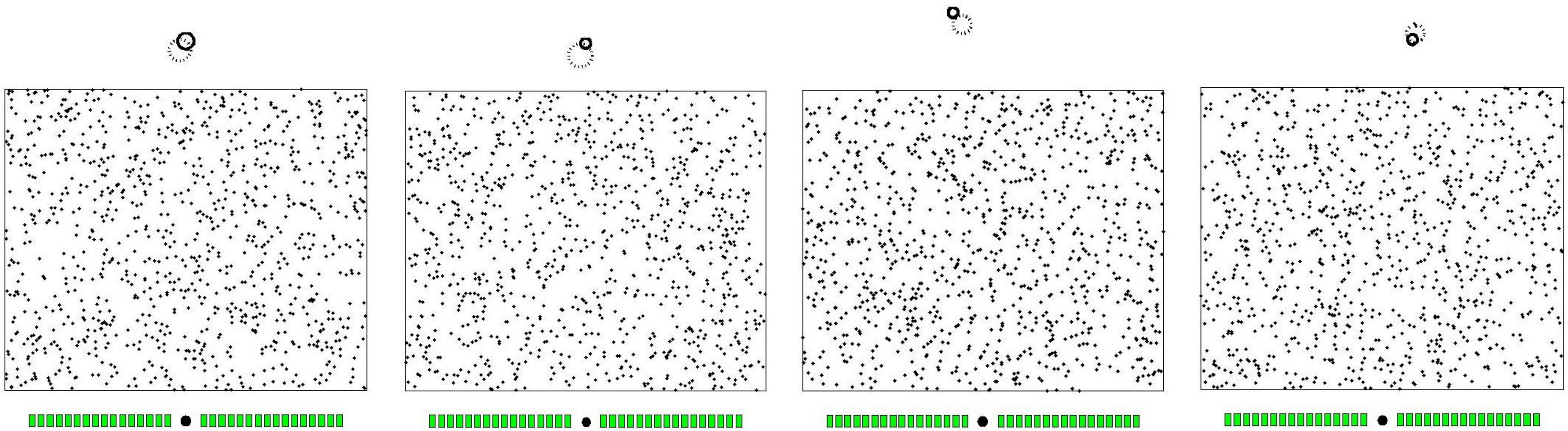


Duke numerical set-up



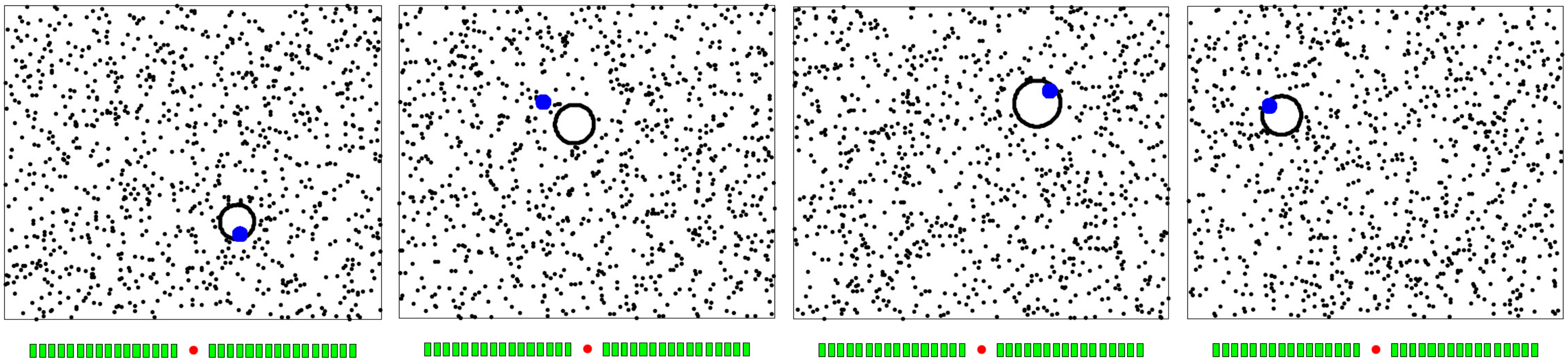
All subsequent reconstructions based on frequency average on the interval $[9GHz, 11GHz]$. Best fit for mean free path is 42cm (91cm for band $[6.5GHz, 8.5GHz]$, showing behavior of mfp in k^{-3}).

