Classical inequalities for the Boltzmann collision operator.

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Outline

- Boltzmann equation: Collision operator and notation.
- Results:
  1. Young's Inequality for Hard potentials.
  2. Hardy-Littlewood-Sobolev inequality for soft potentials.
- Radial symmetrization and proofs.
- Applications to the Boltzmann equation.

Boltzmann equation

- Cauchy problem: Find $f(t,x,v) \geq 0$

$$\frac{\partial f}{\partial t} + v \cdot \nabla f = Q(f,f) \ \text{ in } \ (0,+\infty) \times \mathbb{R}^{2n}$$

$$f(0,x,v) \ = \ f_0(x,v) \ \geq \ 0$$

- Strong form of collision operator

$$\int_{\mathbb{R}^n} \int_{S^{n-1}} \left(f('v) g('v_*) - f(v) g(v_*) \right) B(|u|, \hat{u} \cdot \omega) \, d\omega \, dv_*$$

- Action on observables or weak form

$$\int_{\mathbb{R}^n} Q(f,g)(v) \psi(v) \, dv := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(v) g(v_*) \int_{S^{n-1}} (\psi(v') - \psi(v)) \, B(|u|, \hat{u} \cdot \omega) \, d\omega \, dv_* \, dv.$$

Gain Boltzmann collision operator

- Action on observables

$$\int_{\mathbb{R}^n} Q^+(f,g)(v)\psi(v)\,\mathrm{d}v := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)g(v_*)\int_{S^{n-1}} \psi(v')B(|u|,\hat{u}\cdot\omega)\,\mathrm{d}\omega\,\mathrm{d}v_*\,\mathrm{d}v$$

$$\boxed{|u|^{\lambda} b(\hat{u}\cdot\omega) \quad \text{with} \quad -n<\lambda}$$

- Variables

$$u = v - v_* \quad , \quad v' = v - \frac{\beta}{2}(u - |u|\omega) \quad \text{and} \quad v + v_* = v' + v'_*$$

- Restitution coefficient

$$\beta : [0,\infty) \to (0,1] \text{ defined by } \beta(z) := \frac{1+e(z)}{2} \qquad \boxed{z = |u|\sqrt{\frac{1-\hat{u}\cdot\omega}{2}}}$$

(i) $z \mapsto e(z)$ is absolutely continuous and non-increasing.
(ii) $z \mapsto ze(z)$ is non-decreasing.

Bobylev operator and relation with the Boltzmann collision operator

- Bobylev variables and operator

$$\mathcal{P}(\psi, \phi)(u) := \int_{S^{n-1}} \psi(u^-)\phi(u^+)b(\hat{u}\cdot\omega)\,\mathrm{d}\omega\,,$$

$$u^- := \tfrac{\beta}{2}(u - |u|\omega) \ \text{ and } \ u^+ := u - u^- = (1-\beta)u + \tfrac{\beta}{2}(u + |u|\omega)$$

- Gain collision operator

$$\int_{\mathbb{R}^n} Q^+(f,g)(v)\psi(v)dv = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)g(v-u)\mathcal{P}(\tau_v\mathcal{R}\psi, 1)(u)\ |u|^\lambda du dv$$

$$\tau_v\psi(x) := \psi(x-v) \ \text{ and } \ \mathcal{R}\psi(x) := \psi(-x)$$

- Interesting property in Maxwell molecules

$$\widehat{Q^+(f,g)} = \mathcal{P}(\hat{f}, \hat{g})$$

Main results

- Notation $\|f\|_{L^p_k(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(v)|^p \, (1+|v|^{pk}) \, dv\right)^{1/p}$
- Young's inequality (Alonso-Carneiro 08 and Alonso-Carneiro-Gamba 09)

Theorem 1. *Let $1 \le p,\ q,\ r \le \infty$ with $1/p + 1/q = 1 + 1/r$. Assume that*

$$B(|u|, \hat{u}\cdot\omega) = |u|^{\lambda} b(\hat{u}\cdot\omega)\,,$$

with $\lambda \ge 0$. For $\alpha \ge 0$, the bilinear operator Q^+ extends to a bounded operator from $L^p_{\alpha+\lambda}(\mathbb{R}^n) \times L^q_{\alpha+\lambda}(\mathbb{R}^n) \to L^r_{\alpha}(\mathbb{R}^n)$ via the estimate

$$\left\|Q^+(f,g)\right\|_{L^r_\alpha(\mathbb{R}^n)} \le C\,\|f\|_{L^p_{\alpha+\lambda}(\mathbb{R}^n)}\,\|g\|_{L^q_{\alpha+\lambda}(\mathbb{R}^n)}.$$

