On a Class of Transport Equations for the Dynamics and the Interaction of TCP Flows

F. Baccelli

ENS-INRIA

IPAM, April 27, 2009

Summary

- Isolated TCP flows
  joint work with D. Mc Donald, K.B. Kim & J. Reynier
  - Persistent Flows
  - On-Off Flows
- Competition of Parallel TCP Flows
  - Persistent Flows (AQM)
  - On-Off Flows (TD), with A. Chaintreau, D. De Vleeschauwer & D. Mc-Donald, IHMN, 2006
- Interaction of TCP Flows in Series (AQM)
  ongoing work with G. Carofiglio & S. Foss

Persistent Flows

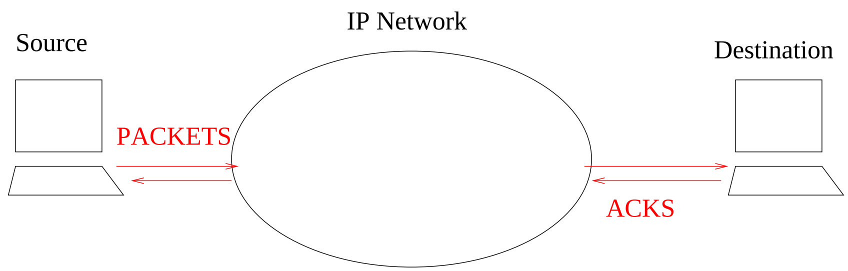
– Dynamics of TCP

– Square root formula

– Markov analysis

– Distributions

Congestion Control



TCP Transmission Control Protocol of the Internet: distributed, end-to-end

– Error control Each packet received by the destination is acknowledged;

– Congestion control The number of unacknowledged packets in transit in the network is limited by the source to a maximal value W called the window.

If the Round Trip Time (RTT) is R, the throughput of the connection is

$$X = \frac{W}{R}$$

TCP

- TCP dynamic window size (updated when acks are received):

$$w_{n+1} = g(w_n, F(n)),$$

$F(n)$: feedback signal on the state of congestion,

- Reno-congestion avoidance: AIMD

$$g(w_n, \mathrm{OK}) = w_n + 1 \text{ every } w_n \text{ acks}, \quad g(w_n, \mathrm{LOSS}) = \left\lfloor \frac{w_n}{2} \right\rfloor$$

- Fast: MIMD

$$g(w_n, \mathrm{OK}) = w_n + 1, \quad g(w_n, \mathrm{LOSS}) = \left\lfloor \frac{w_n}{2} \right\rfloor$$

Hybrid Model for AIMD Dynamics of TCP RENO

In Congestion Avoidance phase, the evolution of the TCP Reno congestion window is described through the differential equation:

$$dW(t) = \frac{dt}{R} - \frac{W(t-)}{2} N(dt), \quad dX(t) = \frac{dt}{R^2} - \frac{X(t-)}{2} N(dt)$$

- the window increase between two loss events is linear with slope $\frac{1}{R}$;
- loss events produce jumps of congestion windows which is cut by half.
- $N(t)$ is the loss point process.
- First studied by T. Ott and then by Altman et al. 00, Adjih et al. 02, Robert et al. 02, F.B. Mc Donald and Reynier 02

Rationale

- AI: TCP stipulates that in Congestion Avoidance, the window is increased of 1 unit every W ack:
  - In dt, the number of acks that arrive is $X(t)dt$;
  - Hence the window increases of $X(t)dt/W(t) = dt/R$.
- MD: TCP stipulates that in case of a loss event, the congestion windows which is cut by half.

Loss Point Processes

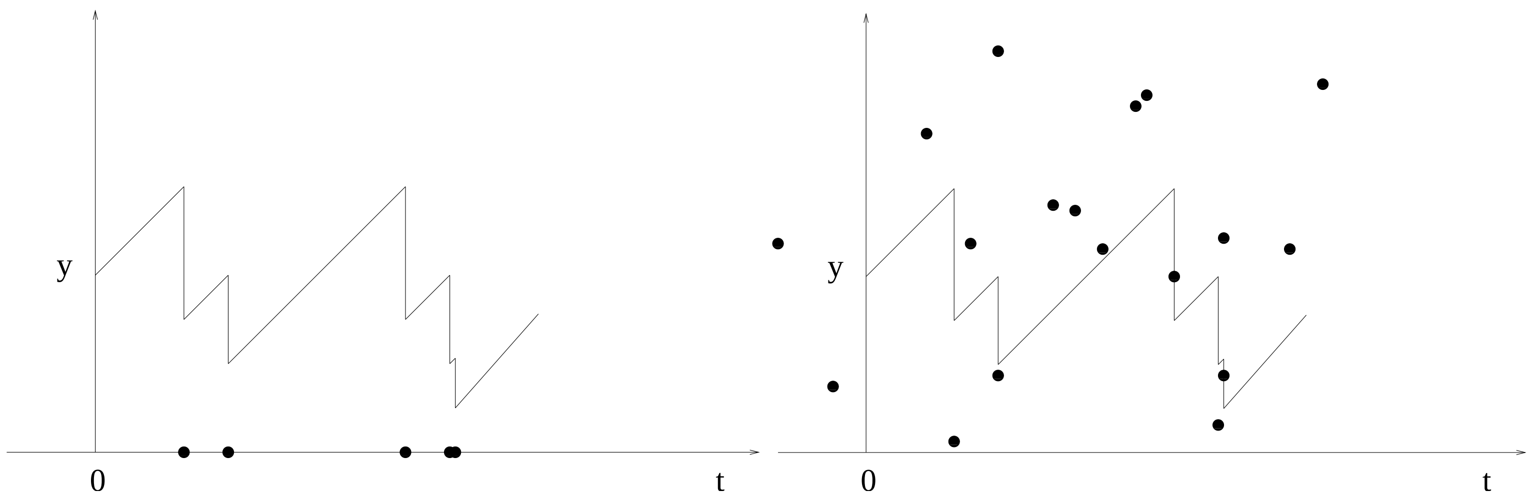
- Losses are modeled by two kinds of point processes:

1. rate independent (RI) case: homogeneous Poisson point process with intensity λ
2. rate dependent (RD) case: point process with a stochastic intensity $pX(t)$

- Rationale

1. RI: losses caused by physical layer events arising on wireless links (fast fading) or DSL links (impulse noise)
2. RD: PER (packet error rate) due to congestion or transmission errors.

Loss Point Processes (*continued*)



The Square Root Formula in 3 Lines - RI case

- If there exists a stationary regime with X integrable, the Rate Conservation Principle gives
$$\frac{1}{R^2} = \frac{\lambda}{2}\mathbb{E}_N^0[X(0-)]$$
with $\mathbb{E}_N^0$ the Palm probability of N;

- Pasta implies $\mathbb{E}_N^0[X(0-)] = \mathbb{E}[X(0)]$, which gives
$$\mathbb{E}[X(0)] = \frac{2}{\lambda R^2}$$

- The packet loss probability p is such that $p\mathbb{E}[X(0)] = \lambda$; Hence
$$\mathbb{E}[X(0)] = \sqrt{\frac{2}{pR^2}}.$$

RD Case

- If there exists a stationary regime where N has intensity λ and X is integrable, the Rate Conservation Principle gives

$$\frac{1}{R^2} = \frac{\lambda}{2}\mathbb{E}_N^0[X(0-)]$$

- From Papangelou's theorem $\mathbb{E}_N^0[X(0-)] = \mathbb{E}[X(0)\frac{pX(0)}{\lambda}]$ so that

$$\mathbb{E}[X(0)^2] = \frac{2}{pR^2}$$

- No simple identification of the mean TCP throughput.

Markov Analysis

- $X(t)$ is a Markov Process
  – with continuous time
  – with continuous state space
- It falls in the Piecewise Deterministic Process framework of Davis.
- The embedded chain (at "discontinuities") is geometrically ergodic.

Markov Analysis (*continued*)

- $\{X_n\}$: throughput sampled just after loss times, forms a discrete time Markov chain on $\mathbb{R}_+$.
- $\{X_n\}$ satisfies the AR-like equation

$$X_{n+1} = X_n/2 + W_{n+1}$$

and with $W_{n+1} = \frac{1}{2}\tau_{n+1}$

– RI: independent of $\{X_k\}_{k\leq n}$ and exponential

– RD: conditionally independent of $\{X_k\}_{k\leq n}$ given X_n (RD) with

$$\begin{aligned} P(\tau_{n+1} > t \mid X_n = x) &= \exp\left(-\int_0^t \alpha\,(x+u)\,du\right) \\ &= \exp\left(-\alpha\left(xt + \frac{t^2}{2}\right)\right), \end{aligned}$$

Markov Analysis (*continued*)

- ϕ_x: restriction of the Lebesgue measure to $[x, 2x]$.
- For all $x > 0$, $[0, x]$ is a petite set w.r.t. ϕ_x
- For all $x > 0$, $\{X_n\}$ is ϕ_x-irreducible
- $\{X_n\}$ is positive recurrent and geometrically ergodic so that one can construct jointly stationary and ergodic versions of X and N.

Analytical Results: Distributions - RI Case

- For all $u > 0$ for all continuity point of $X(t)$,

$$\dot{X}^u(t) = uX^{u-1}(t)\dot{X}(t) = uX^{u-1}(t)/R^2$$

so that

$$X^u(t) = X^u(0) + \frac{uX^{u-1}(t)}{R^2} - \left(1 - \frac{1}{2^u}\right)\int_0^t X^u(v-)N(dv)$$

Thus

$$M(t) = X^u(t) - X^u(0) - \frac{u}{R^2}\int_0^t X^{u-1}(v)dv + \lambda\left(1 - \frac{1}{2^u}\right)\int_0^t X^u(v-)dv$$

is a martingale s.t. $M(0) = 0$ so that whenever moments are finite

$$\frac{\partial}{\partial t}E[X^u(t)] = \frac{u}{R^2}E[X^{u-1}(t)] - \lambda\left(1 - \frac{1}{2^u}\right)E[X^u(t)].$$

Analytical Results: Distributions - RI Case (*continued*)

- Mellin transforms of the density of X at time t:

$$E[X^u(t)] = \int_0^\infty x^u f(t,z)dz = \widehat{f}_t(u+1).$$

- Functional equation:

$$\frac{\partial}{\partial t}\widehat{f}_t(u+1) = \frac{u}{R^2}\widehat{f}_t(u) - \lambda\left(1-\frac{1}{2^u}\right)\widehat{f}_t(u+1)$$

- PDE

$$\frac{\partial f(z,t)}{\partial t} + \frac{1}{R^2}\frac{\partial f(z,t)}{\partial x} + \lambda\left(f(z,t) - 2f\left(2z,t\right)\right) = 0$$

Analytical Results: Distributions - RI Case (*continued*)

- Stationary ODE:

$$\frac{df(z)}{dz} + \xi \left(f(z) - 2f\left(2z\right)\right) = 0$$

with $\xi = \lambda R^2$.

- Stationary functional equation:

$$u\widehat{f}(u) = \xi \left(1 - \frac{1}{2^u}\right) \widehat{f}(u+1)$$

- $\widehat{f}(u) = g(u)\Gamma(u)\xi^{-u}$. Then

$$g(u) = g(u+1)(1-2^{-u}), \quad i.e. \quad g(u) = g(\infty) \prod_{k \geq 0} (1 - 2^{-u-k}),$$

Analytical Results: Distributions - RI Case *(continued)*

- **Theorem**
  The unique stationary distribution solution of this functional equation has for Mellin transform

$$\widehat{f}(u) = \phi\Gamma(u)\xi^{-u}\prod_{k\geq 0}(1-2^{-u-k})$$

with $\xi = \lambda R^2$ and $\phi = \xi\left(\prod_{k\geq 1}(1-2^{-k})\right)^{-1}$.
The associated probability density is

$$f(z) = \phi\sum_{n\geq 0} b_n e^{-(\xi 2^n)z}$$

with $b_0 = 1$ and $b_n = (-1)^n \prod_{k=1}^{n} \frac{2}{(2^k-1)}$.

Distributions - RD

- Formal proof of PDE by the same martingale approach:

$$\frac{\partial f}{\partial t}(z,t) + \frac{1}{R^2}\frac{\partial f}{\partial z}(z,t) \;=\; p(4zf(2z,t) - zf(z,t)), \quad z \geq 0,$$

– mass leaves the interval $[z, z+dz]$ at rate $pzf(z,t)dz$ approximately.

– mass enters this interval because of losses among throughputs in the interval $[2z, 2(z+dz)]$ at rate $p2zf(2z,t) \cdot 2dz$

- Functional equation for the stationary Mellin transform of f:

$$u\widehat{f}(u) = \xi\widehat{f}(u+2)\left(1 - 2^{-u}\right).$$

with $\xi = pR^2$.

