From DiPerna-Lions to Leray

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The Boltzmann Equation: DiPerna-Lions Plus 20 Years
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I. INTRODUCTION

I heard Ron DiPerna presented his new result with Pierre-Louis Lions on the existence of global weak solutions of the Boltzmann in spring of 1988. The result was stunning for several reasons.

First, no similar result was known for the equations of gas dynamics.

Second, they considered all physically meaningful initial data.

Third, they introduced the concept of *renomalized solutions*.

Fourth, they considered a larger class of collision kernels than had been treated in classical mathematical studies of the Boltzmann equation.

It was clear that the mathmatical landscape of the Boltzmann equation had been radically altered. There would be opportunities for many new results.

However, their work was also criticized by many for a number of reasons.

First, because they has used a compactness argument to obtain existence, they had no uniqueness result.

Second, because the notion of renormalized solution was so weak, these solutions did not satisfy formally expected conservation laws that were the starting point for all formal derivations of gas dynamics.

Third, because their result was clearly an analog of Leray's 1934 result on the incompressible Navier-Stokes equations, and because there had been little major progress in that subject for over 50 years, it was argued there would not be much progress built upon the DiPerna-Lions theory. Indeed, Ron DiPerna had made it clear that he had studied the Boltzmann equation as route to obtain a theory of global weak solutions for the Euler equations of gas dynamics, and that this program was far from succeeding.

In May of 1988 Claude Bardos, Francois Golse, and I began to study the direct connection between the DiPerna-Lions theory and that of Leray. Our program quickly broadened to include derivations of other fluid dynamical systems, such as the acoustic system and the incompressible Stokes system. It has achieved considerable success. Pierre-Louis Lions, Nader Masmoudi, and Laure Saint-Raymond have made major contributions to this success. A list of references that covers much (but not all) of this work is given at the end of these slides.

This talk presents work with Nader Masmoudi (2008). We establish the Boussinesq-balance incompressible Navier-Stokes limit for solutions of the Boltzmann equation considered over any periodic spatial domain $\mathbb{T}^D$ of dimension $D \geq 2$. We do this for a broad class of collision kernels that relaxes the Grad small deflection cutoff condition for hard potentials made by Golse and Saint-Raymond (2004 and 2008), and includes for the first time kernels arising from soft potentials.

We show that all appropriately scaled families of DiPerna-Lions renormalized solutions have fluctuations that are compact, and that every limit point of such a family is governed by a Leray solution of the Navier-Stokes system for all time. Key tools include the relative entropy cutoff method of Saint Raymond, the L^1 velocity averaging result of Golse and Saint Raymond, and some new estimates.

II. BOLTZMANN EQUATION PRELIMINARIES

The state of a fluid composed of identical point particles confined to a spatial domain $\Omega \subset \mathbb{R}^D$ is described at the kinetic level by a mass density F over the single-particle phase space $\mathbb{R}^D \times \Omega$. More specifically, $F(v,x,t)\,\mathrm{d}v\,\mathrm{d}x$ gives the mass of the particles that occupy any infinitesimal volume $\mathrm{d}v\,\mathrm{d}x$ centered at the point $(v,x) \in \mathbb{R}^D \times \Omega$ at the instant of time $t \geq 0$. To remove complications due to boundaries, we take Ω to be the periodic domain $\mathbb{T}^D = \mathbb{R}^D/\mathbb{L}^D$, where $\mathbb{L}^D \subset \mathbb{R}^D$ is any D-dimensional lattice. Here $D \geq 2$.

The evolution of $F = F(v,x,t)$ is governed by the Boltzmann equation:

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F,F)\,, \qquad F(v,x,0) = F^{in}(v,x) \geq 0\,. \tag{1}$$

The Boltzmann collision operator $\mathcal{B}$ models binary collisions. It acts only on the v argument of F. It is formally given by

$$\mathcal{B}(F,F) = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} (F_1' F' - F_1 F)\, b(\omega, v_1 - v)\, \mathrm{d}\omega\, \mathrm{d}v_1 \,, \tag{2}$$

where v_1 ranges over $\mathbb{R}^D$ endowed with its Lebesgue measure $\mathrm{d}v_1$ while ω ranges over the unit sphere $\mathbb{S}^{D-1} = \{\omega \in \mathbb{R}^D \;:\; |\omega| = 1\}$ endowed with its rotationally invariant measure $d\omega$. The F_1', F', F_1, and F appearing in the integrand designate $F(\cdot, x, t)$ evaluated at the velocities v_1', v', v_1, and v respectively, where the primed velocities are defined by

$$v_1' = v_1 - \omega\, \omega\cdot(v_1 - v)\,, \qquad v' = v + \omega\, \omega\cdot(v_1 - v)\,, \tag{3}$$

for any given $(\omega, v_1, v) \in \mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D$.