- HLS inequality (Alonso-Carneiro-Gamba 09)

Theorem 2. *Let $1 < p,\ q,\ r < \infty$ with $-n < \lambda < 0$ and $1/p + 1/q = 1 + \lambda/n + 1/r$. For the kernel*

$$B(|u|, \hat{u}\cdot\omega) = |u|^{\lambda}\, b(\hat{u}\cdot\omega),$$

the bilinear operator Q^+ extends to a bounded operator from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ via the estimate

$$\left\|Q^+(f,g)\right\|_{L^r(\mathbb{R}^n)} \le C\,\|f\|_{L^p(\mathbb{R}^n)}\,\|g\|_{L^q(\mathbb{R}^n)}.$$

Rotation property and symmetrization process

- The Bobylev operator satisfies: Let *R* a rotation, then

$$\mathcal{P}(f,g)(Ru) = \mathcal{P}(f \circ R, g \circ R)(u)$$

- This property makes this operator suitable to be analyzed using radial functions.
- Let $G = SO(n)$ be the group of rotations and let $d\mu$ its Haar measure. Define the radial function

$$f_p^\star(x) := \left(\int_G |f(Rx)|^p d\mu\right)^{\frac{1}{p}}, \quad \text{if} \;\; 1 \leq p < \infty$$

- Conservation of norms: $\|f\|_{L^p(\mathbb{R}^n)} = \left\|f_p^\star\right\|_{L^p(\mathbb{R}^n)}$

Majorization by radial functions

- Essential Lemma (Alonso-Carneiro 08 and Alonso-Carneiro-Gamba 09)

Lemma 3. *Let $f, g, \psi \in C_0(\mathbb{R}^n)$ and $1/p + 1/q + 1/r = 1$, with $1 \le p, q, r \le \infty$. Then*

$$\left| \int_{\mathbb{R}^n} \mathcal{P}(f,g)(u)\psi(u)\,\mathrm{d}u \right| \le \int_{\mathbb{R}^n} \mathcal{P}(f_p^\star, g_q^\star)(u)\psi_r^\star(u)\,\mathrm{d}u.$$

- First,

$$\left| \int_{\mathbb{R}^n} \mathcal{P}(f,g)(u)\,\psi(u)\,\mathrm{d}u \right| \le \int_{\mathbb{R}^n} \int_{S^{n-1}} |f(Ru^-)|\,|g(Ru^+)|\,|\psi(Ru)| b(\hat{u}\cdot\omega)\,\mathrm{d}\omega\,\mathrm{d}u.$$

- Then, integration on $G = SO(n)$

$$\left| \int_{\mathbb{R}^n} \mathcal{P}(f,g)(u)\psi(u)\,\mathrm{d}u \right|$$

$$\le \int_{\mathbb{R}^n} \int_{S^{n-1}} \left(\int_G |f(Ru^-)||g(Ru^+)||\psi(Ru)|\,\mathrm{d}\mu(R) \right) b(\hat{u}\cdot\omega)\,\mathrm{d}\omega\,\mathrm{d}u.$$

$$\le f_p^\star(u^-)\,g_q^\star(u^+)\,\psi_r^\star(u)$$

Reduction to dimension 1

- For radial function set $f(x) = \tilde{f}(|x|)$
- Then,

$$\mathcal{P}(f,g)(u) = \left|S^{n-2}\right| \int_{-1}^{1} \tilde{f}\big(a_1(|u|,s)\big)\tilde{g}\big(a_2(|u|,s)\big) b(s)\,(1-s^2)^{\frac{n-3}{2}}\,\mathrm{d}s$$

$$a_1(x,s) = \beta\, x \left(\tfrac{1-s}{2}\right)^{1/2} \quad \text{and} \quad a_2(x,s) = x\left[\left(\tfrac{1+s}{2}\right) + (1-\beta)^2\left(\tfrac{1-s}{2}\right)\right]^{1/2}$$

- So,

$$\widetilde{\mathcal{P}(f,g)}(x) = \left|S^{n-2}\right| \int_{-1}^{1} \tilde{f}\left(a_1(x,s)\right)\,\tilde{g}\left(a_2(x,s)\right)\,\mathrm{d}\xi_n^b(s)$$

- We define then the 1-D functional,

$$\mathcal{B}(f,g)(x) := \int_{-1}^{1} f\left(a_1(x,s)\right)\,g\left(a_2(x,s)\right)\mathrm{d}\xi_n^b(s)$$