Distributions - RD (*continued*)

- Theorem The unique density satisfying the ODE is

$$f(z) = 2\phi \sum_{n\geq 0} a_n e^{-\left(\frac{\xi}{2}4^n\right)z^2}$$

with

$$\phi = \left(\sqrt{\pi}\left(\frac{2}{\xi}\right)^{\frac{1}{2}} \prod_{k\geq 1}(1-2^{-2k+1})\right)^{-1} \quad \text{and} \quad a_n = (-1)^n \prod_{k=1}^{n} \frac{4}{(4^k-1)}.$$

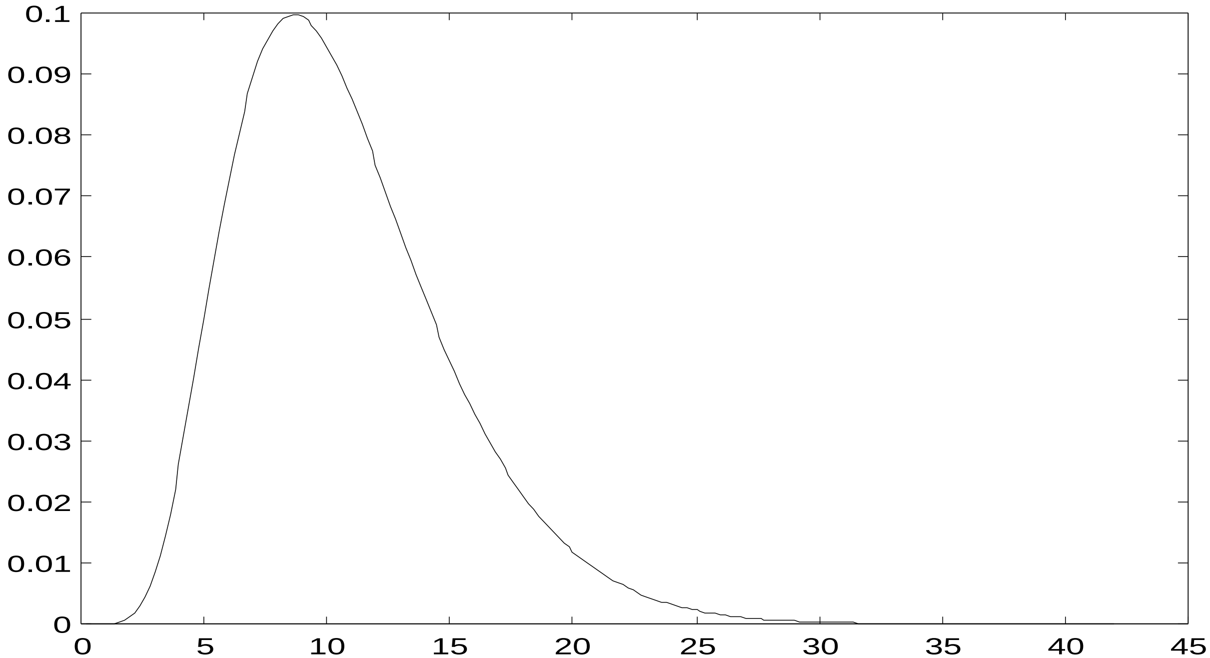
Its Mellin transform is

$$\widehat{f}(u) = \phi\Gamma\left(\frac{u}{2}\right)\left(\frac{2}{\xi}\right)^{\frac{u}{2}} \Pi_\infty(u), \quad \text{with} \quad \Pi(u) = \prod_{k=0}^{\infty}\left(1-2^{-u-2k}\right)$$

Its mean is

$$\mathbb{E}[X(0)] = \sqrt{\frac{2}{pR^2}}\sqrt{\frac{1}{\pi}}\frac{\Pi(2)}{\Pi(1)} \sim \frac{1.309}{R\sqrt{p}}.$$

Distributions - RD (*continued*)

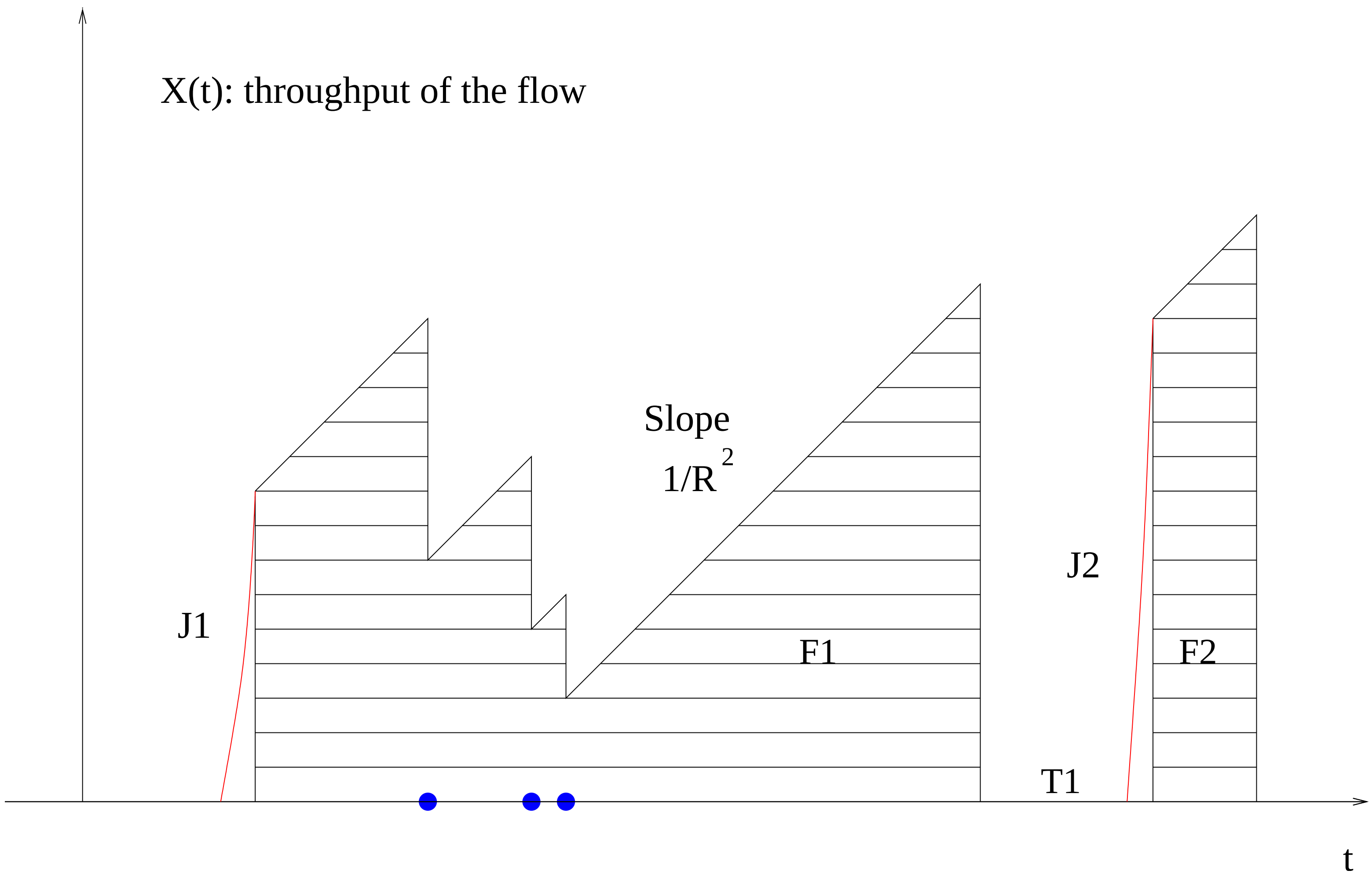


ON-OFF Flows

- On-Off TCP flow
  - PDE for the exponential Model
  - Moments
  - Distributions

Dynamics

- RD model with packet loss probability p and RTT R.
- The flow alternates between document downloads and think times, inducing an ON/OFF flow structure
  - Document sizes F_i are i.i.d. with mean $1/\mu$
  - Think times T_i are i.i.d. with mean $1/\beta$
- Motivation: HTTP 1.1 where the files successively downloaded by a flow use the same TCP-Reno connection:
  - Slow Start jump approximation: jumps J_i are i.i.d.



$N(dt)$: packet loss point process, with a stochastic intensity $pX(t)$.

Exponential Model

- File sizes are exponential with parameter μ
- Think times are exponential with parameter β
- Slow start jump is a bounded random variable with law H (for example with density h).

- $X(t)$ is a Markov Process
  - with continuous time
  - with continuous state space (with an atom)
- It falls in the Piecewise Deterministic Process framework of Davis.
- The embedded chain (at "discontinuities") is geometrically ergodic.

PDE

- $(s(z,t),\nu(t))$ distribution of throughput at time t:
- Non local PDE (e.g. by a martingale argument based on the fact that N admits a stochastic intensity)

$$\frac{\partial s}{\partial t}(z,t)+\frac{1}{R^2}\frac{\partial s}{\partial z}(z,t) \;=\; \beta\nu(t)h(z)-\mu z s(z,t)+4zps(2z,t)-zps(z,t)$$

- Boundary:

$$\frac{d\nu}{dt}(t)=\int_0^\infty \mu z s(z,t)dz-\beta\nu(t).$$

- Normalization:

$$\int_0^\infty s(z,t)dz=1-\nu(t).$$

Non Local ODE for Stationary Regime

$$\begin{aligned}\frac{ds(z)}{dz} &= \beta\nu R^2 h(z) - \mu R^2 z s(z) + 4zpR^2 s(2z) - zpR^2 s(z)\\ &= \mu R^2 \int_0^\infty v s(v) dv h(z) - \mu R^2 z s(z) + 4pzR^2 s(2z) - zpR^2 s(z)\end{aligned}$$

since

$$\int_0^\infty \mu z s(z) dz = \beta\nu.$$

Moments

- $\mathbb{E}T$: mean time to transfer a file
- $\mathbb{E}X(0)$: mean stationary throughput
- Cycle formula (thanks to the regenerative structure):

$$\mathbb{E}X(0) = \frac{\frac{1}{\mu}}{\frac{1}{\beta} + \mathbb{E}T}$$

- Probability that a flow is OFF : $\nu = \mathbb{E}X(0)\frac{\mu}{\beta}$

No Slow Start

- Theorem (F.B., D. Mc Donald 05)

  – The mean time to transfer a file is

$$\mathbb{E}T = \frac{1}{\mu}\sqrt{\frac{\pi}{2}}R\frac{\prod_{l=1}^{\infty}\left(1-\frac{2p}{p+\mu}4^{-l}\right)}{\prod_{l=1}^{\infty}\left(1-\frac{p}{p+\mu}4^{-l}\right)}\sqrt{p+\mu}$$

  – The mean stationary throughput is

$$\mathbb{E}X(0) = \frac{1}{\frac{\mu}{\beta}+\sqrt{\frac{\pi}{2}}R\frac{\prod_{l=1}^{\infty}\left(1-\frac{2p}{p+\mu}4^{-l}\right)}{\prod_{l=1}^{\infty}\left(1-\frac{p}{p+\mu}4^{-l}\right)}\sqrt{p+\mu}}.$$

No Slow Start (*continued*)

– The density $s(z)$ of the throughput at $z > 0$ is

$$s(z) = \frac{\widehat{s}(2)}{\prod_{l=1}^{\infty}(1 - \frac{p}{p+\mu}2^{-2l})}(p+\mu)R^2 \sum_{n\geq 0} a_n e^{-(\frac{(p+\mu)R^2}{2}4^n)z^2}$$

with $\{a_n\}$ the coefficients of the analytic expansion

$$\prod_{l=0}^{\infty}\left(1 - \frac{p}{p+\mu}2^{-2l}x\right) = \sum_n a_n x^n.$$

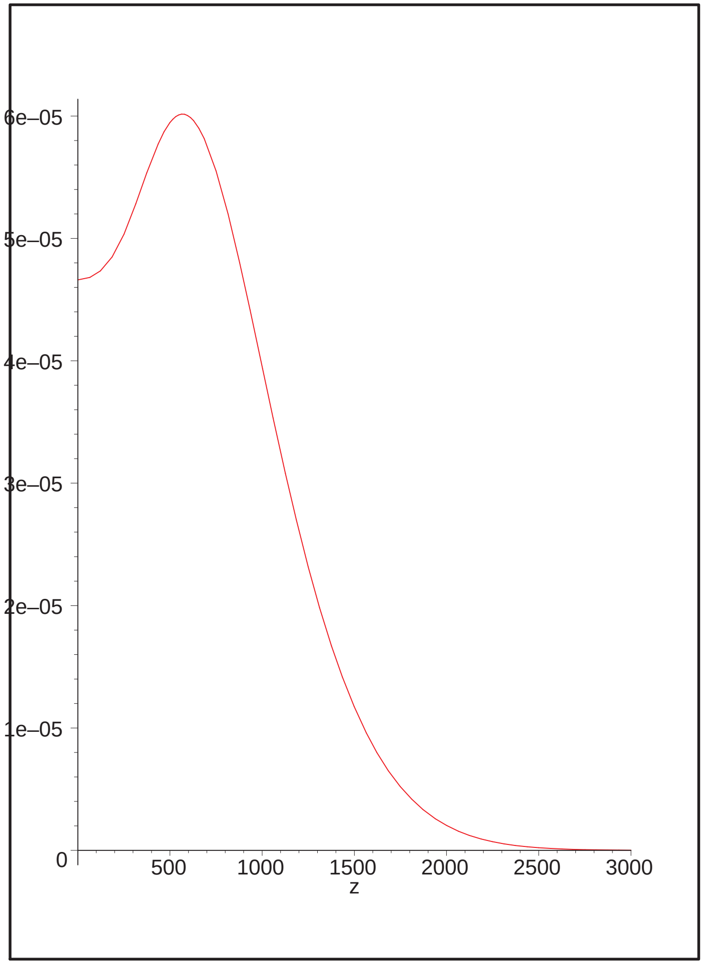


Figure 1: The density of the stationary throughput. Here $R = 0.1$ s., $1/\beta = 2$ s., $1/\mu = 100$ and $p = 1\%$.

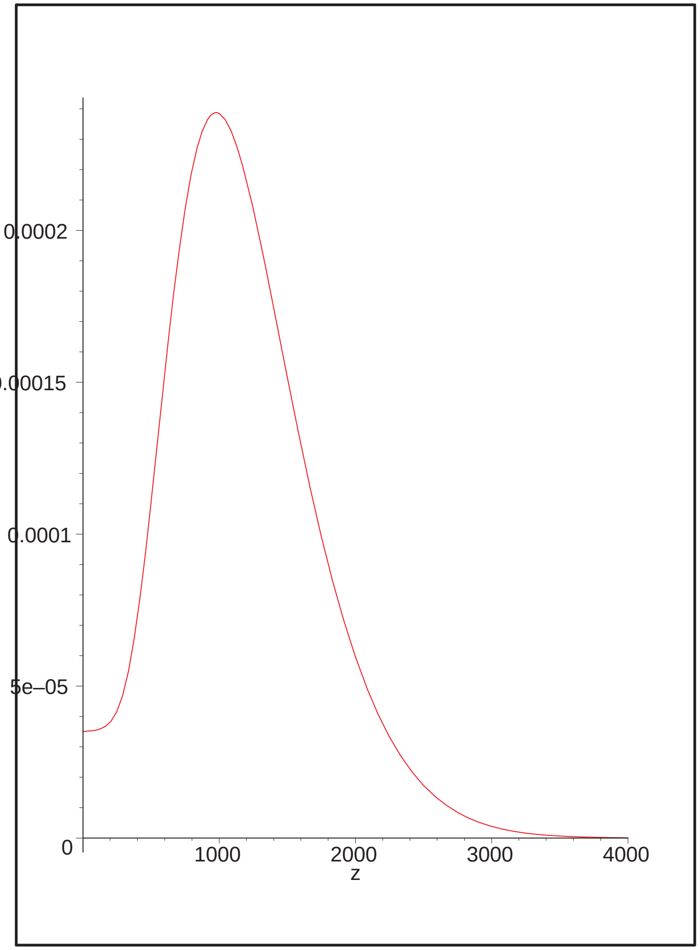


Figure 2: The density of the stationary throughput Here $R = 0.1$ s., $1/\beta = 2$ s., $1/\mu = 1000$ and $p = 1\%$.