Quadratic operators like $\mathcal{B}$ are extended by polarization to be bilinear and symmetric.

The unprimed and primed velocities are possible velocities for a pair of particles either before and after, or after and before, they interact through an elastic binary collision. Conservation of momentum and energy for particle pairs during collisions is expressed as

$$v + v_1 = v' + v_1' \,, \qquad |v|^2 + |v_1|^2 = |v'|^2 + |v_1'|^2 \,.$$

Equation (3) represents the general nontrivial solution of these $D+1$ equations for the $4D$ unknowns v_1', v', v_1, and v in terms of the $3D-1$ parameters (ω, v_1, v).

The collision kernel b is positive almost everywhere. The Galilean invariance of the collisional physics implies that b has the classical form

$$b(\omega, v_1 - v) = |v_1 - v| \, \Sigma(|\omega \cdot n|, |v_1 - v|) \,, \tag{4}$$

where $n = (v_1 - v)/|v_1 - v|$ and Σ is the specific differential cross-section.

Nondimensional Form

The Boltzmann equation can be brought into the nondimensional form

$$\mathsf{St}\,\partial_t F + v \cdot \nabla_x F = \frac{1}{\mathsf{Kn}} \mathcal{B}(F, F)\,,$$

where St is the Strouhal number, Kn is the Knudsen number.

We consider fluid dynamical regimes in which F is close to a spatially homogeneous Maxwellian $M = M(v)$. By an appropriate choice of a Galilean frame and of mass and velocity units, it can be assumed that this so-called absolute Maxwellian M has the form

$$M(v) \equiv \frac{1}{(2\pi)^{D/2}} \exp(-\tfrac{1}{2}|v|^2)\,.$$

Relative Kinetic Density

It is natural to introduce the relative density, $G = G(v, x, t)$, defined by $F = MG$. The initial-value problem for G is

$$\mathrm{St}\, \partial_t G + v \cdot \nabla_x G = \frac{1}{\mathrm{Kn}} \mathcal{Q}(G, G)\,, \qquad G(v, x, 0) = G^{in}(v, x)\,, \quad (5)$$

where the collision operator is now given by

$$\begin{aligned} \mathcal{Q}(G, G) &\equiv \frac{1}{M} \mathcal{B}(MG, MG) \\ &= \iint_{\mathbb{S}^{D-1} \times \mathbb{R}^D} (G_1' G' - G_1 G)\, b(\omega, v_1 - v,)\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1\,. \end{aligned}$$

For simplicity, we consider this problem over the periodic box $\mathbb{T}^D$.

Normalizations

This nondimensionalization has the normalizations

$$\int_{\mathbb{R}^D} M \, \mathrm{d}v = 1 \,, \qquad \int_{\mathbb{T}^D} \mathrm{d}x = 1 \,,$$

associated with the domains $\mathbb{R}^D$ and $\mathbb{T}^D$, the normalization

$$\iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} b(\omega, v_1 - v) \, \mathrm{d}\omega \, M_1 \mathrm{d}v_1 \, M \mathrm{d}v = 1 \,.$$

associated with the collision kernel b, and the normalizations

$$\iint_{\mathbb{R}^D\times\mathbb{T}^D} G^{in} \, M \mathrm{d}v \, \mathrm{d}x = 1 \,, \quad \iint_{\mathbb{R}^D\times\mathbb{T}^D} v \, G^{in} \, M \mathrm{d}v \, \mathrm{d}x = 0 \,,$$
$$\iint_{\mathbb{R}^D\times\mathbb{T}^D} \tfrac{1}{2}|v|^2 G^{in} \, M \mathrm{d}v \, \mathrm{d}x = \frac{D}{2} \,. \tag{6}$$

associated with the initial data G^{in}.