Lemma 4. *Let $1 \le p, q, r \le \infty$ with $1/p + 1/q = 1/r$. For $f \in L^p(\mathbb{R}^+, \mathrm{d}\sigma_n^\alpha)$ and $g \in L^q(\mathbb{R}^+, \mathrm{d}\sigma_n^\alpha)$ we have*

$t^{n-1+\alpha}\mathrm{d}t$

$$\|\mathcal{B}(f,g)\|_{L^r(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} \le C\ \|f\|_{L^p(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)}\ \|g\|_{L^q(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)},$$

where the constant C is given in (2.15). In the case of constant restitution coefficient e, corresponding to a constant parameter $\beta = (1+e)/2$, one can show that

$$C(n,\alpha,p,q,b,\beta) = \beta^{-\frac{n+\alpha}{p}} \int_{-1}^{1} \left(\tfrac{1-s}{2}\right)^{-\frac{n+\alpha}{2p}} \left[\left(\tfrac{1+s}{2}\right) + (1-\beta)^2 \left(\tfrac{1-s}{2}\right)\right]^{-\frac{n+\alpha}{2q}} \mathrm{d}\xi_n^b(s)$$

is sharp.

Theorem 5. *Let $1 \le p, q, r \le \infty$ with $1/p + 1/q = 1/r$, and $\alpha \in \mathbb{R}$. The bilinear operator $\mathcal{P}$ extends to a bounded operator from $L^p(\mathbb{R}^n, \mathrm{d}\nu_\alpha) \times L^q(\mathbb{R}^n, \mathrm{d}\nu_\alpha)$ to $L^r(\mathbb{R}^n, \mathrm{d}\nu_\alpha)$ via the estimate*

$|x|^\alpha \mathrm{d}x$

$$\|\mathcal{P}(f,g)\|_{L^r(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)} \le C\ \|f\|_{L^p(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)}\ \|g\|_{L^q(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)}.$$

Moreover, in the case of constant restitution coefficient e, the constant

$$C = |S^{n-2}|\, \beta^{-\frac{n+\alpha}{p}} \int_{-1}^{1} \left(\tfrac{1-s}{2}\right)^{-\frac{n+\alpha}{2p}} \left[\left(\tfrac{1+s}{2}\right) + (1-\beta)^2 \left(\tfrac{1-s}{2}\right)\right]^{-\frac{n+\alpha}{2q}} \mathrm{d}\xi_n^b(s)$$

is sharp.

Proof Lemma 4

- Direct consequence of Minkowski's inequality.
- The optimality in the constant follows by choosing

$$f_\epsilon(x) = \begin{cases} \epsilon^{1/p}\, x^{-(n+\alpha-\epsilon)/p} & \text{for } \ 0 < x < 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

$$g_\epsilon(x) = \begin{cases} \epsilon^{1/q}\, x^{-(n+\alpha-\epsilon)/q} & \text{for } \ 0 < x < 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

- Theorem 5 follows noticing

$$\|\mathcal{P}(f,g)\|_{L^r(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)} \leq \|\mathcal{P}(f_p^\star, g_q^\star)\|_{L^r(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)}$$

- And

$$\begin{aligned} \|\mathcal{P}(f_p^\star, g_q^\star)\|_{L^r(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)} &= \left|S^{n-1}\right|^{\frac{1}{r}} \ \left\|\widetilde{\mathcal{P}(f_p^\star, g_q^\star)}\right\|_{L^r(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} \\ &= \left|S^{n-1}\right|^{\frac{1}{r}} \ \left|S^{n-2}\right| \ \|\mathcal{B}(\tilde{f_p^\star}, \tilde{g_q^\star})\|_{L^r(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} \\ &\leq C \ \left|S^{n-1}\right|^{\frac{1}{r}} \ \left|S^{n-2}\right| \ \|\tilde{f_p^\star}\|_{L^p(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} \ \|\tilde{g_q^\star}\|_{L^q(\mathbb{R}^+,\,\mathrm{d}\sigma_n^\alpha)} \\ &= C \ \left|S^{n-2}\right| \ \|f\|_{L^p(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)} \ \|g\|_{L^q(\mathbb{R}^n,\,\mathrm{d}\nu_\alpha)}\,, \end{aligned}$$

Sharp constant Maxwell molecules

Alonso-Carneiro-Gamba 09

Corollary 6. *Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$. Then*

$$\left\|Q^+(f,g)\right\|_{L^2(\mathbb{R}^n)} = \left\|\widehat{Q^+(f,g)}\right\|_{L^2(\mathbb{R}^n)} = \left\|\mathcal{P}(\hat{f},\hat{g})\right\|_{L^2(\mathbb{R}^n)}$$

$$\leq C_0 \,\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} \,\|\hat{g}\|_{L^2(\mathbb{R}^n)} \leq C_0 \,\|f\|_{L^1(\mathbb{R}^n)} \,\|g\|_{L^2(\mathbb{R}^n)}.$$