Reconstruction of outside rods



Reconstruction of four (small) rods outside the random medium from differential measurements.

Reconstruction of voids inside random medium



Reconstruction of four holes created by the removal of three rods. Reconstruction based on differential measurements.

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Stability and Imaging

The above derivation for the radiative transfer equation is formal. However, the simplest arguments show that the ensemble average of the energy density solves the (deterministic) radiative transfer equation:

$$\mathbb{E}\{\mathcal{E}_\varepsilon(t,\mathbf{x})\} \xrightarrow{\varepsilon\to 0} \int_{\mathbb{R}^d} a_+(t,\mathbf{x},\mathbf{k})d\mathbf{k}.$$

How about $\mathcal{E}_\varepsilon(t,\mathbf{x})$? Does it converge as well? Is the limit independent of the realization of the random medium?

Answering these questions is crucial to address the inverse problem: we do not have access to the influence of a buried inclusion averaged over (a sample of the) realizations of the random medium.

We thus need to understand the statistical instability of the energy measurements and acknowledge that our lack of knowledge of the random medium translates into inevitable noise).

Paraxial equation and time regularization

In the paraxial approximation, wave propagation is modeled by Schrödinger equation

$$i\kappa\varepsilon\frac{\partial\psi_\varepsilon}{\partial z}+\frac{\varepsilon^2}{2}\Delta_{\mathbf{x}}\psi_\varepsilon-\kappa^2\sqrt{\varepsilon}V(\frac{z}{\varepsilon},\frac{\mathbf{x}}{\varepsilon})\psi_\varepsilon=0,$$
$$\psi_\varepsilon(z=0,\mathbf{x},\kappa)=\psi_{0\varepsilon}(\frac{\mathbf{x}}{\varepsilon},\kappa).$$

Mixing of waves is now simplified: we assume the random field $V(z,\mathbf{x})$ is a Markov process in z with a correlation function $R(z,\mathbf{x})$:

$$\mathbb{E}\left\{V(s,\mathbf{y})V(z+s,\mathbf{x}+\mathbf{y})\right\}=R(z,\mathbf{x})\quad\text{for all } \mathbf{x},\mathbf{y}\in\mathbb{R}^d,\text{ and } z,s\in\mathbb{R}.$$

We now have access to a full machinery (Invariant measures and spectral gaps for Markov processes, perturbed test function method and weak convergence of measures on space of continuous paths) to address the convergence properties of the Wigner transform.

Stability result

Theorem [B. Papanicolaou Ryzhik]. The Wigner distribution W_ε converges *in probability and weakly* in $L^2(\mathbb{R}^{2d})$ to the (deterministic) solution $\overline{W}$ of the transport equation

$$\frac{\partial \overline{W}}{\partial z} + \frac{1}{\kappa}\mathbf{k}\cdot\nabla_{\mathbf{x}}\overline{W} = \kappa\mathcal{L}\overline{W},$$

with $L^2(\mathbb{R}^{2d})$ initial data $W_0(\mathbf{x},\mathbf{k})$ and operator $\mathcal{L}$ defined by

$$\mathcal{L}\lambda = \int_{\mathbb{R}^d}\frac{d\mathbf{p}}{(2\pi)^d}\widehat{R}\Big(\frac{|\mathbf{p}|^2-|\mathbf{k}|^2}{2},\mathbf{p}-\mathbf{k}\Big)(\lambda(\mathbf{p})-\lambda(\mathbf{k})),$$

where $\widehat{R}(\omega,\mathbf{p})$ is the Fourier transform of the correlation function of V.

More precisely, for any test function $\lambda\in L^2(\mathbb{R}^{2d})$ the process $\langle W_\varepsilon(z),\lambda\rangle$ converges to $\langle\overline{W}(z),\lambda\rangle$ in probability as $\varepsilon\to 0$.