Notation

In this lecture $\langle\xi\rangle$ will denote the average over $\mathbb{R}^D$ of any integrable function $\xi = \xi(v)$ with respect to the positive unit measure $M\,\mathrm{d}v$:

$$\langle\xi\rangle = \int_{\mathbb{R}^D} \xi(v)\, M\,\mathrm{d}v\,.$$

Because $\mathrm{d}\mu = b(\omega, v_1 - v)\,\mathrm{d}\omega\; M_1\mathrm{d}v_1\; M\,\mathrm{d}v$ is a positive unit measure on $\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D$, we denote by $\langle\!\langle\Xi\rangle\!\rangle$ the average over this measure of any integrable function $\Xi = \Xi(\omega, v_1, v)$:

$$\langle\!\langle\Xi\rangle\!\rangle = \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} \Xi(\omega, v_1, v)\,\mathrm{d}\mu\,.$$

The measure $d\mu$ is invariant under the coordinate transformations

$$(\omega, v_1, v) \mapsto (\omega, v, v_1)\,, \qquad (\omega, v_1, v) \mapsto (\omega, v_1', v')\,.$$

These, and compositions of these, are called collisional symmetries.

Conservation Properties

The collision operator has the following property related to the conservation laws of mass, momentum, and energy.

For every measurable ζ the following are equivalent:

- $\zeta \in \operatorname{span}\{1, v_1, \cdots, v_D, \frac{1}{2}|v|^2\}$;
- $\langle \zeta \, \mathcal{Q}(G, G) \rangle = 0$ for "every" G;
- $\zeta_1' + \zeta' - \zeta_1 - \zeta = 0$ for every (ω, v_1, v).

Local Conservation Laws

If G solves the scaled Boltzmann equation then G satisfies local conservation laws of mass, momentum, and energy:

$$
\begin{aligned}
\mathsf{St}\,\partial_t\langle G\rangle + \nabla_x\cdot\langle v\,G\rangle &= 0\,,\\
\mathsf{St}\,\partial_t\langle v\,G\rangle + \nabla_x\cdot\langle v\otimes v\,G\rangle &= 0\,,\\
\mathsf{St}\,\partial_t\langle \tfrac{1}{2}|v|^2G\rangle + \nabla_x\cdot\langle v\tfrac{1}{2}|v|^2G\rangle &= 0\,.
\end{aligned}
$$

Global Conservation Laws

When these are integrated over space and time while recalling the normalizations associated with G^{in}, they yield the global conservation laws of mass, momentum, and energy:

$$
\begin{aligned}
\int_{\mathbb{T}^D} \langle G(t) \rangle \, \mathrm{d}x &= \int_{\mathbb{T}^D} \langle G^{in} \rangle \, \mathrm{d}x = 1 \,, \\
\int_{\mathbb{T}^D} \langle v\, G(t) \rangle \, \mathrm{d}x &= \int_{\mathbb{T}^D} \langle v\, G^{in} \rangle \, \mathrm{d}x = 0 \,, \\
\int_{\mathbb{T}^D} \langle \tfrac{1}{2}|v|^2 G(t) \rangle \, \mathrm{d}x &= \int_{\mathbb{T}^D} \langle \tfrac{1}{2}|v|^2 G^{in} \rangle \, \mathrm{d}x = \frac{D}{2} \,.
\end{aligned}
\tag{7}
$$

Dissipation Properties

The collision operator has the following property related to the dissipation of entropy and equilibrium.

$$\langle \log(G)\, \mathcal{Q}(G,G) \rangle \leq 0 \quad \text{for "every" } G\,.$$

Moreover, for "every" G the following are equivalent:

- $\langle \log(G)\, \mathcal{Q}(G,G) \rangle = 0$;
- $\mathcal{Q}(G,G) = 0$;
- G is a local Maxwellian.