The constant is given by

$$C_0 = \left|S^{n-2}\right| \int_{-1}^{1} \left[\left(\tfrac{1+s}{2}\right) + (1-\beta)^2 \left(\tfrac{1-s}{2}\right)\right]^{-\frac{n}{4}} \mathrm{d}\xi_n^b(s)\,.$$

Similarly, for $f \in L^2(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ we have

$$\left\|Q^+(f,g)\right\|_{L^2(\mathbb{R}^n)} \leq C_1 \,\|f\|_{L^2(\mathbb{R}^n)} \,\|g\|_{L^1(\mathbb{R}^n)},$$

where

$$C_1 = \left|S^{n-2}\right| \beta^{-\frac{n}{2}} \int_{-1}^{1} \left(\tfrac{1-s}{2}\right)^{-\frac{n}{4}} \mathrm{d}\xi_n^b(s).$$

Theorem 1: Young's inequality

- Recall that

$$I := \int_{\mathbb{R}^n} Q^+(f,g)(v)\psi(v)\,\mathrm{d}v = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)g(v-u)\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)\,\mathrm{d}u\,\mathrm{d}v.$$

- In addition for $1/p' + 1/q' + 1/r = 1$ one has

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} & f(v)g(v-u)\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)dudv \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(f(v)^{\frac{p}{r}}g(v-u)^{\frac{q}{r}}\right)\left(f(v)^{\frac{p}{q'}}\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)^{\frac{r'}{q'}}\right) \\ &\qquad\left(g(v-u)^{\frac{q}{p'}}\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)^{\frac{r'}{p'}}\right)dudv\end{aligned}$$

Young's inequality

- Apply Holder inequality with these exponents to obtain $I \leq I_1 I_2 I_3$ where

$$I_1 := \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)^p g(v-u)^q \,\mathrm{d}u\,\mathrm{d}v\right)^{\frac{1}{r}}$$

$$I_2 := \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)^p \mathcal{P}(\tau_v \mathcal{R}\psi, 1)(u)^{r'} \,\mathrm{d}u\,\mathrm{d}v\right)^{\frac{1}{q'}}$$

$$I_3 := \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(v-u)^q \mathcal{P}(\tau_v \mathcal{R}\psi, 1)(u)^{r'} \,\mathrm{d}u\,\mathrm{d}v\right)^{\frac{1}{p'}}$$

$$= \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(v)^q \mathcal{P}(1, \tau_{-v}\psi)(u)^{r'} \,\mathrm{d}u\,\mathrm{d}v\right)^{\frac{1}{p'}}$$

- Apply Theorem 5 in each term to conclude

$$I \leq C\, \|f\|_{L^p(\mathbb{R}^n)}\, \|g\|_{L^q(\mathbb{R}^n)}\, \|\psi\|_{L^{r'}(\mathbb{R}^n)}$$

Young's constant

- The constant is proportional to

$$C = \left|S^{n-2}\right| \left(2^{\frac{n}{r'}} \int_{-1}^{1} \left(\tfrac{1-s}{2}\right)^{-\frac{n}{2r'}} \mathrm{d}\xi_n^b(s)\right)^{\frac{r'}{q'}}$$

$$\left(\int_{-1}^{1} \left[\left(\tfrac{1+s}{2}\right) + (1-\beta_0)^2 \left(\tfrac{1-s}{2}\right)\right]^{-\frac{n}{2r'}} \mathrm{d}\xi_n^b(s)\right)^{\frac{r'}{p'}}$$

- In the case $\alpha + \lambda > 0$ is enough to use

$$|v'|^\alpha = |v - u^-|^\alpha \leq \left(|v|^2 + |v_*|^2\right)^{\alpha/2} \leq 2^{\alpha/2}\left(|v|^\alpha + |v-u|^\alpha\right)$$

$$|u|^\lambda \leq \left(|v-u| + |v|\right)^\lambda \leq 2^\lambda\left(|v-u|^\lambda + |v|^\lambda\right)$$

Theorem 2. HLS inequality

- Recall

$$I := \int_{\mathbb{R}^n} Q^+(f,g)(v)\,\psi(v)\,\mathrm{d}v = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)g(v-u)\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)\,|u|^\lambda\,\mathrm{d}u\,\mathrm{d}v$$
$$= \int_{\mathbb{R}^n} f(v)\left(\int_{\mathbb{R}^n} \tau_v\mathcal{R}g(u)\,\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)\;|u|^\lambda\mathrm{d}u\right)\mathrm{d}v.$$