Radiative transfer and diffusion regimes

Consider more generally the following Schrödinger equation

$$i\varepsilon^{1+\delta}\frac{\partial\psi_\varepsilon}{\partial z}+\frac{\varepsilon^2}{2}\Delta\psi_\varepsilon-\varepsilon^{\frac{\beta-\delta}{2}}V\Big(\frac{z}{\varepsilon^\alpha},\frac{\mathbf{x}}{\varepsilon^\beta}\Big)\psi_\varepsilon=0.$$

Theorem [B. Ryzhik] The associated Wigner distribution converges *in probability and weakly* to the deterministic solution of

(i) the radiative transfer equation when $\alpha<\beta=1$ and $\delta=0$:

$$\frac{\partial\overline{W}}{\partial z}+\mathbf{k}\cdot\nabla_{\mathbf{x}}\overline{W}=\int_{\mathbb{R}^d}\frac{d\mathbf{p}}{(2\pi)^d}\delta\Big(\frac{|\mathbf{p}|^2-|\mathbf{k}|^2}{2}\Big)\hat{R}_0(\mathbf{p}-\mathbf{k})(W(\mathbf{p})-W(\mathbf{k}))$$

(ii) the diffusion equation when $\alpha<\beta=1$ and $\delta>0$:

$$\frac{\partial\overline{W}}{\partial z}-\nabla_{\mathbf{x}}D\nabla_{\mathbf{x}}\overline{W}=0,\qquad D=\overline{\mathbf{k}\otimes\mathcal{L}^{-1}\mathbf{k}},$$

(iii) the Fokker-Planck equation when $\alpha < \beta < 1$ and $\delta = 0$.

$$\frac{\partial \overline{W}}{\partial z} + \mathbf{k} \cdot \nabla_{\mathbf{x}} \overline{W} = \frac{1}{2} \nabla_{\mathbf{k}} \cdot \Big(\int_{\mathbb{R}^d} \delta(\mathbf{k} \cdot \mathbf{p}) \hat{R}_0(\mathbf{p}) \mathbf{p} \otimes \mathbf{p} \frac{d\mathbf{p}}{(2\pi)^d} \Big) \nabla_{\mathbf{k}} W$$

The Fokker Planck equation is valid the wavelength is much shorter than the correlation length of the medium. We may see the dynamics as wave packets propagating in a random Hamiltonian.

The Fokker-Planck may be seen as an approximation to radiative transfer when scattering is highly peaked forward so that the direction of the wavepackets follows Brownian dynamics.

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Parabolic equation with large potential

Consider the parabolic equation

$$\frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon - \frac{1}{\varepsilon^\alpha} q\Big(\frac{x}{\varepsilon}\Big) u_\varepsilon = 0, \qquad u_\varepsilon(0,x) = u_0(x),$$

with $q(x)$ mean-zero Gaussian. Assume that the correlation function

$$R(x) = \mathbb{E}\{q(y)q(x+y)\} \sim \frac{\kappa}{|x|^{\mathfrak{p}}}, \qquad 0 < \mathfrak{p} < d.$$

Then for $\boxed{\mathfrak{p} > 2}$ and $\alpha = 1$, u_ε converges strongly to u solution of

$$\frac{\partial u}{\partial t} - \Delta u - \rho u = 0 \qquad t \geq 0, \quad x \in \mathbb{R}^d, \quad \rho = \int_{\mathbb{R}^d} \frac{\hat{R}(\xi)}{|\xi|^2} d\xi.$$

The **fluctuation** (corrector) $u_{1\varepsilon} = \varepsilon^{-\frac{\mathfrak{p}-2}{2}}(u_\varepsilon - \mathbb{E}\{u_\varepsilon\})$ converges in law to u_1 solution of the SPDE with additive (fractional) noise:

$$\frac{\partial u_1}{\partial t} - \Delta u_1 - \rho u_1 = \sqrt{\kappa} u \dot{W}(x).$$

Parabolic equation with large potential

Consider the same parabolic equation

$$\frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon - \frac{1}{\varepsilon^\alpha} q\Big(\frac{x}{\varepsilon}\Big) u_\varepsilon = 0, \qquad u_\varepsilon(0,x) = u_0(x),$$
$$R(x) = \mathbb{E}\{q(y)q(x+y)\} \sim \frac{\kappa}{|x|^{\mathfrak{p}}}, \qquad 0 < \mathfrak{p} < d.$$

Then for $\boxed{\mathfrak{p} < 2}$, $\alpha = \frac{\mathfrak{p}}{2}$ and $d \geq 2$, u_ε converges in law to u solution of

$$\frac{\partial u}{\partial t} - \Delta u = \sqrt{\kappa} u \circ \dot{W}(x), \quad t \geq 0, \quad x \in \mathbb{R}^d.$$

The above SPDE with multiplicative noise has to be understood in the Stratonovich sense and admits a unique square integrable solution in an appropriate functional setting.