Local Entropy Dissipation Law

If G solves the scaled Boltzmann equation then G satisfies local entropy dissipation law:

$$
\begin{aligned}
\mathsf{St}\,\partial_t\langle(G\log(G)-G+1)\rangle + \nabla_x\cdot\langle v\,(G\log(G)-G+1)\rangle
&= \frac{1}{\mathsf{Kn}}\langle\log(G)\,\mathcal{Q}(G,G)\rangle \\
&= -\frac{1}{\mathsf{Kn}}\left\langle\!\!\left\langle \tfrac{1}{4}\log\left(\frac{G_1'G'}{G_1G}\right)(G_1'G'-G_1G)\right\rangle\!\!\right\rangle \\
&\leq 0\,.
\end{aligned}
$$

Global Entropy Dissipation Law

When this is integrated over space and time, it yields the global entropy equality

$$H(G(t)) + \frac{1}{\mathrm{St}\,\mathrm{Kn}} \int_0^t R(G(s))\,\mathrm{d}s = H(G^{in})\,,$$

where the relative entropy functional H is given by

$$H(G) = \int_{\mathbb{T}^D} \langle (G\log(G) - G + 1)\rangle\,\mathrm{d}x \geq 0\,,$$

while the entropy dissipation rate functional R is given by

$$R(G) = \int_{\mathbb{T}^D} \tfrac{1}{4}\left\langle\!\!\left\langle \log\left(\frac{G_1' G'}{G_1 G}\right)(G_1' G' - G_1 G)\right\rangle\!\!\right\rangle \mathrm{d}x \geq 0\,.$$

III. ASSUMPTIONS ON THE COLLISION KERNEL

We now give our additional assumptions regarding the collision kernel b. These assumptions are satiisfied by many classical collision kernels.

For example, they are satisfied by the collision kernel for hard spheres of mass m and radius r_o, which has the form

$$b(\omega, v_1 - v) = |\omega \cdot (v_1 - v)| \frac{(2r_o)^{D-1}}{2m} \, . \tag{8}$$

They are also satisfied by all the classical collision kernels with a small deflection cutoff that derive from a repulsive intermolecular potential of the form c/r^k with $k > 2\frac{D-1}{D+1}$. Specifically, these kernels have the form

$$b(\omega, v_1 - v) = \widehat{b}(\omega \cdot n)\,|v_1 - v|^\beta \quad \text{with } \beta = 1 - 2\frac{D-1}{k}\,, \tag{9}$$

where $\widehat{b}(\omega \cdot n)$ is positive almost everywhere, has even symmetry in ω, and satisfies the small deflection cutoff condition

$$\int_{\mathbb{S}^{D-1}} \widehat{b}(\omega \cdot n)\,\mathrm{d}\omega < \infty\,. \tag{10}$$

The condition $k > 2\frac{D-1}{D+1}$ is equivalent to $\beta > -D$, which insures that $b(\omega, v_1 - v)$ is locally integrable with respect to $v_1 - v$.

The cases $\beta < 0$, $\beta = 0$, and $\beta > 0$ correspond respectively to the so-called “soft”, “Maxwell”, and “hard” potential cases.

DiPerna-Lions Assumption

Our *first assumption* is that b satifies the requirements of the DiPerna-Lions theory. That theory requires that $b(\omega, v_1 - v)$ be locally integrable with respect to $\mathrm{d}\omega\, M_1 \mathrm{d}v_1\, M \mathrm{d}v$, and that it moreover satisfies

$$\lim_{|v|\to\infty} \frac{1}{1+|v|^2} \int_K \bar{b}(v_1 - v)\, \mathrm{d}v_1 = 0\,, \quad \text{for every compact } K \subset \mathbb{R}^D\,, \tag{11}$$

where $\bar{b}$ is defined by

$$\bar{b}(v_1 - v) \equiv \int_{\mathbb{S}^{D-1}} b(\omega, v_1 - v)\, \mathrm{d}\omega\,.$$

Galilean symmetry (4) implies that $\bar{b}(v_1 - v)$ is a function of $|v_1 - v|$ only. The hard sphere kernel (8) and the inverse power kernels (9) with $\beta > -D$ satisfy this assumption.

Attenuation Assumption

A major role will be played by the attenuation coefficient a, which is defined by

$$a(v) \equiv \int_{\mathbb{R}^D} \bar{b}(v_1 - v)\, M_1 \mathrm{d}v_1 = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} b(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1\,.$$

Galilean symmetry (4) implies that $a(v)$ is a function of $|v|$ only.

Our *second assumption* regarding the collision kernal b is that a is bounded below as

$$C_a\Big(1 + |v|\Big)^{\alpha} \leq a(v)\,, \tag{12}$$

for some constants $C_a > 0$ and $\alpha \in \mathbb{R}$. This assumption is satisfied by the hard sphere kernel (8) and by the inverse power kernels (9) with $\beta > -D$.