- For the inner integral we use Holder

$$\int_{\mathbb{R}^n} \tau_v\mathcal{R}g(u)\,\mathcal{P}(\tau_v\mathcal{R}\psi,1)(u)\;|u|^\lambda\mathrm{d}u \leq \;\|\mathcal{P}(\tau_v\mathcal{R}\psi,1)\|_{L^a(\mathbb{R}^n,\,\mathrm{d}\nu_\lambda)}\,\|\tau_v\mathcal{R}g\|_{L^{a'}(\mathbb{R}^n,\,\mathrm{d}\nu_\lambda)}$$
$$\leq C_1\,\|\tau_v\mathcal{R}\psi\|_{L^a(\mathbb{R}^n,\,\mathrm{d}\nu_\lambda)}\,\|\tau_v\mathcal{R}g\|_{L^{a'}(\mathbb{R}^n,\,\mathrm{d}\nu_\lambda)}$$
$$= C_1\left[\left(|\psi|^a * |u|^\lambda\right)(v)\right]^{1/a}\left[\left(|g|^{a'} * |u|^\lambda\right)(v)\right]^{1/a'},$$

- Then

$$\boxed{I \leq C_1\int_{\mathbb{R}^n} f(v)\left[\left(|\psi|^a * |u|^\lambda\right)(v)\right]^{1/a}\left[\left(|g|^{a'} * |u|^\lambda\right)(v)\right]^{1/a'}\mathrm{d}v.}$$

- Applying Holder again

$$I \le C_1 \, \|f\|_{L^p(\mathbb{R}^n)} \, \left\| |\psi|^a * |u|^\lambda \right\|_{L^{b/a}(\mathbb{R}^n)}^{1/a} \, \left\| |g|^{a'} * |u|^\lambda \right\|_{L^{c/a'}(\mathbb{R}^n)}^{1/a'}$$

- HLS for the last two terms

$$\left\| |\psi|^a * |u|^\lambda \right\|_{L^{b/a}(\mathbb{R}^n)} \le C_2 \, \|\psi\|_{L^{ad}(\mathbb{R}^n)}^a$$

$$\left\| |g|^{a'} * |u|^\lambda \right\|_{L^{c/a'}(\mathbb{R}^n)} \le C_3 \, \|g\|_{L^{a'e}(\mathbb{R}^n)}^{a'}$$

- Concluding

$$\boxed{I \le C_1 \, C_2^{1/a} \, C_3^{1/a'} \, \|f\|_{L^p(\mathbb{R}^n)} \, \|g\|_{L^{a'e}(\mathbb{R}^n)} \, \|\psi\|_{L^{ad}(\mathbb{R}^n)}}$$

- Need

$$\frac{1}{a} + \frac{1}{a'} = 1, \qquad 1 \le a \le \infty$$

$$\frac{1}{p} + \frac{1}{b} + \frac{1}{c} = 1, \quad 1 < b, c < \infty$$

$$1 + \frac{a}{b} = \frac{1}{d} - \frac{\lambda}{n}, \qquad b > a, \qquad 1 < d < \infty$$

$$1 + \frac{a'}{c} = \frac{1}{e} - \frac{\lambda}{n}, \qquad c > a', \qquad 1 < e < \infty$$

$$a'e = q$$

$$ad = r'$$

- Last two equations determine *d* and *e* in terms of *a*.
- The remainder linear system is undetermined because of the initial relation

$$\frac{1}{p}+\frac{1}{q}=1+\frac{\lambda}{n}+\frac{1}{r}$$

- The choice $\frac{1}{b}=\frac{1}{r'}-\frac{1}{a}\left(1+\frac{\lambda}{n}\right)$ $\frac{1}{c}=\frac{1}{q}-\frac{1}{a'}\left(1+\frac{\lambda}{n}\right)$
- with

$$\max\left\{\frac{1}{r'(2+\frac{\lambda}{n})},1-\frac{1}{q(1+\frac{\lambda}{n})}\right\}<\frac{1}{a}<\min\left\{\frac{1}{r'(1+\frac{\lambda}{n})},1-\frac{1}{q(2+\frac{\lambda}{n})}\right\}$$

Explicit constant

- By Theorem 5 we have the optimal

$$C_1 = \left|S^{n-2}\right| \, 2^{\frac{n+\lambda}{a}} \int_{-1}^{1} \left(\tfrac{1-s}{2}\right)^{-\frac{n+\lambda}{2a}} \, \mathrm{d}\xi_n^b(s)$$

- Constants C2 and C3 are given by the HLS constant.
- The overall constant is as good as the HLS constant.