With Slow Start

- Mean time to transfer a file:

$$
\begin{aligned}
\mathbb{E}T &= \frac{1}{\mu}\sqrt{\frac{\pi}{2}}R\frac{\Pi_\infty(1)}{\Pi_\infty(2)}\sqrt{p+\mu} \\
&+ \frac{\sqrt{\pi}R^2}{2}\sum_{k=0}^{\infty}\left(\Pi_k(2)\frac{\Pi_\infty(1)}{\Pi_\infty(2)}\widehat{h}(2k+3)\frac{\left(\frac{(p+\mu)R^2}{2}\right)^{k+\frac{1}{2}}}{(k+1)!}\right. \\
&\qquad \left. -\Pi_k(1)\widehat{h}(2k+2)\frac{\left(\frac{(p+\mu)R^2}{2}\right)^{k}}{\Gamma(k+\frac{3}{2})}\right).
\end{aligned}
$$

with

$$
\Pi_k(u) = \prod_{l=0}^{k-1}\left(1 - \frac{p}{p+\mu}2^{-u-2l}\right).
$$

Proof

- Functional equation:

$$u\widehat{s}(u) \;=\; pR^2\widehat{s}(u+2)\left(1-2^{-u}\right)+\mu R^2\widehat{s}(u+2)-\mu R^2\widehat{s}(2)\widehat{h}(u+1).$$

  Let

$$\widehat{s}(u) = f(u)\Gamma(\frac{u}{2})\left(\frac{2}{(p+\mu)R^2}\right)^{\frac{u}{2}}.$$

Then

$$f(u) \;=\; f(u+2)(1-\frac{p}{p+\mu}2^{-u})-\frac{\mu R^2}{2}\widehat{s}(2)\widehat{h}(u+1)\frac{\left(\frac{(p+\mu)R^2}{2}\right)^{\frac{u}{2}}}{\Gamma(\frac{u}{2}+1)}$$

Proof (*continued*)

Which implies

$$f(u) \;=\; f(\infty)\Pi_\infty(u) - \frac{\mu R^2 \widehat{s}(2)}{2} \sum_{k=0}^{\infty} \Pi_k(u)\widehat{h}(u+2k+1) \frac{\left((p+\mu)R^2/2\right)^{\frac{u}{2}+k}}{\Gamma(\frac{u}{2}+k+1)}$$

$$\widehat{s}(u) \;=\; f(\infty) \frac{\Gamma(\frac{u}{2})}{\left(\frac{(p+\mu)R^2}{2}\right)^{\frac{u}{2}}} \Pi_\infty(u) - \frac{\mu R^2 \widehat{s}(2)}{2} \sum_{k=0}^{\infty} \Pi_k(u)\widehat{h}(u+2k+1) \left(\frac{(p+\mu)R^2}{2}\right)^k \frac{\Gamma(\frac{u}{2})}{\Gamma(\frac{u}{2}+k+1)}.$$

- Specializing to $u = 2$, we get a first linear relation between $\widehat{s}(2)$ and $f(\infty)$
- Specializing to $u = 1$ and using normalization, we get a second independent linear relation between $\widehat{s}(2)$ and $f(\infty)$

Distribution of the Throughput - Slow Start Case

- Methodology:

1. Solve first an auxiliary Rate Independent Equation
2. Express the solution of the original ODE in function of that of the RI Equation.

A More General Class of ODEs

We consider the more general equation

$$\frac{df(z)}{dz} = \delta T(f)A(z) - \mu z^{\gamma-1} f(z) + \beta z^{\gamma-1}(\rho^\gamma f(\rho z) - f(z)), \quad z \geq 0,$$

where

$$T(f) = \int_0^\infty z^{\gamma-1} f(z) dz,$$

$\delta \geq \mu$,
A is a probability density function,
$\gamma \geq 1$,
$\rho > 1$,
$\beta \geq 0$,
$\mu \geq 0$,
$\mu + \beta > 0$.

RD Special Case

$\gamma = 2,$
$\rho = 2,$
$pR^2 \longrightarrow \beta,$
$\mu R^2 \longrightarrow \delta,$
$h(z) \longrightarrow A(z),$
$\mu R^2 \longrightarrow \mu$

- One gets back the initial RD ODE

$$\frac{ds(z)}{dz} = \mu R^2 \int_0^\infty vs(v)dvh(z) - \mu R^2 zs(z) + 4pzR^2 s(2z) - zpR^2 s(z)$$

RI Special Case

$\gamma = 1,$
$\rho \to \theta,$
$\lambda R^2 \to \beta,$
$\mu R^2 \to \delta,$
$h(z) \to A(z),$
$\mu R^2 \to \mu$

- One gets the RI ODE

$$\frac{df(z)}{dz} = \mu R^2 h(z) - \mu R^2 f(z) + \lambda R^2(\theta f(\theta z) - f(z)), \quad z \geq 0,$$

which represents the AIMD on-off dynamics when

– losses occur according to a Poisson point process of intensity λ, leading to a division of the throughput by θ;

– file lifetime (on-time) is exponentially distributed with parameter μ.

Analysis of the RI Equation

$$\frac{df(z)}{dz} = \delta A(z) - \mu f(z) + \beta(\theta f(\theta z) - f(z)), \quad z \geq 0,$$

- Laplace transform $\widetilde{f}(s)$:

$$s\widetilde{f}(s) - f(0) \; = \; \delta\widetilde{A}(s) - \mu\widetilde{f}(s) \; + \; \beta\left(\widetilde{f}\left(\frac{s}{\theta}\right) - \widetilde{f}(s)\right);$$

that is

$$\widetilde{f}(s) \; = \; \frac{f(0) + \delta\widetilde{A}(s)}{\mu + \beta + s} \; + \; \frac{\beta}{\mu + \beta + s}\widetilde{f}\left(\frac{s}{\theta}\right)$$

Analysis of the RI Equation (*continued*)

By iteration it follows that

$$\widetilde{f}(s) = \sum_{n=0}^{N} \frac{\beta^n}{\prod_{k=0}^{n}(\mu+\beta+s/\theta^k)} \cdot \left(f(0) + \delta \widetilde{A}\left(\frac{s}{\theta^n}\right)\right)$$

$$+\prod_{n=0}^{N}\left(\frac{\beta}{\mu+\beta+s/\theta^k}\right)\widetilde{f}\left(\frac{s}{\theta^{N+1}}\right).$$

Letting N go to infinity, we get that the solution is necessarily

$$\widetilde{f}(s) = \sum_{n\geq 0} \frac{\beta^n}{\prod_{k=0}^{n}(\mu+\beta+s/\theta^k)} \left(f(0) + \delta \widetilde{A}\left(\frac{s}{\theta^n}\right)\right)$$

$$= \sum_{n\geq 0}\left(1-\frac{\delta}{\mu}+\frac{\delta}{\mu}\widetilde{A}\left(\frac{s}{\theta^n}\right)\right)\left(1-\frac{\beta}{\mu+\beta}\right)\left(\frac{\beta}{\mu+\beta}\right)^n\prod_{k=0}^{n}\frac{\mu+\beta}{\mu+\beta+s/\theta^k}$$

which has a probabilistic interpretation in terms of the AR process.

General ODE

- Theorem (F.B., K.B. Kim & D. Mc Donald 06)
  Assume that $\gamma \geq 1$, $\rho > 1$, $\mu \geq \delta \geq 0$, $\beta \geq 0$, $\mu + \beta > 0$.
  Let $\theta = \rho^\gamma$ and let A be a density such that

$$\int_0^\infty A(z) e^{\frac{(\mu+\beta)}{\gamma} z^\gamma} dz \, < \, \infty.$$

  Then the unique density solution to the ODE is the function

$$f(z) \, = \, \frac{1}{C\gamma} \sum_{n \geq 0} \left(\frac{\beta}{\mu + \beta} \right)^n b_n d_n(z) e^{-\left(\frac{\beta+\mu}{\gamma}\right) \theta^n z^\gamma}$$

General ODE (*continued*)

with the b_n's the coefficients of the expansion

$$\prod_{k\geq 0}(1-\theta^{-k}x) \;=\; \sum_{n\geq 0} b_n x^n.$$

i.e.

$$b_n \;=\; (-1)^n \prod_{k=1}^{n} \frac{\theta}{(\theta^k - 1)}$$

with

$$d_n(z) \;=\; \sum_{m\geq 0} c_m \left(\delta \int_0^{z\theta^{\frac{(n+m)}{\gamma}}} A(x) e^{\frac{\mu+\beta}{\gamma}\theta^{-m}x^{\gamma}} dx + (\mu - \delta) \right)$$

$$c_m \;=\; \left(\frac{\beta}{\mu+\beta}\right)^m \prod_{i=1}^{m} \frac{1}{1-\theta^{-i}}$$

and with C the constant which normalizes f (known in closed form).

PROOF

Let A be some density. Then $\frac{A(z^{1/\gamma})}{\gamma z^{1-1/\gamma}}$ is a density too.
Let k be the density solution of

$$\frac{dk}{dz}(z) = \frac{1}{\gamma}\left(\left(\frac{A(z^{1/\gamma})}{\gamma z^{1-1/\gamma}}\right) - \mu k(z) + \beta(\theta k(\theta z) - k(z))\right).$$

where we assume $k(0) = \mu - \delta \geq 0$.
Let $f(z) = C^{-1}k(z^\gamma)$, where C normalizes f to a density, and $\rho^\gamma = \theta$. Then

$$\begin{aligned}\frac{df(z)}{dz} &= C^{-1}\delta\gamma^{-1}A(z) - \mu z^{\gamma-1}f(z) + \beta z^{\gamma-1}\left(\rho^\gamma f(\rho z) - f(z)\right)\\ &= \delta T(f)A(z) - \mu z^{\gamma-1}f(z) + \beta z^{\gamma-1}\left(\rho^\gamma f(\rho z) - f(z)\right),\end{aligned}$$

where we used

$$T(f) = \int_0^\infty z^{\gamma-1}C^{-1}h(z^\gamma)dz = \frac{1}{C\gamma}.$$

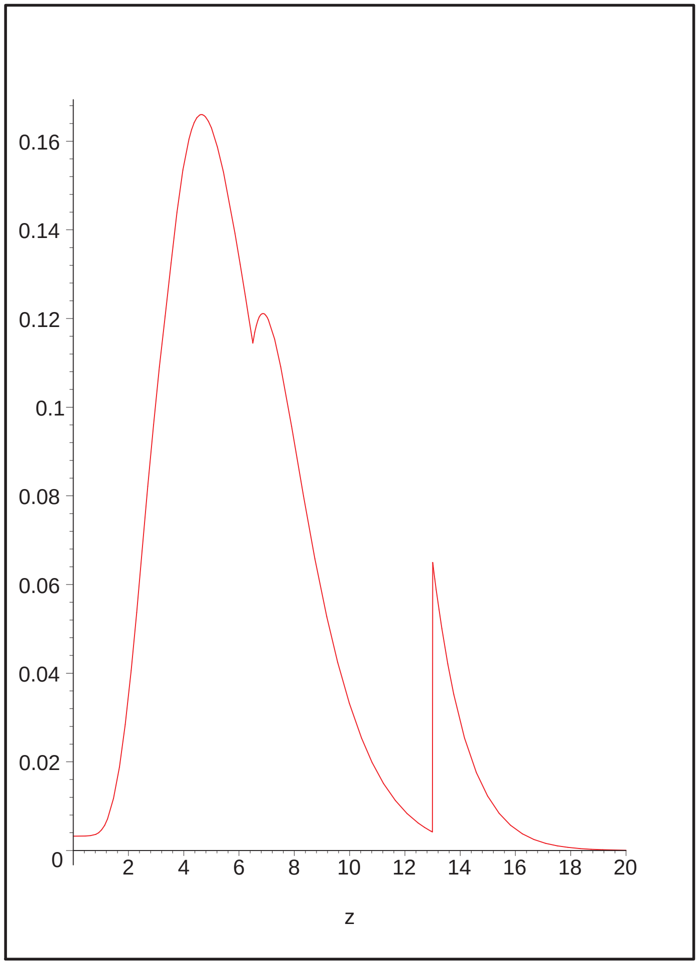


Figure 3: The density of the stationary throughput conditioned by the fact that the flow is ON. Here $R = 1$ s., $1/\mu = 100$, the loss probability is $p = 5/100$ and $h = \delta_\eta$ with $\eta = 13$.