Loss and Gain Operators

Another major role in what follows will be played by the linearized collision operator $\mathcal{L}$, which is defined formally by

$$\mathcal{L}\tilde{g} = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} \Big(\tilde{g} + \tilde{g}_1 - \tilde{g}' - \tilde{g}_1'\Big)\, b(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1 \,. \tag{13}$$

One has the decomposition

$$\frac{1}{a}\mathcal{L} = \mathcal{I} + \mathcal{K}^- - 2\mathcal{K}^+ \,,$$

where the loss operator $\mathcal{K}^-$ and gain operator $\mathcal{K}^+$ are formally defined by

$$\mathcal{K}^-\tilde{g} \equiv \frac{1}{a}\int_{\mathbb{R}^D} \tilde{g}_1\, \bar{b}(v_1 - v)\, M_1 \mathrm{d}v_1 \,,$$
$$\mathcal{K}^+\tilde{g} \equiv \frac{1}{2a}\iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} \Big(\tilde{g}' + \tilde{g}_1'\Big)\, b(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1 \,.$$

Loss Operator Assumption

Our *third assumption* regarding the collision kernel b is that there exists $s \in (1, \infty]$ and $C_b \in (0, \infty)$ such that

$$\left(\int_{\mathbb{R}^D} \left| \frac{\bar{b}(v_1 - v)}{a(v_1)\, a(v)} \right|^s a(v_1)\, M_1 \mathrm{d}v_1 \right)^{\frac{1}{s}} \leq C_b \, . \tag{14}$$

Because this bound is uniform in v, we may take C_b to be the supremum over v of the left-hand side of (14). This assumption is satisfied by the hard sphere kernel (8) and by the inverse power kernels (9) with $\beta > -D$. The case $s < \infty$ allows the treatment of soft potential kernels.

Because $\mathcal{K}^- : L^p(aM\mathrm{d}v) \rightarrow L^p(aM\mathrm{d}v)$ is bounded with $\|\mathcal{K}^-\|_{L^p} \leq 1$ for every $p \in [1, \infty)$, this assumption implies that

$$\mathcal{K}^- : L^p(aM\mathrm{d}v) \rightarrow L^p(aM\mathrm{d}v) \quad \text{is compact for every } p \in (1, \infty)\, .$$

Gain Operator Assumption

Our *fourth assumption* regarding b is that

$$\mathcal{K}^+ : L^2(aM\,\mathrm{d}v) \to L^2(aM\,\mathrm{d}v) \quad \text{is compact}\,. \tag{15}$$

This assumption is satisfied by the hard sphere kernel (8) and by the cutoff inverse power kernels (9) with $\beta > -D$. For general D this fact was demonstrated by Sun.

Because $\mathcal{K}^+ : L^p(aM\,\mathrm{d}v) \to L^p(aM\,\mathrm{d}v)$ is bounded with $\|\mathcal{K}^+\|_{L^p} \leq 1$ for every $p \in [1, \infty)$, this assumption implies that

$$\mathcal{K}^+ : L^p(aM\,\mathrm{d}v) \to L^p(aM\,\mathrm{d}v) \quad \text{is compact for every } p \in (1, \infty)\,.$$

Saturated Kernel Assumption

Our *fifth assumption* regarding b is that for every $\delta > 0$ there exists C_δ such that $\overline{b}$ satisfies

$$\frac{\overline{b}(v_1 - v)}{1 + \delta \, \dfrac{\overline{b}(v_1 - v)}{1 + |v_1 - v|^2}} \le C_\delta \Big(1 + a(v_1)\Big)\Big(1 + a(v)\Big) \quad \text{for every } v_1, v \in \mathbb{R}^D \,. \tag{16}$$

This assumption is satisfied by the hard sphere kernel (8) and by the cutoff inverse power kernels (9) with $\beta > -D$.

This technical assumption only enters into our key nonlinear equi-integrablity estimate. It allows the treatment of soft potential kernels.