This shows a totally different behavior of u_ε depending on the decorrelation properties of $R(x)$: **stochastic behavior** for slow decorrelation and **homogenization** for fast decorrelation.

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Itô Schrödinger approximation

Assume now that the fluctuations in z are faster than in the transverse variables $\mathbf{x}$:

$$\frac{\partial \psi_\varepsilon}{\partial z} - \frac{i\varepsilon}{2}\Delta_{\mathbf{x}}\psi_\varepsilon = i\frac{1}{\varepsilon^{\frac{1+\alpha}{2}}}V\Big(\frac{z}{\varepsilon^{1+\alpha}},\frac{\mathbf{x}}{\varepsilon}\Big)\psi_\varepsilon, \qquad \alpha > 0.$$

Then we can formally replace

$$\frac{1}{\varepsilon^{\frac{1+\alpha}{2}}}V\Big(\frac{z}{\varepsilon^{1+\alpha}},\frac{\mathbf{x}}{\varepsilon}\Big) \qquad \text{by} \qquad B(\frac{\mathbf{x}}{\varepsilon}, dz),$$

where $B(\mathbf{x}, dz)$ is the usual Brownian motion in z with statistics $\langle B(\mathbf{x}, z)B(\mathbf{y}, z')\rangle = Q(\mathbf{y}-\mathbf{x})z \wedge z'$, and obtain the Itô-Schrödinger equation

$$d\psi_\varepsilon(\mathbf{x}, z) = \frac{i}{2\varepsilon}\Delta_{\mathbf{x}}\psi_\varepsilon(\mathbf{x}, z)dz + i\kappa\psi_\varepsilon(\mathbf{x}, z) \circ B(\frac{\mathbf{x}}{\varepsilon}, dz).$$

Kinetic models

Upon using the Itô formula, we obtain that the average Wigner transform

$$a_\varepsilon(t, \mathbf{x}, \mathbf{k}) = \mathbb{E}\{W_\varepsilon(t, \mathbf{x}, \mathbf{k})\},$$

solves the following kinetic model

$$\begin{aligned} &\frac{\partial a_\varepsilon}{\partial t} + \mathbf{k} \cdot \nabla_{\mathbf{x}} a_\varepsilon + R_0 a_\varepsilon = \int_{\mathbb{R}^d} \widehat{R}(\mathbf{k} - \mathbf{q}) a_\varepsilon(t, \mathbf{x}, \mathbf{q}) \frac{d\mathbf{q}}{(2\pi)^d}, \\ &a_\varepsilon(0, \mathbf{x}, \mathbf{k}) = W_\varepsilon(0, \mathbf{x}, \mathbf{k}). \end{aligned}$$

This equation provides a kinetic model for the ensemble average of the Wigner transform. The kinetic model is here exact.

Kinetic model for the scintillation function

A natural object in the study of the statistical stability of W_ε is the following covariance (scintillation) function:

$$J_\varepsilon(t,\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{p}) = \mathbb{E}\{W_\varepsilon(t,\mathbf{x},\mathbf{k})W_\varepsilon(t,\mathbf{y},\mathbf{p})\} - \mathbb{E}\{W_\varepsilon(t,\mathbf{x},\mathbf{k})\}\mathbb{E}\{W_\varepsilon(t,\mathbf{y},\mathbf{p})\}.$$

Another application of the Itô formula shows that

$$\Big(\frac{\partial}{\partial t} + \mathcal{T}_2 + 2R_0 - \mathcal{Q}_2 - \mathcal{K}_\varepsilon\Big) J_\varepsilon = \mathcal{K}_\varepsilon a_\varepsilon \otimes a_\varepsilon,$$