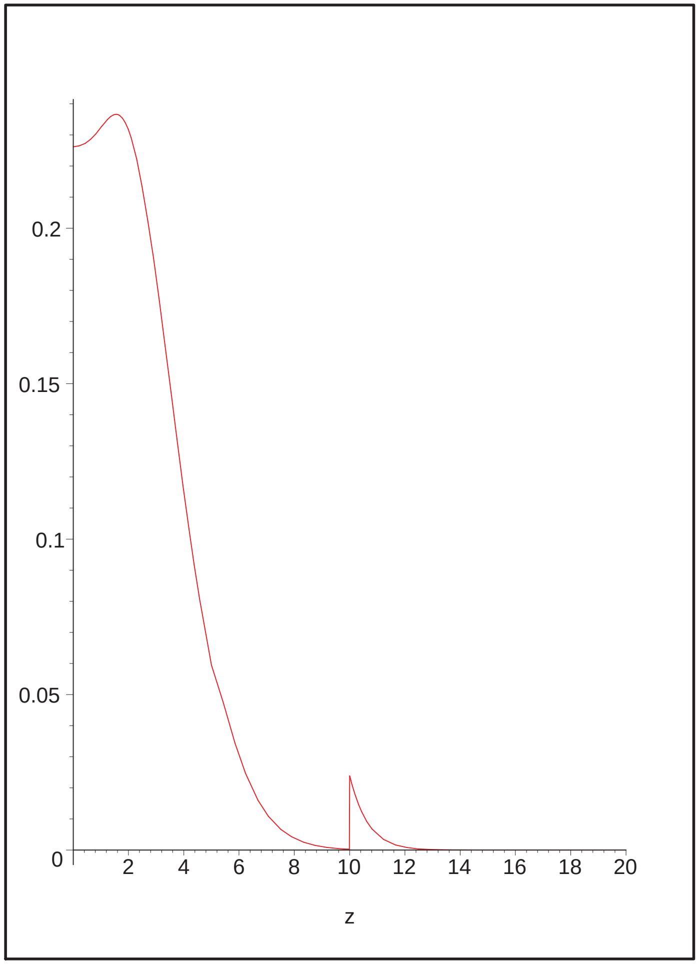
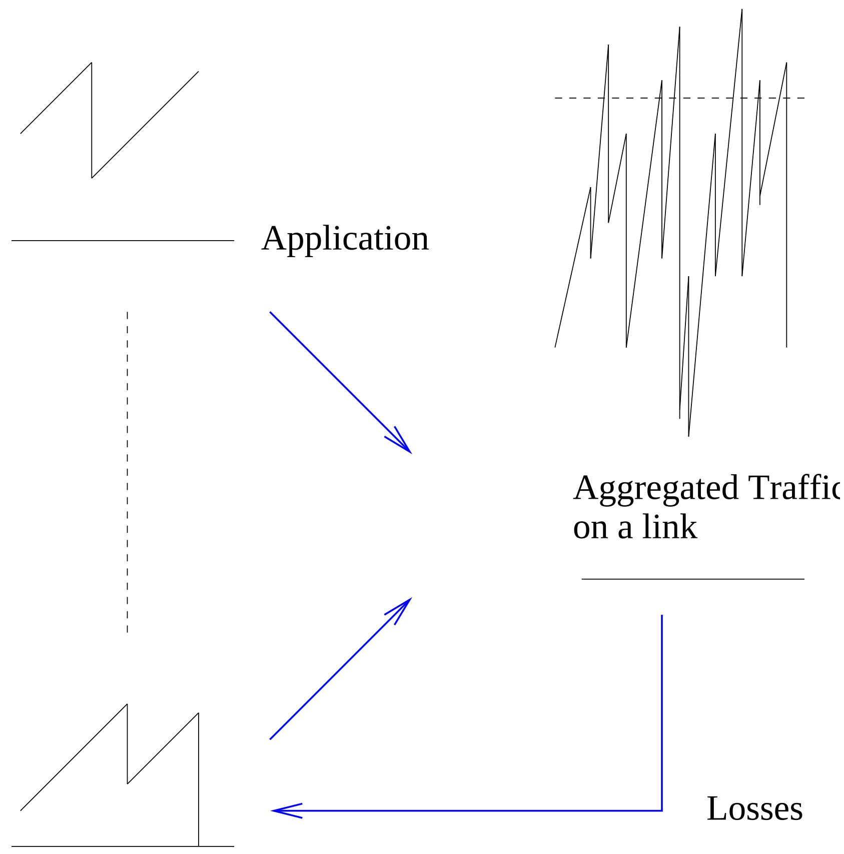


Figure 4: The density of the stationary throughput conditioned by the fact that the flow is ON. Here $R = 1$ s., $1/\mu = 5$ and $p = 5\%$ and $\eta = 13$.

Flow Competition

- The instantaneous throughput of a flow depends on losses, which result from the competition with other flows sharing the same links.
- Proof of the mean field by Mc Donald & Reynier 06 in the AQM setting.



Mean Field Fixed Points for AQM Bandwidth Sharing

- N statistically identical ON-OFF flows with parameters (μ, β, h, R) that share a common AQM link with capacity $C \cdot N$.
- Let

$$\rho = \frac{1/\mu}{\frac{1}{\beta} + \mathbb{E}T}$$

with

$$\mathbb{E}T = \int_{z=0}^{\infty} \int_{u=0}^{\infty} h(z) \mu e^{-\mu u} \left(\sqrt{z^2 R^4 + 2uR^2} - zR^2 \right) du dz$$

mean value of the throughput of one flow in the absence of packet loss

The Two Mean Field Regimes

- The stabilized congestion regime : when $\rho > C$, a positive drop probability is required to match the load brought by the flows and the capacity of the link. The system stabilizes to a constant buffer content b, to a constant packet loss probability p and to a mean throughput per flow $\overline{X}[p]$, s.t.

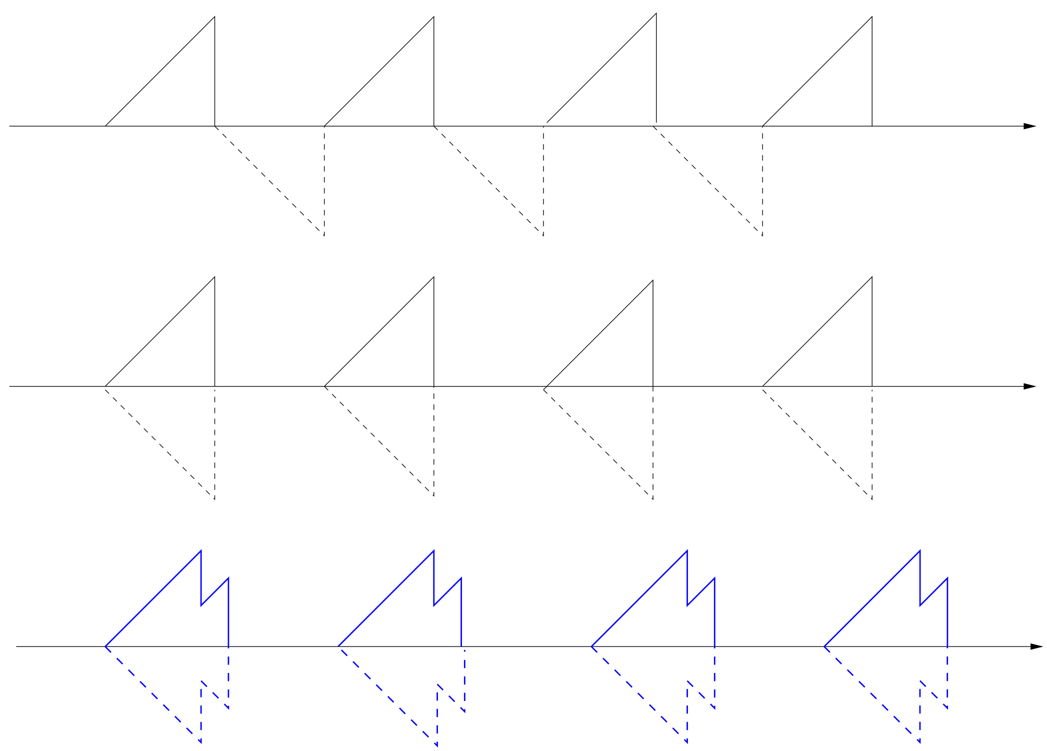
$$\overline{X}[p](1-p) = C$$

Since the function $p \to \overline{X}[p](1-p)$ is decreasing in p and tends to ρ when p tends to 0, the above equation defines a unique equilibrium point p^* whenever $\rho > C$.

- The congestion-less regime: when $\rho < C$; the load brought by the flows is less than the link rate, and each flow gets a mean throughput of ρ.

- Other and in particular oscillating regimes are possible.

In Phase and Out of Phase AIMD, On-Off Flows in Parallel



Interaction of On-Off TCP Flows on a TD Link

– N homogeneous HTTP users share a link of capacity NC

– Each HTTP user alternates between document downloads and think times, inducing an ON/OFF flow structure

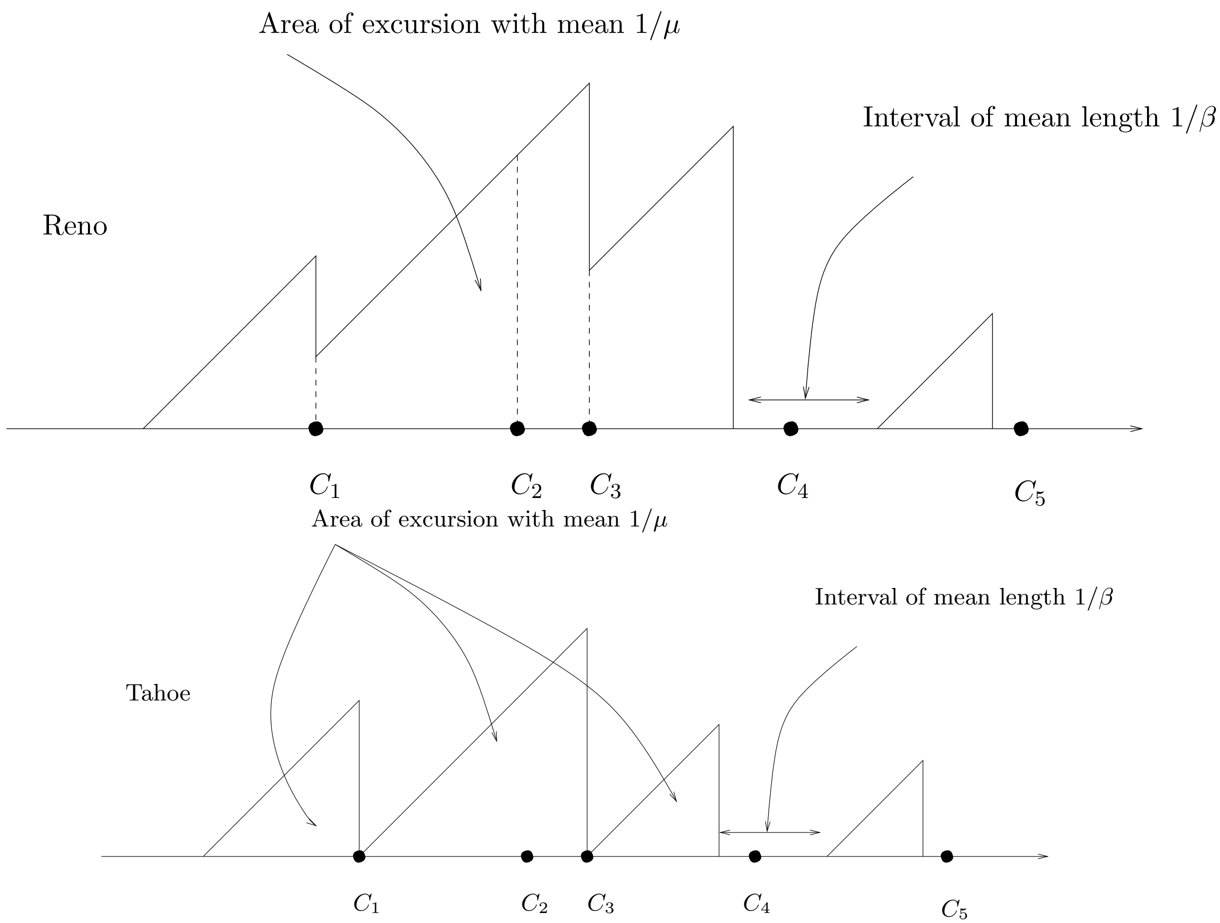
– Document sizes are i.i.d. with mean $1/\mu$

– Think times are i.i.d. $1/\beta$

– All connections have the same RTT R

– Congestion takes place as soon as the sum of the rates is equal to or exceeds NC

– Congestions result in an instantaneous halving of rate for a proportion p of the flows (synchronization rate).



Sample paths of the rate $X_n(t)$ of flow n

Mean Field Limit

- We let the population parameter go to ∞
- We analyze

– the limiting aggregated rate

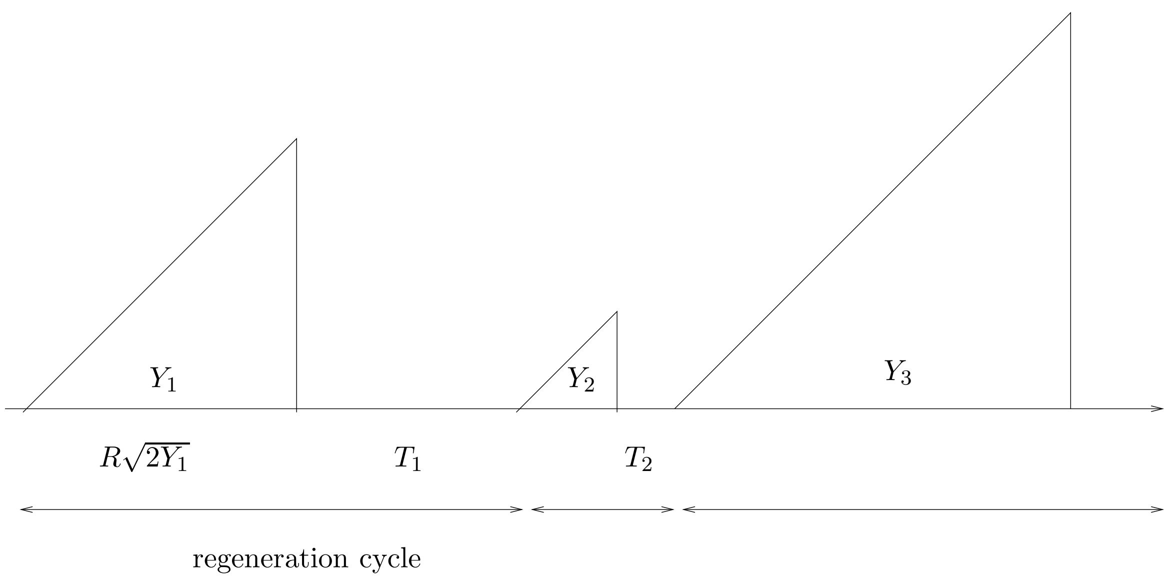
$$\alpha(t) = \lim_N \frac{1}{N} \sum X_n^N(t)$$

– the limiting distribution of rates

$$s(dz,t) = \lim_N \frac{1}{N} \sum \delta_{X_n^N(t) \in dz}$$

The Free Regime Mean Field Limit ($C = \infty$)

$\{Y_i\}$ i.i.d. sequence of file sizes for tagged flow
$\{T_i\}$ i.i.d. sequence of think times for tagged flow
The rate $X(t)$ of the tagged flow is a regenerative process.



The Free Regime Mean Field Limit ($C = \infty$) (*continued*)

- The mean field aggregated rate

$$\alpha(t) = \lim_N \frac{1}{N} \sum X_n(t)$$

exists and is deterministic as well as

– $s(z, t)$ the proportion of flows active and with rate z at time t

– $\nu(t)$ the proportion of flows inactive at time t

The Free Regime Mean Field Limit ($C = \infty$) (*continued*)

- Example of analytical characterization in the exponential Y and T case: PDE for the congestion-less mean field measure:

$$
\begin{aligned}
\frac{\partial}{\partial t} s(z,t) + \frac{1}{R^2}\frac{\partial}{\partial z} s(z,t) &= -\mu z s(z,t) \\
\frac{d}{dt}\nu(t) &= -\beta\nu(t) + \mu \int_0^\infty z s(z,t) dz
\end{aligned}
$$

with

$$
s(0,t)/R^2 = \beta\nu(t) \qquad \text{and} \qquad \int_0^\infty s(z,t)dz = 1 - \nu(t).
$$

- Explicit solution via either Laplace transform in time or Renewal theory.