Application 1.

- Let $\mathcal{M}_a(v) := \exp\left(-a|v|^2\right)$, then

Theorem 7. *Let $1 \le p,\, q,\, r \le \infty$ with $1/p + 1/q = 1 + 1/r$. Assume that*

$$B(|u|, \hat{u}\cdot\omega) = |u|^{\lambda}\, b(\hat{u}\cdot\omega)\,,$$

with $\lambda \ge 0$. Then, for $a > 0$,

$$\left\|Q^{+}(f,g)\ \mathcal{M}_a^{-1}\right\|_{L^r(\mathbb{R}^n)} \le C\, \|f\ \mathcal{M}_a^{-1}\|_{L^p_\lambda(\mathbb{R}^n)}\, \|g\ \mathcal{M}_a^{-1}\|_{L^q(\mathbb{R}^n)}.$$

- For this theorem we assume *b(s)* vanishes for s<0.

- *Proof:* Recall the weak formulation

$$I := \int_{\mathbb{R}^n} Q^+(f,g)(v)\left(\mathcal{M}_a^{-1}\psi\right)(v)\,\mathrm{d}v$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v)g(v-u)\int_{S^{n-1}}\left(\mathcal{M}_a^{-1}\psi\right)(v')\,|u|^{\lambda}\,b(\hat{u}\cdot\omega)\,\mathrm{d}\omega\,\mathrm{d}u\,\mathrm{d}v.$$

- Also, due to dissipation of energy

$$\mathcal{M}_a^{-1}(v') \le \mathcal{M}_a^{-1}(v)\,\mathcal{M}_a^{-1}(v_*)\,\mathcal{M}_a(v_*').$$

- Then

$$I \le$$
$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\mathcal{M}_a^{-1}f\right)(v)\left(\mathcal{M}_a^{-1}g\right)(v-u)\int_{S^{n-1}}\psi(v')\,\mathcal{M}_a(v_*')\,|u|^{\lambda}\,b(\hat{u}\cdot\omega)\,\mathrm{d}\omega\,\mathrm{d}u\,\mathrm{d}v.$$

- But *b* vanishes in (-1,0), then

$$\int_{S^{n-1}} \psi(v')\,\mathcal{M}_a(v'_*)\,|u|^\lambda\, b(\hat{u}\cdot\omega)\,\mathrm{d}\omega$$
$$= \int_{S^{n-2}} \psi(v-u^-)\mathcal{M}_a(v-u^+)\,|u|^\lambda\, b(\hat{u}\cdot\omega)\,\mathrm{d}\omega$$
$$\le 2^{\lambda/2}\int_{S^{n-2}} \psi(v-u^-)\mathcal{M}_a(v-u^+)\,\left|u^+\right|^\lambda b(\hat{u}\cdot\omega)\,\mathrm{d}\omega$$

- Next we use the Maxwellian to balance the growth in the kernel.

$$\mathcal{M}_a(v-u^+)|u^+|^\lambda \le 2^\lambda\,\mathcal{M}_a(v-u^+)\left(|v-u^+|^\lambda + |v|^\lambda\right) \le C_{\lambda,a}\,(1+|v|^\lambda)$$

- Then,

$$I \leq$$

$$2^{\lambda/2} C_{\lambda,a} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(\mathcal{M}_a^{-1} f\right)(v)(1+|v|^{\lambda}) \left(\mathcal{M}_a^{-1} g\right)(v-u) \mathcal{P}(\tau_v \mathcal{R}\psi, 1)(u) \, \mathrm{d}u \, \mathrm{d}v.$$

- Proceed with the proof as before for Young's inequality in the Maxwell molecules case.
- Note in the case (r=q=∞ and p=1) we obtain the classical L1-L∞ comparison. See Gamba-Panferov-Villani (2008) for the proof of a slightly different version using the Carleman representation (in the elastic case).

Application 2.