Fredholm Operators and Null Spaces

The decomposition $\frac{1}{a}\mathcal{L} = \mathcal{I} + \mathcal{K}^- - 2\mathcal{K}^+$ and the fact that $\mathcal{K}^-$ and $\mathcal{K}^+$ are compact combine to show that

$$\frac{1}{a}\mathcal{L} : L^p(aM\,\mathrm{d}v) \to L^p(aM\,\mathrm{d}v) \quad \text{is Fredholm for every } p \in (1,\infty)\,. \tag{17}$$

Moreover, these operators are symmetric in the sense that

$$\left(\frac{1}{a}\mathcal{L}\Big|_{L^p(aM\,\mathrm{d}v)}\right)^* = \frac{1}{a}\mathcal{L}\Big|_{L^{p^*}(aM\,\mathrm{d}v)} \qquad \text{for every } p \in (1,\infty)\,.$$

These combine with classical L^2 results to show that for every $p \in (1,\infty)$

$$\mathrm{Null}\left(\frac{1}{a}\mathcal{L}\Big|_{L^p(aM\,\mathrm{d}v)}\right) = \mathrm{Null}(\mathcal{L}) = \mathrm{span}\{1, v_1, \cdots, v_D, |v|^2\}\,.$$

Coercivity of $\mathcal{L}$

One can show that for some $\ell > 0$ the operator $\mathcal{L}$ satisfies the coercivity estimate

$$\ell \left\langle a \, (\mathcal{P}^{\perp}\tilde{g})^2 \right\rangle \leq \langle \tilde{g} \mathcal{L} \tilde{g} \rangle \quad \text{for every } \tilde{g} \in L^2(aM\,\mathrm{d}v)\,. \tag{18}$$

Here $\mathcal{P}^{\perp} = \mathcal{I} - \mathcal{P}$ and $\mathcal{P}$ is the orthogonal projection from $L^2(M\,\mathrm{d}v)$ onto $\mathrm{Null}(\mathcal{L})$, which is given by

$$\mathcal{P}\tilde{g} = \langle \tilde{g} \rangle + v \cdot \langle v\, \tilde{g} \rangle + \left(\tfrac{1}{2}|v|^2 - \tfrac{D}{2}\right) \left\langle (\tfrac{1}{D}|v|^2 - 1)\, \tilde{g} \right\rangle .$$

Assumption (12) ensures that $\mathcal{P}$ and $\mathcal{P}^{\perp}$ are bounded as linear operators from $L^2(aM\,\mathrm{d}v)$ into itself.

Psuedoinverse of $\mathcal{L}$

We use a pseudoinverse of $\mathcal{L}$ defined as follows. For every $p \in (1,\infty)$ the Fredholm property (17) implies that for every $\xi \in L^p(a^{1-p}M\,\mathrm{d}v)$ there exists a unique $\widehat{\xi} \in L^p(aM\,\mathrm{d}v)$ such that

$$\mathcal{L}\widehat{\xi} = \mathcal{P}^{\perp}\xi\,, \qquad \mathcal{P}\widehat{\xi} = 0\,. \tag{19}$$

For every $\xi \in L^p(a^{1-p}M\,\mathrm{d}v)$ we define $\mathcal{L}^{-1}\xi = \widehat{\xi}$ where $\widehat{\xi}$ is determined above. This defines an operator $\mathcal{L}^{-1}$ such that

$$\begin{aligned} \mathcal{L}^{-1} : L^p(a^{1-p}M\,\mathrm{d}v) \to L^p(aM\,\mathrm{d}v) \quad & \text{is bounded}\,, \\ \mathcal{L}^{-1}\mathcal{L} = \mathcal{P}^{\perp} \quad & \text{over } L^p(aM\,\mathrm{d}v)\,, \\ \mathcal{L}\mathcal{L}^{-1} = \mathcal{P}^{\perp} \quad & \text{over } L^p(a^{1-p}M\,\mathrm{d}v)\,, \end{aligned}$$

and $\mathrm{Null}(\mathcal{L}^{-1}) = \mathrm{Null}(\mathcal{L})$. The operator $\mathcal{L}^{-1}$ is the unique pseudoinverse of $\mathcal{L}$ with these properties.

IV. FORMAL NAVIER-STOKES DERIVATION

The Boussinesq-balance incompressible Navier-Stokes system governs (ρ, u, θ), the fluctuations of mass density, bulk velocity, and temperature about their spatially homogeneous equilibrium values. Specifically, these fluctuations satisfy the incompressibility and Boussinesq relations

$$\nabla_x \cdot u = 0 \,, \qquad \rho + \theta = 0 \,; \tag{20}$$

while their evolution is governed by the motion and heat equations

$$\begin{aligned} &\partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \Delta_x u \,, && u(x,0) = u^{in}(x) \,, \\ &\tfrac{D+2}{2}\Big(\partial_t \theta + u \cdot \nabla_x \theta\Big) = \kappa \Delta_x \theta \,, && \theta(x,0) = \theta^{in}(x) \,, \end{aligned} \tag{21}$$

where ν is the kinematic viscosity and κ is the thermal conductivity.