The Free Regime Mean Field Limit ($C = \infty$) (*continued*)

- Results via Laplace transform in time and Tauberian theorems

$$
\begin{aligned}
\nu(\infty) &= \frac{\frac{1}{\beta}}{\frac{1}{\beta} + R\sqrt{\frac{\pi}{2\mu}}} \\
s(z,\infty) &= \frac{R^2 e^{-R^2\mu z^2/2}}{\frac{1}{\beta} + R\sqrt{\frac{\pi}{2\mu}}} \\
\alpha(\infty) &= \frac{1}{\mu}\frac{1}{\frac{1}{\beta} + R\sqrt{\frac{\pi}{2\mu}}} = \rho.
\end{aligned}
$$

- ρ: load per user in the steady state congestion-less regime.

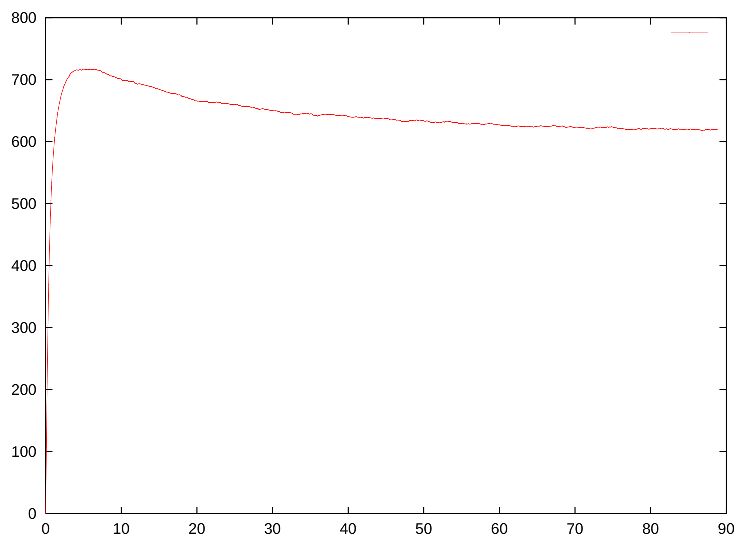
The Free Regime Mean Field Limit ($C = \infty$) (*continued*)

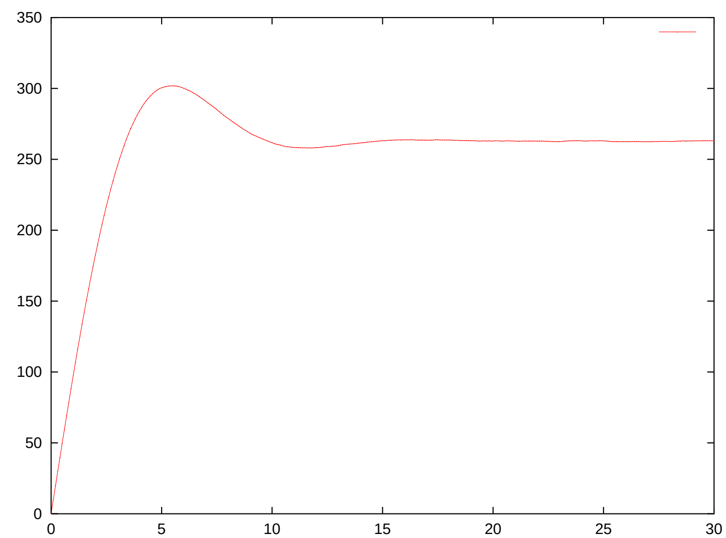
- Reduction to a Fredholm integral equation of the second kind

$$s(z,t)=s(z-\frac{t}{R^2},0)\ e^{-\mu\left(tz-\frac{t^2}{2R^2}\right)}+e^{-\mu R^2\frac{z^2}{2}}R^2\beta\left(1-\int_0^\infty s(x,t-zR^2)dx\right)$$

- Handy for numerical exploitation: reduction to linear matrix equations after discretization.

Examples of Aggregated Rates

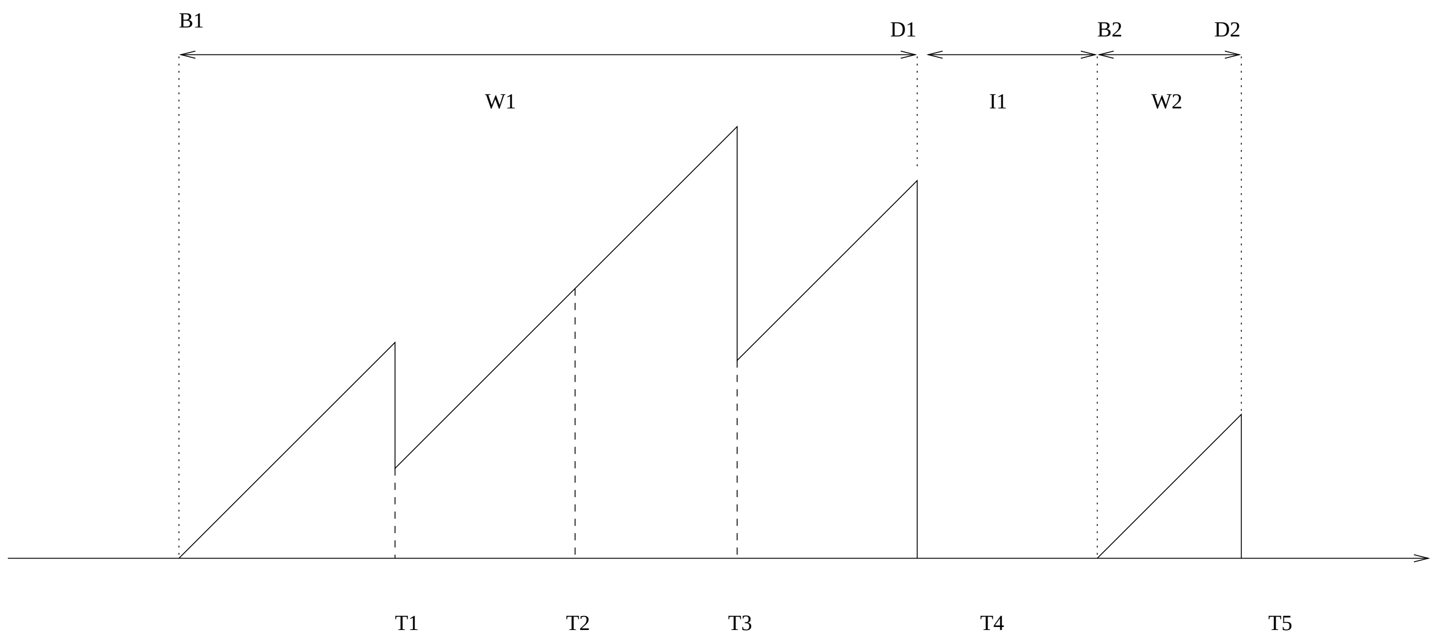




All flows are initially active and with 0 rate;
Mean file size 2000; mean think time 2 sec.
Left: lognormal with standard deviation 4 times the mean and RTT 30 ms.
Right: exponential with RTT 100 ms.

The Finite Capacity Mean Field Limit ($C < \infty$)

- For all finite C, there exists a deterministic mean-field limit with a sequence of intercongestion times τ_1, τ_2, ... (finite or not).
- Proof of D. Mc Donald & J. Reynier based on the method of T. Kurtz "Bring Back the Particles" .



The Finite Capacity Mean Field Limit ($C < \infty$) (*continued*)

- If one of the τ_i's is infinite, the stationary mean field limit for C is an interaction-less regime (similar to the free regime);
- If all τ_i's are finite, the stationary mean field limit for C is an interaction regime; of special interest: periodic interaction regimes with $\tau_i = \tau$.

Necessary Condition for a Periodic Congestion Interaction Mean Field Limit Regime

- Necessary condition for the existence of a periodic interaction mean-field regime with intercongestion time $\tau < \infty$: τ should solve the Rate Conservation Principle equation:

$$\frac{\mathbb{P}(X(0) > 0)}{R^2} = \frac{pC}{2\tau} + \lambda_\delta \mathbb{E}_0^\delta[X(0^-)].$$

- In the exponential case, all terms in this fixed point equation are computable thanks to the regenerative structure.
- Regeneration when tagged flow is inactive at a congestion epoch.

Example of Computation: Mean Number of Files Transmitted in a Regeneration Cycle

- $f(z)$: expected number of files that will be transmitted by the end of the current cycle given that the tagged flow is inactive at time $0 \leq z < \tau$.
- $g(v)$: expected number of files that will be transmitted by the end of the current cycle given that the tagged flow is active at some congestion epoch and that the current transmission rate of the tagged flow is v.

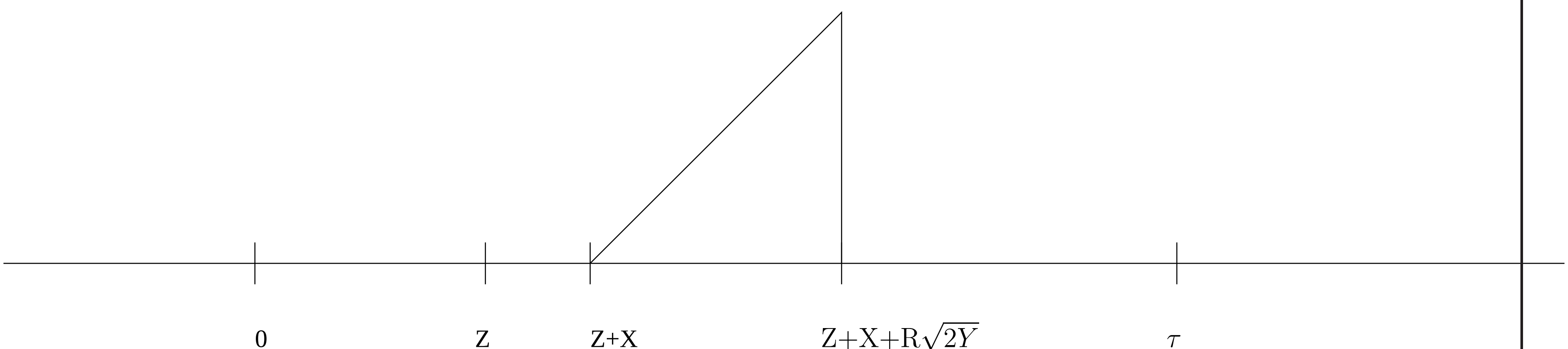
$$E(N) = f(0)$$

- Fredholm type equation for (f, g).

Example of Fredholm Equations for $f(.)$

- Case 0: no file transmitted within the end of the current cycle: contributes 0;
- Case 1: at least one file transmitted and we can transmit this file before the next congestion epoch. Then the expected number of files to transfer until end of cycle is:

$$\int_0^{\tau-z} \beta e^{-\beta x} dx \int_0^{(\tau-z-x)^2/2R^2} \mu e^{-\mu y} dy (1 + f(z + x + R\sqrt{2y}).$$



Example of Fredholm Equations for $f(.)$ (*continued*)

- Case 2: same as 1 but the file cannot be transmitted by the the next congestion epoch.
- This occurs with probability $\exp(-\mu(\tau - z - x)^2/2R^2)$.
- By the congestion epoch the transmission rate of the tagged flow is $(\tau - z - x)/R^2$.
- Then the expected number of files to transfer until end of cycle is:

$$\exp(-\mu(\tau - z - x)^2/2R^2)\left(pg(\frac{\tau - z - x}{2R^2}) + (1-p)g(\frac{\tau - z - x}{R^2})\right).$$

Example of Fredholm Equations for $f(.)$ (*continued*)

- We conclude

$$f(z) = \int_0^{\tau-z} \beta e^{-\beta x} dx \int_0^{(\tau-z-x)^2/2R^2} \mu e^{-\mu y} dy (1 + f(z + x + R\sqrt{2y})$$
$$+ \exp(-\mu(\tau - z - x)^2/2R^2) \left(pg(\frac{\tau - z - x}{2R^2}) + (1-p)g(\frac{\tau - z - x}{R^2}) \right).$$

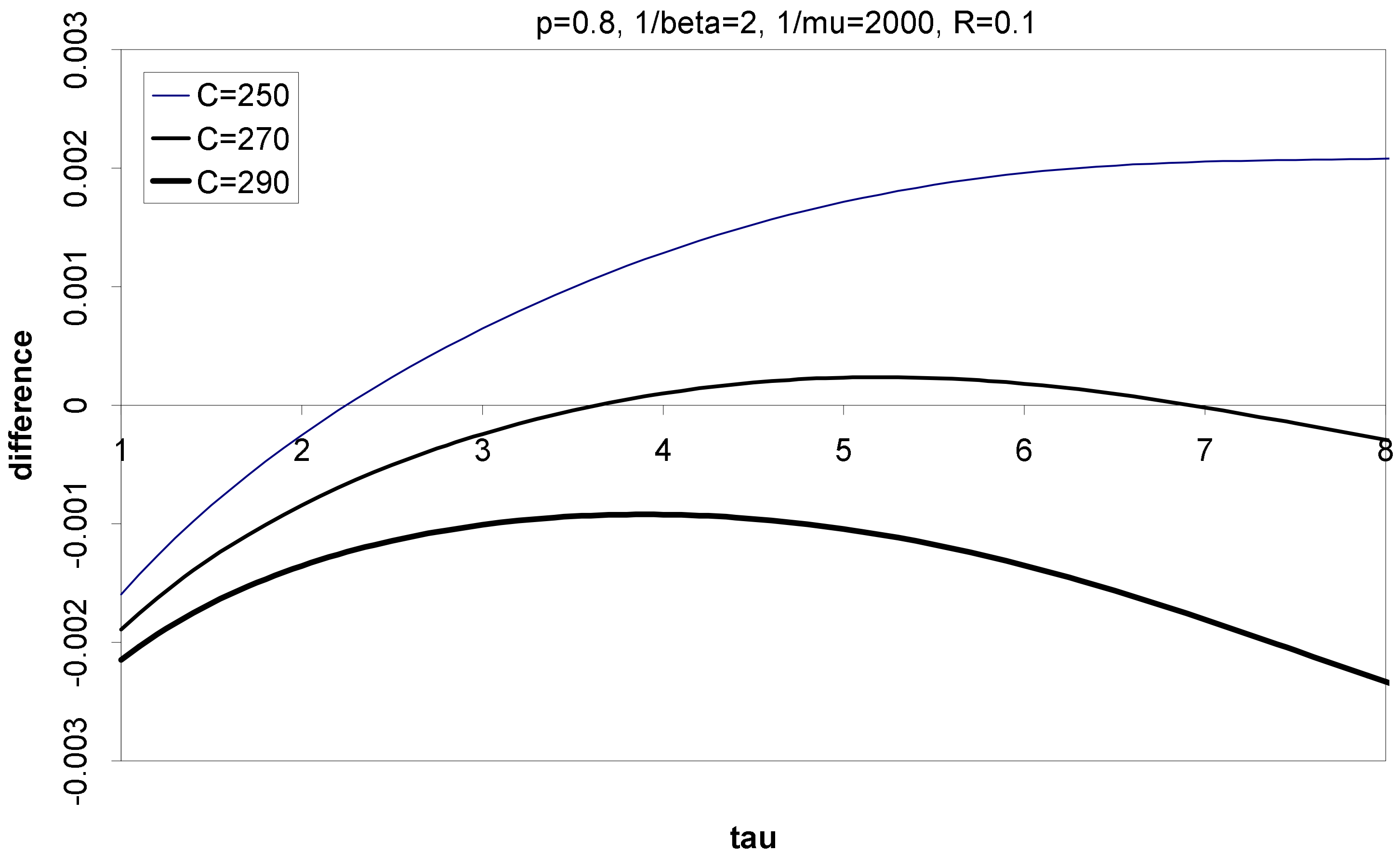


Figure 5: The 3 Cases for the Rate Conservation Principle Fixed Point Equation

Congestion Regime: the Invariant Measure Equation

– $s_0(z)$ the proportion of flows active and with rate z at a congestion epoch

– ν_0 the proportion of flows inactive at a congestion epoch

- Invariant measure equation for τ:

$$s_0(z) = (1-p)s_0(z,\tau) + ps_0(2z,\tau),$$

where $s_0(z,t)$ is the solution of the congestion-less PDE with the initial condition s_0.