where

$$\begin{aligned}
\mathcal{T}_2 &= \mathbf{k}\cdot\nabla_{\mathbf{x}} + \mathbf{p}\cdot\nabla_{\mathbf{y}} \\
\mathcal{Q}_2 J &= \int_{\mathbb{R}^{2d}} \Big(\hat{R}(\mathbf{k}-\mathbf{k}')\delta(\mathbf{p}-\mathbf{p}') + \hat{R}(\mathbf{p}-\mathbf{p}')\delta(\mathbf{k}-\mathbf{k}')\Big) J(\mathbf{k}',\mathbf{p}') \frac{d\mathbf{k}'d\mathbf{p}'}{(2\pi)^d} \\
\mathcal{K}_\varepsilon h &= \sum_{\epsilon_i,\epsilon_j=\pm 1} \int_{\mathbb{R}^{2d}} \hat{R}(\mathbf{u}) e^{i\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{u}}{\varepsilon}} \epsilon_i\epsilon_j h(\mathbf{x},\mathbf{k}+\epsilon_i\frac{\mathbf{u}}{2},\mathbf{y},\mathbf{p}+\epsilon_j\frac{\mathbf{u}}{2}) \frac{d\mathbf{u}}{(2\pi)^d}.
\end{aligned}$$

Stability estimates for the Wigner transform

Define domains of measurements:

$$\varphi_{\varepsilon,s_1,s_2}(\mathbf{x},\mathbf{k}) = \frac{1}{\varepsilon^{d(s_1+s_2)}}\varphi\Big(\frac{\mathbf{x}}{\varepsilon^{s_1}}, \frac{\mathbf{k}-\mathbf{k}_1}{\varepsilon^{s_2}}\Big).$$

By using the Chebyshev inequality, we obtain the following estimate on the probability that W_ε deviate from its ensemble average a_ε:

$$\mathbb{P}\Big(|\langle W_\varepsilon(t), \varphi_{\varepsilon,s_1,s_2}\rangle - \langle a_\varepsilon(t), \varphi_{\varepsilon,s_1,s_2}\rangle| \geq \delta\Big) \leq \frac{1}{\delta^2}\langle J_\varepsilon(t), \varphi_{\varepsilon,s_1,s_2}\otimes\varphi_{\varepsilon,s_1,s_2}\rangle.$$

We are therefore interested in estimating the above right-hand side.

Main stability result

Theorem.[B. Pinaud] Let $\psi_\varepsilon(\mathbf{x},0)$ be a sequence of functions uniformly bounded in $L^2(\mathbb{R}^d)$, compact at infinity, and ε-oscillatory. Let $a_\varepsilon(0,\mathbf{x},\mathbf{k})$ be the corresponding sequence of Wigner transforms. We assume that

$$\|\mathcal{F}_{\mathbf{x}} a_\varepsilon(0,\mathbf{u},\mathbf{k})\|_{L^1(\mathbb{R}^{2d})} \lesssim \varepsilon^{-\alpha d} \quad \text{and} \quad \|\mathcal{F}_{\mathbf{k}} a_\varepsilon(0,\mathbf{x},\boldsymbol{\xi})\|_{L^1(\mathbb{R}^{2d})} \lesssim \varepsilon^{-\beta d}.$$

This means the source concentrates at the scale ε^α in space and ε^β in wavenumbers Physically, $\alpha+\beta=1$. Then we find that

$$\langle J_\varepsilon(t), \varphi_{\varepsilon,s_1,s_2}\otimes\varphi_{\varepsilon,s_1,s_2}\rangle \lesssim g_\varepsilon$$

$$g_\varepsilon = \varepsilon^{d(1-\alpha)-2d(s_1+s_2)}\Big[\varepsilon^{2(1-\alpha)-s_1-s_1\vee s_2+(\alpha-\beta)\vee 0}\Big] \vee \varepsilon^{1-\beta+((\alpha-\beta)\vee 0)\wedge((d-1)(1-\alpha-\beta)+\alpha)},$$

when $d\geq 3$ (with a modified expression when $d=2$).

Small support of the sources large detectors

Let us assume that the spatial support of the domain of measurements is large. Then we find that

$$\langle J_\varepsilon(t), \varphi \otimes \varphi \rangle \lesssim \varepsilon^{\alpha + d(1-\alpha)}.$$

The scintillation is of order $O(\varepsilon^d)$ when $\alpha = 0$, which corresponds to a large support of the initial source term. This corresponds to the ideal case where the scintillation is smallest. In such a setting, we obtain that $\langle W_\varepsilon - a_\varepsilon, \varphi \rangle$ is of order $\varepsilon^{\frac{d}{2}}$.