- Note that by Theorem 5. (1/p+1/q=1)

$$Q^+(f,g)(v) = \int_{\mathbb{R}^n} \mathcal{P}(\tau_v \mathcal{R} f, \tau_v \mathcal{R} g)(u)\ |u|^\lambda du$$

$$\leq C \left(\int_{\mathbb{R}^n} f(u)^p |v-u|^\lambda du \right)^{1/p} \left(\int_{\mathbb{R}^n} g(u)^q |v-u|^\lambda du \right)^{1/q}.$$

- Therefore (p=∞, q=1)

$$Q^+(f,g)(v) \leq C \|f\|_\infty \int_{\mathbb{R}^n} g(u)|v-u|^\lambda du$$

- Compare with $Q^-(f,g)(v) = f(v) \int_{\mathbb{R}^n} g(u)|v-u|^\lambda du$

- This shows that the gain and the loss are somehow comparable, but, the semi-local nature of the loss operator and global of the gain make it hard to compare them. Note that if $C<1$ previous inequality proves that solutions of the B.E. would be bounded.
- Let us note that previous inequality can be helpful in some instances. For example assume for simplicity elastic collisions ($\lambda \leq 0$) and that

$$f^{\#}(x,v,t) \leq C\ (1+|x|^2)^{-\alpha/2}\ (1+|v|^2)^{-\beta/2} := C\ h(x)\ k(v).$$

- *Classical result*: Let **z** and **y** are orthogonal then

$$h(x+z)h(x+y) \le h(x)\left(h(x+z)+h(x+y)+h(x+z+y)\right)$$

- Also note that

$$Q_+^{\#}(f,f)(x,v,t) =$$

$$\int_{\mathbb{R}^n}\int_{S^{n-1}} f^{\#}(x+tu^-, v-u^-, t) f^{\#}(x+tu^+, v-u^+, t)\, |u|^\lambda\, b(\hat{u}\cdot\omega)\, d\omega du$$

- Using previous inequality with $z = u^-$ and $y = u^+$ we obtain

$$Q_+^{\#}(f,f)(x,v,t) \le$$

$$C^2\, h(x)\, k(v) \left(\int_{\mathbb{R}^n}\int_{S^{n-1}} h(x+tu^-)k(v-u^+)\, |u|^\lambda\, b(\hat{u}\cdot\omega)\, d\omega du + \cdots\right)$$

- Using Theorem 5. we conclude

$$Q_+^{\#}(f,f)(x,v,t) \le C_1\, C^2\, h(x)\, k(v) \min\left\{1, \int_{\mathbb{R}^n} h(x+tu)\, |u|^\lambda du\right\}$$

$$\le \frac{C_1\, C^2}{(1+t)^{n+\lambda}}\, h(x)\, k(v)$$

- And therefore for 1-n<λ≤0,

$$\int_0^\infty Q_+^{\#}(f,f)(x,v,s)ds \le C_1\, C^2\, h(x)\, k(v)$$

- *Find more applications in Irene presentation.*

Previous work

- *Previous L^p estimates done by Gustafsson 1988, Gamba-Panferov-Villani 2003, Mouhot –Villani 2004, Gamba-Panferov-Villani 2008.*
- *Some ideas for the short application 2. taken from Bellomo-Palczewski (1988) and Ha (2005).*
- *Radial symmetrization can be found in Alonso-Carneiro 2008 and Alonso-Carneiro-Gamba 2009.*
- *Other estimates for nice collision kernels (moments and compactness) can be found in the work of P-L. Lions 1993, Wennberg 1993-94, Desvillettes 1993.*
- *Applications to the homogeneous case Mouhot-Villani 2004*

Application: Inhomogeneous Boltzmann. (Alonso-Gamba 09)

- For this application we consider the distributional solution of the Cauchy inhomogeneous Boltzmann problem in the near vacuum case or the near local Maxwellian case. These solutions are proven to satisfy *($0 \le \lambda < n-1$)*

$$f(t,x,v) \le C \, \exp\left(|x - vt|^2 + |v|^2\right)$$

- We use the standard notation $f^{\#}(t,x,v) := f(t, x+tv, v)$

$$\boxed{\frac{df^{\#}}{dt}(t) = Q^{\#}(f,f)(t) \ \text{ with } \ f(0) = f_0}$$

- We want to study the propagation of regularity in order to find classical solutions in the soft potential case.
- Technical difficulty: Do the above without regularizing or pw cutting-off the kernel.
- Finite difference: $(D_{h,\hat{x}} f)(x) := \dfrac{f(x+h\hat{x}) - f(x)}{h}$
- Translation: $(\tau_{h,\hat{x}} f)(x) := f(x+h\hat{x})$

- Applying the difference operator to the equation

$$\boxed{\frac{d(Df)^{\#}}{dt}(t) = (DQ(f,f))^{\#}(t) = Q^{\#}(Df,f)(t) + Q^{\#}(\tau f, Df)(t)}$$