Congestion Regime: the Invariant Measure Equation (*continued*)

- The existence of a "good" solution to the invariant measure equation, i.e. of a probability measure $(\nu_0, s_0(z))$
  - solution of the invariant measure equation for τ
  - such that the

$$\alpha_0(\tau) = C \quad \text{and} \quad \alpha_0(t) < C \quad \text{for all} \quad t < C$$

is necessary and sufficient for the existence of a congestion periodic regime of period τ

- The time average mean rate of a flow and the time average rate distribution of a flow can be expressed from this (cycle formula).

Multiplicity of Stationary Mean Field Regimes

- If $\rho > C$, the congestion-less regime is impossible.
- Main Finding (proved in the Tahoe case, numerical evidence in the Reno case):
  1. The condition $\rho < C$ is not sufficient for having an interaction-less mean-field regime only
  2. There exist values of C such that depending on the initial condition, one enters either in an interaction-less or in an interaction stationary regime.

HTTP Turbulence

- We call turbulent regime the periodic congestion regime when $\rho < C$.
- Rationale:
  - for an appropriate phasing of the flow (e.g. stationary), there would be no congestion
  - for other initial conditions, in phasing and synchronization jointly lead to the perpetuation of a periodic congestion regime

Turbulence: Scenario 1 – Numerical Evidence

- Exponential model, $1/\mu = 2000$ Pkts, $1/\beta = 2$, $p = 0.8$ and $R = 0.1s$. The load factor ρ is then around 263 Pkts/sec. We take $C = 270$ Pkts/sec.
- When the initial condition is the stationary law of the interactionless regenerative rate process, no congestions occur at all since $\rho < C$.
- When the initial condition is with all sources initially active and with 0 rate, periodic congestion regime with $\tau \sim 3.7s$.
- Backed by the following numerical evidence:

– τ is one of the two solutions for the RCP

– the invariant measure equation has a "good" solution for τ

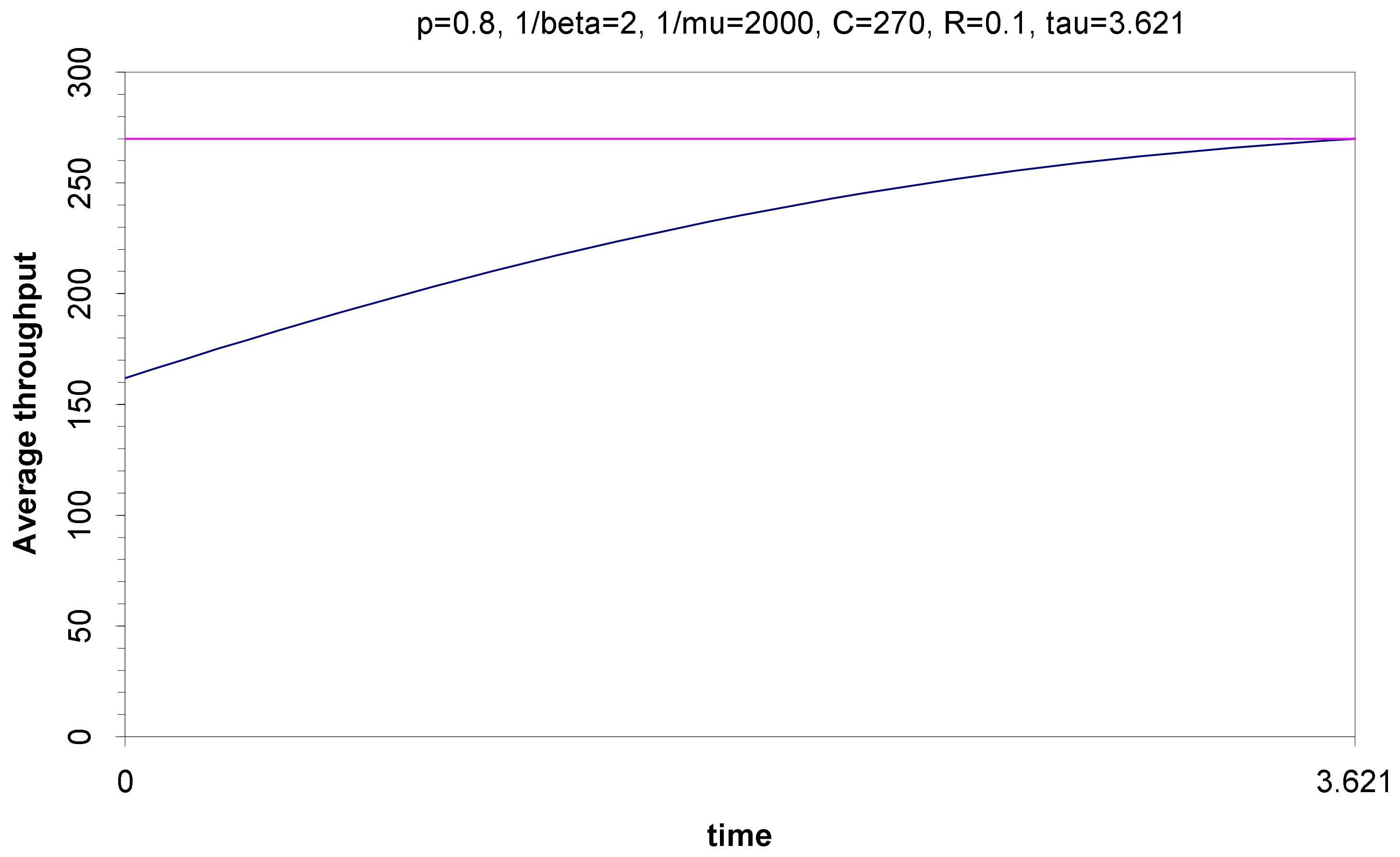


Figure 6: Evolution of the congestion-less aggregated rate with the time

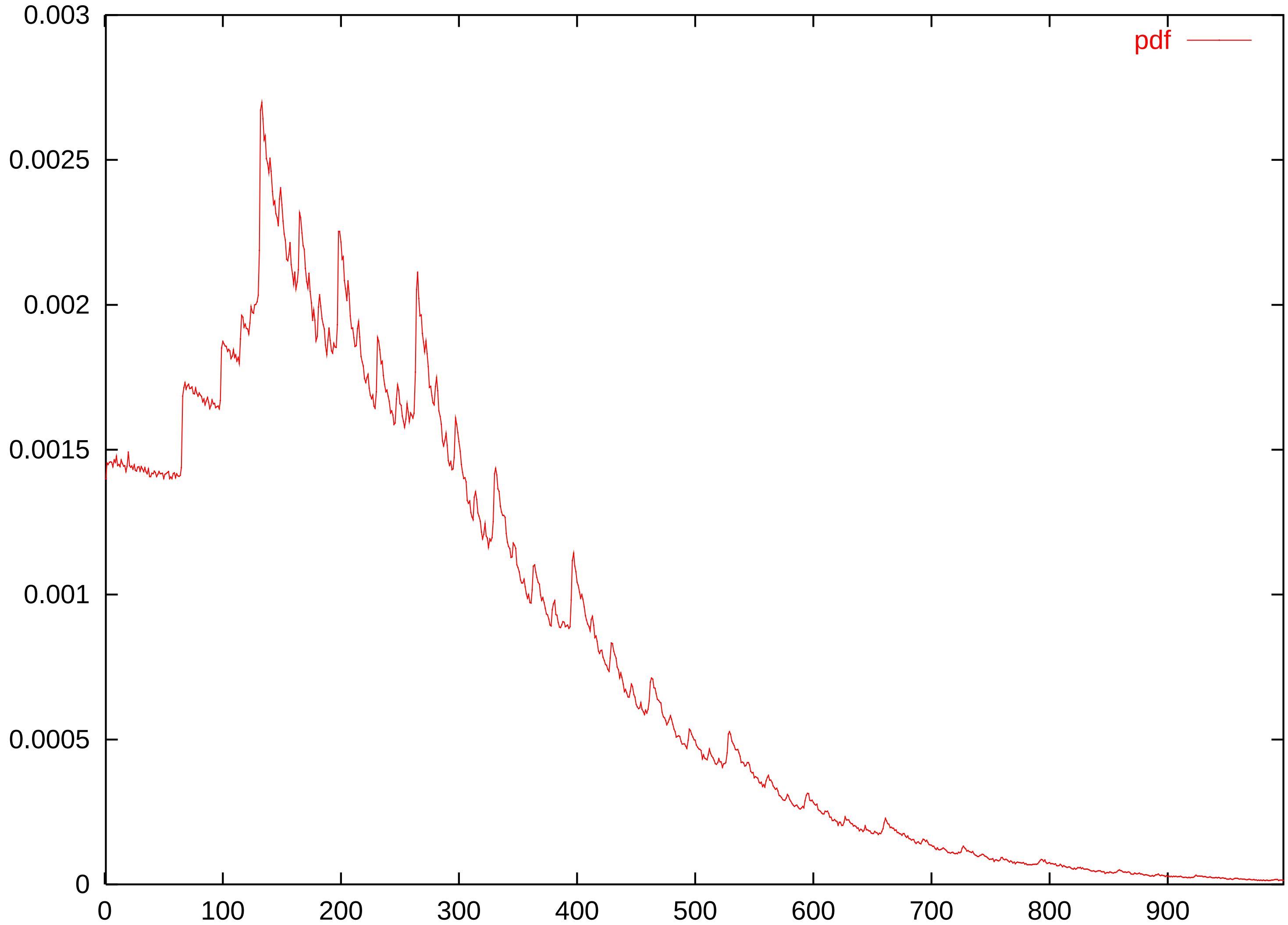


Figure 7: Invariant rate pdf

Turbulence: Scenario 1 – Simulation Evidence

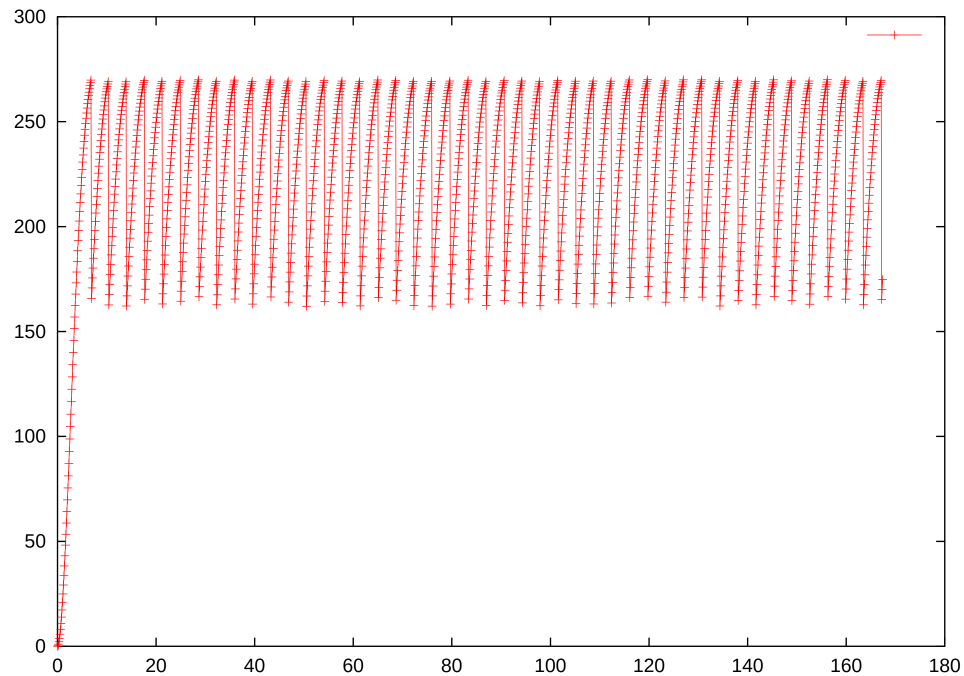
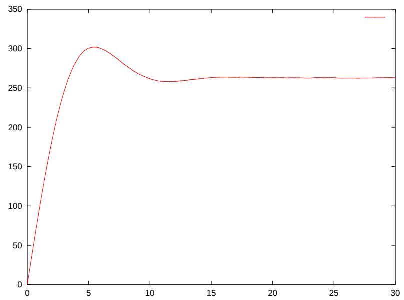


Figure 8: Evolution of aggregated rate when all flows are initially active and with null rate for $C = 270$ Pkts/sec.