For a very small support of the initial source with $\alpha = 1$, we obtain that the scintillation is of order $O(\varepsilon)$ so that $\langle W_\varepsilon - a_\varepsilon, \varphi \rangle$ is now of order $\varepsilon^{\frac{1}{2}}$.

Small domain of measurements.

Conversely, we consider the case of a source term with a large support and a small spatial measurement domain of size ε^{ds_1}. In this setting, we find that

$$\langle J_\varepsilon, \varphi_{\varepsilon,s_1} \otimes \varphi_{\varepsilon,s_1} \rangle \lesssim \varepsilon^{d(1-s_1)}.$$

The energy density becomes asymptotically statistically stable as soon as it is measured over an area that is large compared to the correlation length of the medium. This is an optimal result of self-averaging as we cannot expect the energy density to be statistically stable on sub-wavelength domains.

Convergence of the scintillation function

Theorem. Consider initial conditions of the form $a_\varepsilon(0,\mathbf{x},\mathbf{k}) = \delta(\mathbf{x})f(\mathbf{k})$ for some smooth function $f(\mathbf{k})$ in dimension $d \geq 2$. (Small domain of the source in space.) Then $\varepsilon^{-1}J_\varepsilon(t)$ converges in the space of distributions uniformly in time to the limit $J(t) = J(t,\mathbf{x},\mathbf{k},\mathbf{y},\mathbf{p})$, which solves an explicit kinetic equation.

This result shows that the $O(\varepsilon)$ estimate obtained earlier is optimal for sources supported on small domains.

Single scattering Full kinetic model

Let us come back to the kinetic model for the Schrödinger equation

$$i\varepsilon\frac{\partial u_\varepsilon}{\partial t}+\varepsilon^2\Delta u_\varepsilon-\sqrt{\varepsilon}V\Big(\frac{x}{\varepsilon}\Big)u_\varepsilon=0,$$

with highly oscillatory initial conditions. Let $R(x)$ be the correlation function of V. The scintillation function has two contributions: one linear in $R(x)$, and the other one involving higher-order moments of V. We analyze the former contribution to scintillation. Consider an initial condition for u_ε that is localized in space at 0 and wavenumber around k_0. The Wigner transform of such an initial condition is of the form

$$a_\varepsilon(x,k)=\frac{1}{\varepsilon^d}a\Big(\frac{x}{\varepsilon^\alpha},\frac{k-k_0}{\varepsilon^\beta}\Big),$$

with $\beta=1-\alpha$ (uncertainty principle).

Single scattering scintillation

For φ a (large) detector array characteristic, we want

$$I_\varepsilon := \int J_{11\varepsilon}(t,x,k,y,q)\varphi(x,k)\varphi(y,q)dxdkdydq.$$

Long range correlations (slow decay of $R(x)$) are modeled by singular behavior of power spectrum $\hat{R}(\xi)$ at $\xi = 0$:

$$\hat{R}(\xi) = \frac{\hat{S}(\xi)}{|\xi|^\delta}, \qquad \delta = d - \mathfrak{p}, \qquad R(x) \sim \frac{\kappa}{|x|^{\mathfrak{p}}}, \quad x \to \infty.$$

The size of I_ε depends in a fairly complex way on α, $\beta = 1 - \alpha$, and δ:

$$0 \le I_\varepsilon \le C\varepsilon^{(d-\delta)(1-\alpha)+\alpha\vee(1-\alpha)}, \qquad d \ge 1(!).$$

When $\delta = 0$ (integrable correlation function), the maximum scintillation is of order ε and $\alpha = 1$ (localized source term in space).

When $\delta \to d$ (very long range $\mathfrak{p} \to 0$), the maximum scintillation approaches $\varepsilon^{\frac{1}{2}}$ and $\alpha = \beta = \frac{1}{2}$.

Conclusions

Kinetic models for waves in random media are **homogenization** models. They work great in many practical settings.

Limitations are to be found in resulting scintillation. Scintillation is a complex functional of the wave propagation model, the singular structure of the **initial conditions**, and the **long range properties** of the randomness.

In the parabolic Anderson model, homogenization is replaced by a **stochastic** description for sufficiently long range correlations (more constraining in high dimensions).

Same should occur for waves with a **transition** from homogenization to localization. Waves always localize in 1D (at least for strictly time-independent randomness). They may localize in higher dimensions for long range power spectra, though the model will presumably depend on the initial conditions.