- Multiplying by $p\,|(Df)^{\#}|^{p-1}\,\mathrm{sgn}((Df)^{\#})$ we obtain

$$\frac{d\,\|Df\|_{L^p}^p}{dt} = p\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |Df|^{p-1}\mathrm{sgn}(Df)\,(Q(Df,f)+Q(\tau f,Df))\,dv dx$$

- Holder inequality and HLS theorem imply

$$\frac{d\,\|Df\|_{L^p}^p}{dt} \le p\int_{\mathbb{R}^n} \|Df\|_{L_v^p(\mathbb{R}^n)}^{p-1}\left(\|Q(Df,f)\|_{L_v^p(\mathbb{R}^n)} + \|Q(\tau f,Df)\|_{L_v^p(\mathbb{R}^n)}\right)dx$$

$$\le p\,C\int_{\mathbb{R}^n} \|Df\|_{L_v^p(\mathbb{R}^n)}^{p}\left(\|f\|_{L_v^a(\mathbb{R}^n)} + \|\tau f\|_{L_v^a(\mathbb{R}^n)}\right)dx. \qquad \boxed{a=\frac{n}{n-\lambda}}$$

- Since *f* is control by travelling Maxwellian

$$\|f\|_{L_v^a(\mathbb{R}^n)} \le \frac{C}{(1+t)^{n/a}} = \frac{C}{(1+t)^{n-\lambda}}$$

- Therefore, using Gronwall's argument

$$\|Df\|_{L^p(\mathbb{R}^{2n})}(t) \le \|Df_0\|_{L^p(\mathbb{R}^{2n})} \exp\left(\int_0^t \frac{C}{(1+s)^{n-\lambda}}\, ds\right)$$

- Recall that $C = C(n, p, \lambda, \|b\|_{L^1(S^{n-1})})$
- Then (after sending h $\longrightarrow$ 0)

$$\|\nabla f\|_{L^p(\mathbb{R}^{2n})}(t) \le C\, \|\nabla f_0\|_{L^p(\mathbb{R}^{2n})} \quad \textit{for all } t \in [0, T], \quad 1 < p < \infty.$$

- This *produces the classical solution* of the Cauchy problem. Moreover, the same process for the velocity variable gives

$$\boxed{\frac{d(Df)}{dt}(t) + v \cdot \nabla(Df)(t) + \hat{v} \cdot \nabla(\tau f)(t) = DQ(f, f)(t) = Q(Df, f)(t) + Q(\tau f, Df)(t)}$$

- Leads to the equation

$$\frac{d\,\|Df\|^p_{L^p}}{dt}(t) \le \frac{p\,C}{(1+t)^{n-\lambda}}\,\|Df\|^p_{L^p(\mathbb{R}^{2n})} + p\,\|Df\|^{p-1}_{L^p(\mathbb{R}^{2n})}\,\|\nabla f\|_{L^p(\mathbb{R}^{2n})}$$

- That implies

$$\|Df\|_{L^p(\mathbb{R}^{2n})}(t) \le \left(\|Df_0\|_{L^p(\mathbb{R}^{2n})} + t\,\|\nabla f_0\|_{L^p(\mathbb{R}^{2n})}\right)\exp\left(\int_0^t \frac{C}{1+s^{n-\lambda}}\,ds\right)$$

- So (after sending h $\longrightarrow$ 0),

$$\|(\nabla_v f)(t)\|_{L^p(\mathbb{R}^{2n})} \le C\left(\|\nabla_v f_0\|_{L^p(\mathbb{R}^{2n})} + t\,\|\nabla_x f_0\|_{L^p(\mathbb{R}^{2n})}\right)$$

- Same argument can be used to obtain the stability result

$$\|f-g\|_{L^p} \le C\;\|f_0-g_0\|_{L^p} \quad \textit{with} \quad 1<p<\infty$$

Previous work

- *Previous L^p estimates done by Gustafsson 1988, Gamba-Panferov-Villani 2003, Mouhot –Villani 2004 .*
- *Radial symmetrization can be found in Alonso-Carneiro 2008 and Alonso-Carneiro-Gamba 2009.*
- *Other estimates (moments and compactness) can be found in the work of P-L. Lions 1993, Wennberg 1993-94, Desvillettes 1993.*
- *Existence of distributional solutions near vacuum is due to Kaniel-Shinbrot 1977, Shinbrot-Illner 1981, Toscani 1985 and others.*
- *Existence local Maxwellian is due to Toscani 1988, Goudon 1997, Mischler-Perthame 1997, Alonso - Gamba 2009.*
- *Existence classical solutions in the aforementioned cases can be found in Alonso-Gamba 2009. Previous due to Boudin-Desvillettes 2000.*
- *Stability in L1 done by Ha 2004 for different potentials.*