Turbulence: – Mathematical proof (Tahoe Case)



– M: maximum of $\alpha(t)$ over all $t > 0$;

– m: minimum of $\alpha(t)$ over all $t > \tau$;

– γ: minimum of $1 - \nu(t)$ over all $t > 0$

for the initial condition with all flows active and with 0 rate.

Turbulence: – Mathematical proof (Tahoe Case) (*continued*)

Let

$$C_T = p\gamma M + (1 - p\gamma)m.$$

Lemma If $C_T > \rho$, then the Tahoe version of the model has turbulence for all C in the interval $\rho \leq C \leq C_T$ for this initial condition.

- No proof for Reno at this stage.

Turbulence: Scenario 2 – Simulation

- Lognormal file size and off-periods; file size has mean value 2000 Pkt and standard deviation 8669 Pkts, and the off-period has a mean value of 2 sec and a standard deviation of 8.7 sec
- TCP Reno, $R = 0.03$ s., $p = 0.8$;
- We observe the same phenomenon concerning α as in the exponential case, with a first maximum at 717 Pkts/s, significantly larger than the horizontal asymptote at $\rho = 620$ Pkts/s.
- The turbulence region goes from $C = 620$ to $C = 680$ Pkts/s.

Refinements

- These phenomena are also present when taking into account
  - Slow start (extension of the PDE approach)
  - Maximal window

Bistability of the Finite Population Model – Simulation

- The fact that the mean field limit has two stationary regimes for some values of the parameters translates into the existence of two stable regimes for any finite stochastic system with the same mean parameters, with rare oscillations from one stable regime to the other.
- Ongoing analysis with M. Lelarge & D. McDonald of the rarity of the transitions using Kiffer's discrete version of Wentzell-Freidlin's theory.

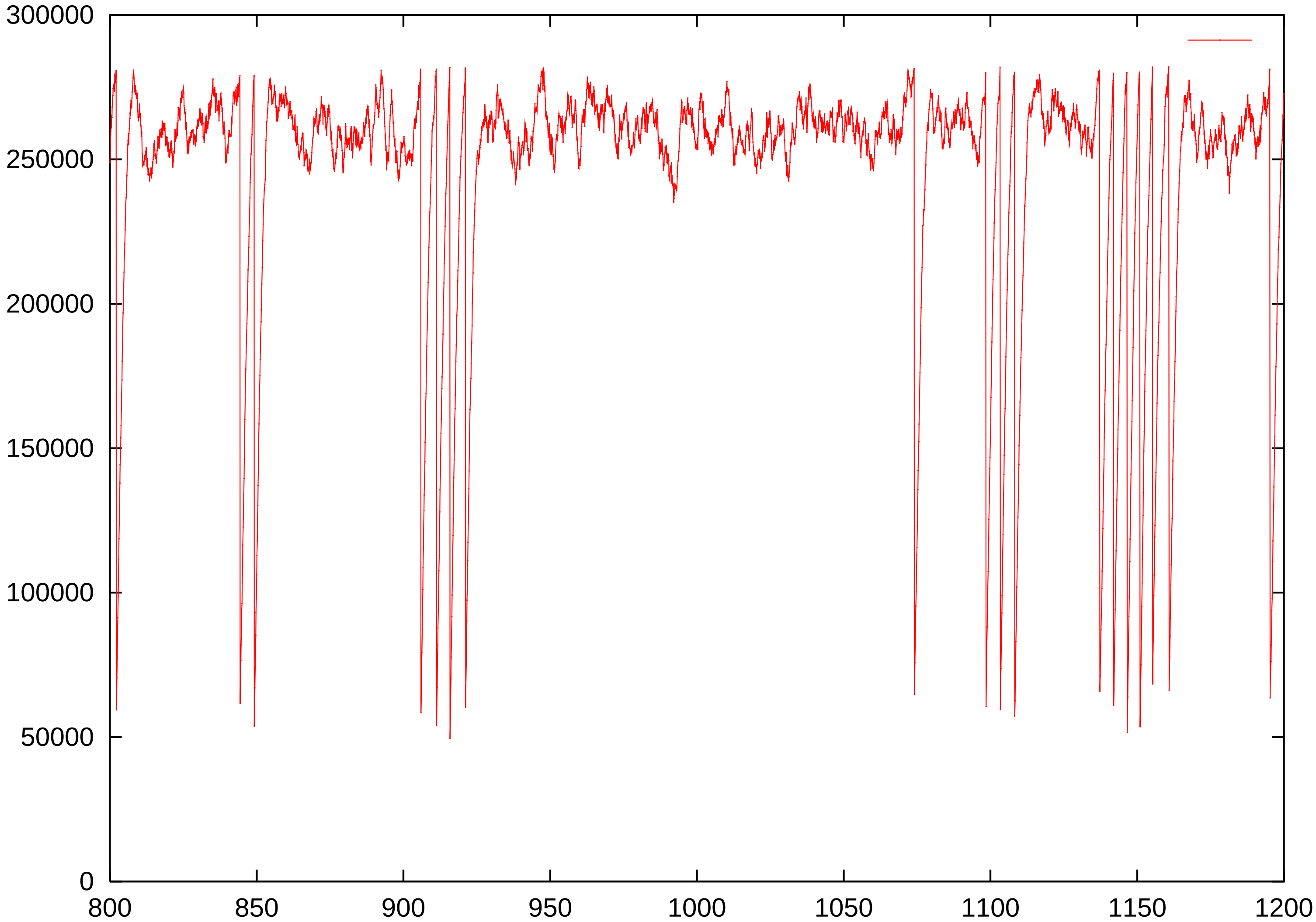


Figure 9: Bi-stability: 1000 Tahoe flows with $C = 282$.

Split TCP

– Dynamics

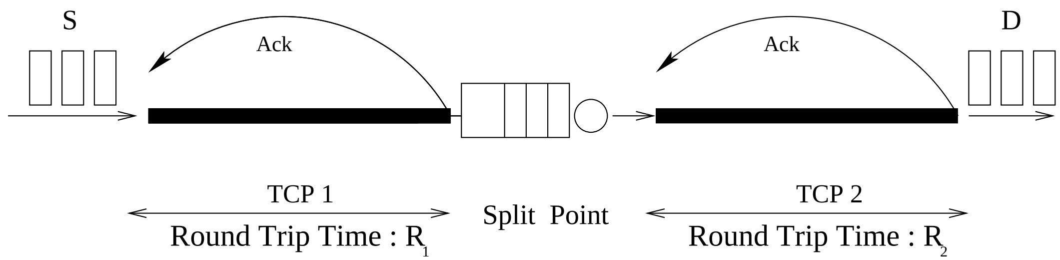
– Stability

– Tails

– PDEs

Split TCP Dynamics

- The split of a multihop TCP connection consists in replacing a plain end-to-end TCP connection by a series of TCP connections.



- Used in overlay networks; dominant in wireless networks (separation of the wireless and the wired parts);
- Used either with infinite buffer or with backpressure.

Notation

- $X(t)$, $Y(t)$: the throughputs of $TCP1$, $TCP2$ time t
- $M(dt)$, $N(dt)$: the loss point process on $TCP1$, $TCP2$
- λ, μ: the loss point process intensity on $TCP1$, $TCP2$ in the RI case
- p, q: the packet error rate on $TCP1$, $TCP2$ in the RD case
- R_1, R_2: the local Round Trip Times
- $Q(t)$: the proxy buffer content at time t
- B: the proxy buffer size

Phases

- Phase 1 or the free phase: the buffer is neither empty nor full, and $X(t)$ and $Y(t)$ evolve independently;
- Phase 2 or the starvation phase: the buffer is empty and Y is limited by the input throughput X
- Phase 3 or the backpressure phase: the buffer has reached it storage capacity B and X is forced by the backpressure algorithm to slow down to the rate Y at which the buffer is drained off.

No phase 3 if $B = \infty$.

Dynamics - Phase 1

- In the free phase:

$$\text{on } \{0 < Q(t) < B\} \quad \begin{cases} dX(t) = \alpha dt - \frac{X(t)}{2} M(dt) \\ dY(t) = \beta dt - \frac{Y(t)}{2} N(dt). \\ dQ(t) = X(t) - Y(t) \end{cases}$$

  where $\alpha = 1/R_1^2$, $\beta = 1/R_2^2$.

- Rationale: the RENO AIMD rule + fluid dynamics for the queue.

Dynamics - Phase 2

- Potential rate of TCP2: $Y(t) = W_2(t)/R_2$
- In phase 2, the buffer is empty, which requires that $X(t) \leq Y(t)$,

$$\text{on} \quad \{Q(t) = 0\} \quad \begin{cases} dX(t) = \alpha dt - \frac{X(t)}{2} M(dt) \\ dY(t) = \beta \frac{X(t)}{Y(t)} dt - \frac{Y(t)}{2} N(dt). \end{cases}$$

- Rationale for a diff. increase of $Y(t)$ proportional to $\frac{X(t)}{Y(t)} \leq 1$:
  - when the buffer is empty, since $X(t) < Y(t)$, the rate at which packets are injected in $TCP2$ and hence the rate at which $TCP2$ acks arrive is $X(t)$.
  - the window of $TCP2$, W_2, increases of $X(t)dt/W_2(t) = dt\frac{X(t)}{R_2 Y(t)}$ in the interval $(t, t + dt)$
  - the potential rate of $TCP2$ thus increases of $\beta dt \frac{X(t)}{Y(t)}$ during this interval.

Dynamics - Phase 3

- In the backpressure phase, which lasts until the buffer is saturated (this requires that $X(t) \geq Y(t)$):

$$\text{on } \{Q(t) = B\} \quad \begin{cases} dX(t) = \alpha \frac{Y(t)}{X(t)} dt - \frac{X(t)}{2} M(dt) \\ dY(t) = \beta dt - \frac{Y(t)}{2} N(dt). \end{cases}$$

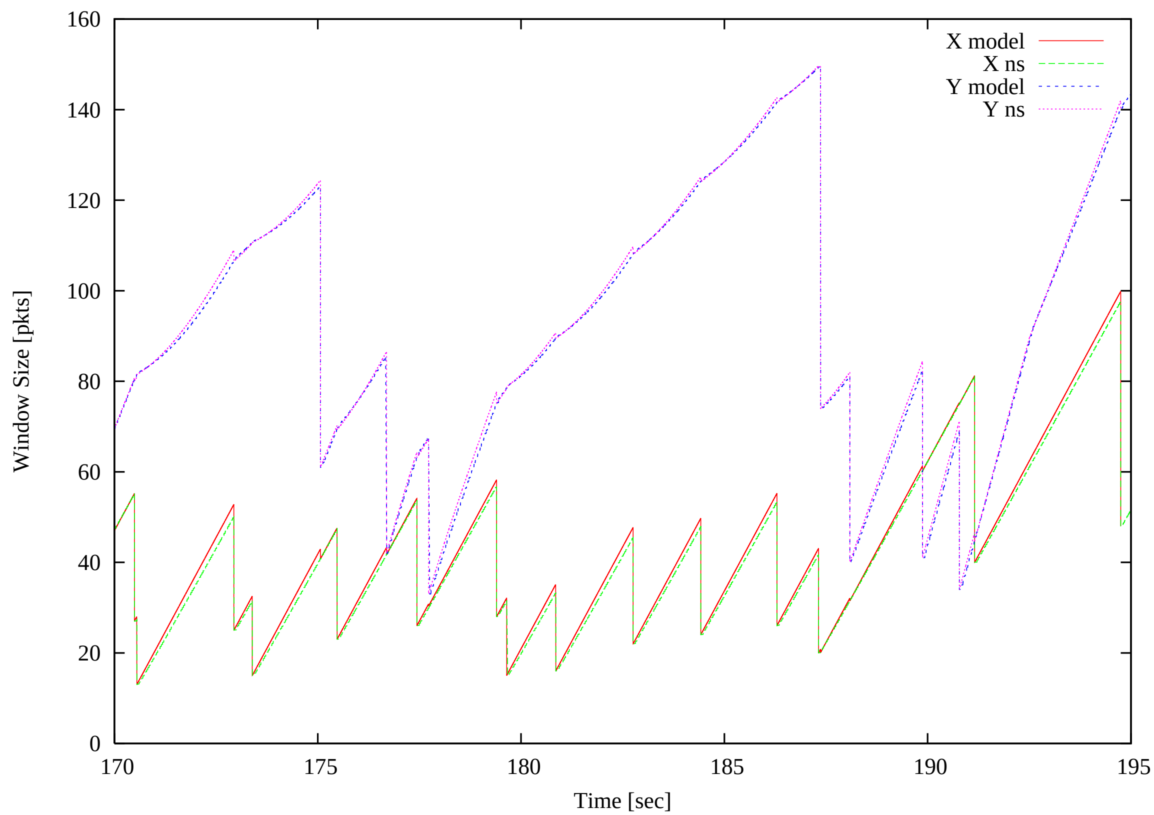
- Rationale: acks of $TCP1$ now come back at a rate of $Y(t)$. Hence the congestion window, $W_1(t)$ of $TCP1$ grows at the rate $Y(t)/W_1(t)$.

Global Queue Dynamics

$$Q(t) = \max\left(\sup_{0\le u\le t}\int_u^t (X(v)-Y(v))dv,\ Q(0)+\int_0^t (X(u)-Y(u))du\right).$$

- Fluid queue with input $X(.)$ and drain $Y(.)$.

Global Queue Dynamics (*continued*)



Congestion windows: comparison between the model and *ns2*

Observations on Dynamics

- The triple $(X(t), Y(t), Q(t))$ forms a Markov process on $\mathbb{R}_+^3$.
- Example of interaction with $B = \infty$
  - $X(.)$ evolves freely.
  - $Y(.)$ is slowed down by X whenever phase 2 is visited
  - This slow down in turn affects the building of $Q(.)$....

Monotonicity in the RI, $B = \infty$ Case

- Denote by Y^f some fictitious process which evolves according to the dynamics of phase 1 only, built from the same realization of N.

1. If $Y^f(t), \widehat{Y}^f(t)$ are built on the same realization of N, but depart from different initial conditions,
$$Y^f(0) \leq \widehat{Y}^f(0) \quad \text{implies} \quad Y^f(t) \leq \widehat{Y}^f(t) \quad \forall t \geq 0.$$

2. Let $Y_v^f(t), t \geq v$ be the process which starts from 0 at time v; then
$$Y_{v_1}^f(t) \geq Y_{v_2}^f(t), \quad \forall v_1 < v_2 \leq t.$$

3. If $Y^f(0) = Y(0)$, then
$$Y(t) \leq Y^f(t), \quad \forall t \geq 0.$$

Monotonicity in the RI, $B < \infty$ Case

- Lemma
  In the RI case with $B < \infty$, when backpressure is used, the stationary rate of Split TCP is strictly less than the minimum of that of $TCP1$ and $TCP2$ in isolation.
- Proof based on monotonicity and a coupling argument.

Backward Construction

- $Q_t(0)$: queue size at time 0 when departing from the following condition at time $t < 0$:

  – Queue size: $Q(t) = 0$,

  – TCP1: the stationary rate $\widetilde{X}(t)$ of $TCP1$ at time t in isolation,

  – TCP2: the stationary rate $\widetilde{Y}^f(t)$ of $TCP2$ at time t in isolation.

$$Q_t(s) = \sup_{t \le u \le s} \int_u^s (\widetilde{X}(v) - Y_t(v))dv, \forall s \ge t,$$

  with $Y_t(v)$ the rate of TCP2 in at time v in Split TCP under the above assumptions.

- Stability: Does $Q_t(0)$ have an a.s. finite limsup when t tends to $-\infty$?

Lower Bound

- From monotonicity property 3,

$$Q_t(0) \geq L_t = \sup_{t \leq u \leq 0} \int_u^0 (\widetilde{X}(v) - \widetilde{Y}^f(v))dv,$$

with $\widetilde{Y}^f(.)$ the stationary free process for TCP2.

- This is a fluid input and fluid drain queue with
input $(\widetilde{X}(.))$
drain $(\widetilde{Y}^f(.))$

- The stochastic process $(\widetilde{X}(t), \widetilde{Y}^f(t))$ forms a stationary and geometrically ergodic Harris chain.

Upper Bound

- $\tau(t)$: the beginning of the last busy period of $Q_t(s)$ before time 0 (0 if $Q_t(0) = 0$ and t if $Q_t(s) > 0$ for all $t < s \leq 0$).

$$Q_t(0) = \int_{\tau(t)}^{0} (\widetilde{X}(v) - Y_t(v))dv \leq \int_{\tau(t)}^{0} (\widetilde{X}(v) - Y^f_{\tau(t)}(v))dv$$
$$\leq U_t = \sup_{t \leq u \leq 0} \int_{u}^{0} (\widetilde{X}(v) - Y^f_t(v))dv,$$

- the first inequality follows from the fact that the dynamics on $(\tau(t), 0)$ is that of the free phase and from the fact that $Y^f_{\tau(t)}(.)$ is the minimal value for the free TCP2 process (monotonicity property 1).

Stability - RI

- Lemma If $\rho < 1$, where $\rho = \frac{\alpha\mu}{\beta\lambda}$, then the RI system is stable. If $\rho > 1$, then it is unstable.
- If $\rho > 1$, the lower bound queue is unstable
- Assume $\rho < 1$ and $\limsup Q_t(0) = \infty$ with a positive probability.Then $\limsup U_t = \infty$ with a positive probability too. This implies that there exists a sequence t_n tending to $-\infty$ and such that a.s.

$$\int_{t_n}^{0} (\widetilde{X}(v) - Y_{t_n}^{f}(v))dv \rightarrow_{n\to\infty} \infty.$$

Stability - RI (*continued*)

- The pointwise ergodic theorem implies that

$$\frac{1}{t}\int_{-t}^{0} \widetilde{X}(v)dv = \frac{1}{t}\int_{-t}^{0} \widetilde{X}(0)\circ\theta_v dv \rightarrow_{t\to\infty} \mathbb{E}[\widetilde{X}(0)],$$

- If we show that the following a.s. limit also holds:

$$\frac{1}{t}\int_{-t}^{0} Y^{f}_{-t}(v)dv \rightarrow_{t\to\infty} \mathbb{E}[\widetilde{Y}^{f}(0)]$$

this will conclude the proof by contradiction.

Stability - RI (*continued*)

- The function

$$\phi_t = \int_{-t}^{0} Y_{-t}^{f}(v)dv$$

is super-additive: $\phi_{t+s} \geq \phi_t \circ \theta_{-s} + \phi_s$

- Thanks to the sub-additive ergodic theorem, this together with the fact that ϕ_t is integrable imply that a.s.

$$\exists \lim_{t\to\infty} \frac{1}{t} \int_{-t}^{0} Y_{-t}^{f}(v)dv = K,$$

for some constant K which may be finite or infinite.

- The fact that K is finite follows from the bound $0 < Y_{-t}^{f}(v) \leq \widetilde{Y}^{f}(v)$ and from the pointwise ergodic theorem applied to $\{\widetilde{Y}^{f}(v)\}$.

Stability - RI (*continued*)

- Since K is finite, the last limit holds both a.s. and in L^1
- By the same arguments

$$K = \lim_{t} \frac{1}{t} \int_{-t}^{0} Y_{-t}^{f}(v) dv = \lim_{t} \mathbb{E} \left(\frac{1}{t} \int_{0}^{t} Y_{0}^{f}(v) dv \right) .$$

- From the fact that $Y_0^f(v)$, $v \geq 0$ is a geometrically ergodic Markov chain,

$$\exists \lim_{t \to \infty} \frac{1}{t} \int_{0}^{t} Y_{0}^{f}(v) dv = \mathbb{E}[\widetilde{Y}^{f}(0)] \quad \text{a.s.}$$

Hence $K = \mathbb{E}[Y^{\tilde{f}}(0)]$

Stability - RD

- Lemma If $\rho < 1$, where $\rho = \frac{\alpha q}{\beta p}$, then the RD system is stable. If $\rho > 1$, then it is unstable.
- Uses the coupling based on the 2-d point Poisson point process and the optimization problem:

What is the infimum over all $y \geq 0$ of the integral

$$\int_u^0 Y^f_{u,y}(v) dv$$

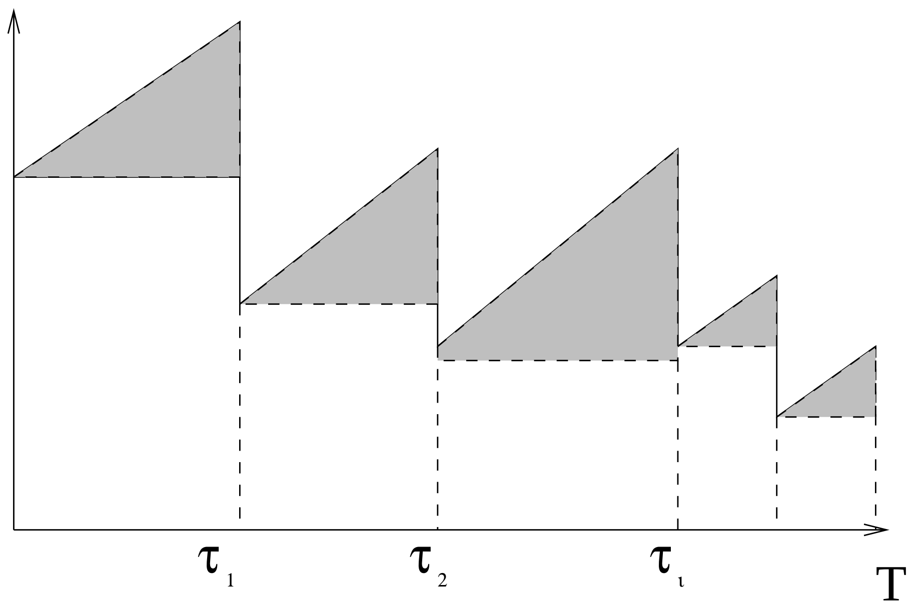
where $Y^f_{u,y}(v)$ is the value of the free process of $TCP2$ at time $v \geq u$ when starting from an initial value of y at time u?

Tails - RI

- Lemma
  In the stable RI case, the queue distribution is heavier than a Weibull distribution of shape parameter $k = 0.5$.
- relies on the lower bound queue L_t
- relies on the fact that the fluid input process and the fluid draining process of this queue are jointly stationary and ergodic and have renewal cycles
- T the length of the renewal cycle and

$$\Delta = \int_0^T \widetilde{X}(t) - \widetilde{Y}^f(t) dt = I_x - I_y.$$

Tails - RI (*continued*)



Tails - RI (*continued*)

$$\Pr\left(\sum_0^{N_T} Trap_i > q\right) \geq \Pr\left(\sum_0^{N_T} \alpha\frac{\tau_i^2}{2} > q\right),$$

where N_T denotes the number of losses in the cycle.

- All triangular areas are i.i.d and heavy tailed:

$$\Pr\left(\alpha\frac{\tau^2}{2} > x\right) = \Pr\left(\tau > \sqrt{\frac{2x}{\alpha}}\right) = e^{-\mu\sqrt{\frac{2x}{\alpha}}},$$

  which is Weibull with shape parameter $k = 0.5$.
- propagates to Δ (Foss & Zachary 03)
- propagates to stationary L_t (Veraverbeke's theorem)

PDE - RI - $B = \infty$

- Evolution of the triple $X(t), Y(t), Q(t)$ over phase 1 via Mellin: pick u, v, z positive.

  – at each loss event on TCP1

$$X_t^u Y_t^v Q_t^z \longrightarrow \frac{X_t^u}{2^u} Y_t^v Q_t^z$$

  – at each loss event on TCP2

$$X_t^u Y_t^v Q_t^z \longrightarrow \frac{Y_t^v}{2^v} X_t^u Q_t^z$$

  – between losses:

$$d(X_t^u Y_t^v Q_t^z) = u X_t^{u-1} Y_t^v Q_t^z dX_t + v Y_t^{v-1} X_t^u Q_t^z dY_t + z Q_t^{z-1} X_t^u Y_t^v dQ_t$$
$$= u X_t^{u-1} Y_t^v Q_t^z \alpha dt + v Y_t^{v-1} X_t^u Q_t^z \beta dt + z Q_t^{z-1} X_t^u Y_t^v (X_t - Y_t) dt$$

PDE - RI - $B = \infty$ (*continued*)

- τ duration of phase 1. ϕ duration of phase 2. Martingale:

$$\mathbb{E}^0[X_\tau^u Y_\tau^v Q_\tau^z] - \mathbb{E}^0[X_0^u Y_0^v Q_0^z] =$$

$$\mathbb{E}^0[\int_0^{\tau+\phi} u X_t^{u-1} Y_t^v Q_t^z \alpha dt + v Y_t^{v-1} X_t^u Q_t^z \beta dt + z Q_t^{z-1} X_t^u Y_t^v (X_t - Y_t) dt] +$$

$$-\left[\lambda\left(1 - \frac{1}{2^u}\right) + \mu\left(1 - \frac{1}{2^v}\right)\right] \mathbb{E}^0[\int_0^{\tau+\phi} X_t^u Y_t^v Q_t^z]$$

- Palm inversion formula:

$$\frac{\mathbb{E}^0[X_\tau^u Y_{v\tau} Q_\tau^z] - \mathbb{E}^0[X_0^u Y_0^v Q_0^z]}{\mathbb{E}^0[\tau + \phi]} = \alpha u \mathbb{E}[X^{u-1} Y^v Q^z] + \beta v \mathbb{E}[X^u Y^{v-1} Q^z] +$$

$$z(\mathbb{E}[X^{u+1} Y^v Q^{z-1}] - \mathbb{E}[X^u Y^{v+1} Q^{z-1}]) + \left[\lambda\left(\frac{1}{2^u} - 1\right) + \mu\left(\frac{1}{2^v} - 1\right)\right] \mathbb{E}[X^u Y^v Q^z]$$

PDE - RI - $B = \infty$ (*continued*)

- Functional equation for Mellin transforms

- Phase 1: $F(u,v,z) = \mathbb{E}(X^u Y^v Q^z 1_{\text{phase 1}})$

$$\frac{\mathbb{E}^0[X_\tau^u Y_\tau^v Q_\tau^z] - \mathbb{E}^0[X_0^u Y_0^v Q_0^z]}{\mathbb{E}^0[\tau + \phi]} = \alpha u F(u-1,v,z) + \beta v F(u,v-1,z) +$$
$$z(F(u+1,v,z-1) - F(u,v+1,z-1)) + [\lambda\left(\frac{1}{2^u} - 1\right) + \mu\left(\frac{1}{2^v} - 1\right)]F(u,v,z)$$

- Phase 2: $G(u,v) = \mathbb{E}(X^u Y^v 1_{\text{phase 2}})$

$$\frac{\mathbb{E}^0[X_{\tau+\phi} Y_{\tau+\phi}] - \mathbb{E}^0[X_\tau Y_\tau]}{\mathbb{E}^0[\tau + \phi]} = \alpha u G(u-1,v) + \beta v G(u+1,v-2) +$$
$$+ \left[\lambda\left(\frac{1}{2^u} - 1\right) + \mu\left(\frac{1}{2^v} - 1\right)\right] G(u,v)$$

PDE - RI - $B = \infty$ (*continued*)

Hence

$$
\begin{aligned}
0 &= \alpha u F(u-1,v,z) + \beta v F(u,v-1,z) \\
&+ z(F(u+1,v,z-1) - F(u,v+1,z-1)) \\
&+ \left[\lambda\left(\frac{1}{2^u}-1\right) + \mu\left(\frac{1}{2^v}-1\right)\right] F(u,v,z) \\
&+ \alpha u G(u-1,v) + \beta v G(u+1,v-2) \\
&+ \left[\lambda\left(\frac{1}{2^u}-1\right) + \mu\left(\frac{1}{2^v}-1\right)\right] G(u,v)
\end{aligned}
$$

- Equivalent to PDE obtained by the PDP approach.

References

– F. Baccelli, D. Mc Donald and J. Reynier, “A Mean-field Model for Multiple TCP Connections Through a Buffer Implementing RED”, Performance Evaluation, 49, pp. 77-97, 2002.

– F. Baccelli, A. Chaintreau, D. De Vleeschauwer and D. Mc Donald, “HTTP Turbulence”, Networks and Heterogeneous Media (AIMS), Vol. 1 No. 1, March 2006.

– F. Baccelli, K.B. Kim and D. McDonald, ”Equilibria of a Class of Transport Equations Arising in Congestion Control”, Queueing Systems, Volume 55, Number 1, pp. 1–8, January, 2007.

– F.Baccelli and D. McDonald, “A Stochastic Model for the Throughput of Non-Persistent TCP Flows”, Performance Evaluation, Volume 65, Number 6-7, Pages 512-530, June 2008.

– F. Baccelli, G. Carofiglio and S. Foss, “Proxy Caching in Split TCP: Dynamics, Stability and Tail Asymptotics”, Proc. of IEEE INFOCOM, Phoenix, Arizona, April 2008.