The Navier-Stokes system (20–21) can be formally derived from the Boltzmann equation through a scaling in which the fluctuations of the kinetic density F about the absolute Maxwellian M are scaled to be on the order of ϵ. More precisely, we consider families of initial data G_ϵ^{in} for and families of solutions G_ϵ to the scaled Boltzmann initial-value problem (5) that are parametrized by $\mathsf{St} = \mathsf{Kn} = \epsilon$ and have the form

$$G_\epsilon^{in} = 1 + \epsilon\, g_\epsilon^{in}\,, \qquad G_\epsilon = 1 + \epsilon\, g_\epsilon\,, \tag{22}$$

One sees from the Boltzmann equation (5) satisfied by G_ϵ that the fluctuations g_ϵ satisfy

$$\epsilon\, \partial_t g_\epsilon + v \cdot \nabla_x g_\epsilon + \frac{1}{\epsilon} \mathcal{L} g_\epsilon = \mathcal{Q}(g_\epsilon, g_\epsilon)\,,$$

where $\mathcal{L}$ is the linearized collision operator defined by (13).

A moment-based formal derivation can be carried out by assuming that $g_\epsilon \to g$ with $g \in L^\infty(\mathrm{d}t; L^2(M\,\mathrm{d}v\,\mathrm{d}x))$, and that all formally small terms vanish. One finds that g has the infinitesimal Maxwellian form

$$g = v \cdot u + \left(\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2}\right)\theta\,, \tag{23}$$

where (u, θ) solves the Navier-Stokes system (20–21) with the coefficients of kinematic viscosity ν and thermal conductivity κ given by

$$\nu = \tfrac{1}{(D-1)(D+2)}\left\langle \widehat{A} : \mathcal{L}\widehat{A} \right\rangle, \qquad \kappa = \tfrac{1}{D}\left\langle \widehat{B} \cdot \mathcal{L}\widehat{B} \right\rangle. \tag{24}$$

Here $\widehat{A} = \mathcal{L}^{-1}A$ and $\widehat{B} = \mathcal{L}^{-1}B$ where the matrix-valued function A and the vector-valued function B are defined by

$$A(v) = v \otimes v - \tfrac{1}{D}|v|^2 I\,, \qquad B(v) = \tfrac{1}{2}|v|^2 v - \tfrac{D+2}{2}v\,.$$

Because $\mathcal{P}A = 0$ and $\mathcal{P}B = 0$, it follows from (19) that $\widehat{A}$ and $\widehat{B}$ are respectively the unique solutions of

$$\mathcal{L}\widehat{A} = A\,, \quad \mathcal{P}\widehat{A} = 0\,, \qquad \text{and} \qquad \mathcal{L}\widehat{B} = B\,, \quad \mathcal{P}\widehat{B} = 0\,. \tag{25}$$

Because each entry of A and B is in $L^p(a^{1-p}M\mathrm{d}v)$ for every $p \in (1,\infty)$, each entry of $\widehat{A}$ and $\widehat{B}$ is therefore in $L^p(aM\mathrm{d}v)$ for every $p \in (1,\infty)$.

V. GLOBAL SOLUTIONS

In order to justify the Navier-Stokes limit of the Boltzmann equation, we must make precise: (1) the notion of solution for the Boltzmann equation, and (2) the notion of solution for the fluid dynamical systems. Ideally, these solutions should be global while the bounds should be physically natural.

We therefore work in the setting of DiPerna-Lions renormalized solutions for the Boltzmann equation, and in the setting of Leray solutions for the Navier-Stokes system. These theories have the virtues of considering physically natural classes of initial data, and consequently, of yielding global solutions.

DiPerna-Lions Theory

The DiPerna-Lions theory gives the existence of a global weak solution to a class of formally equivalent initial-value problems that are obtained by multiplying the Boltzmann equation by $\Gamma'(G)$, where Γ' is the derivative of an admissible function Γ:

$$\begin{aligned} \Big(\epsilon\,\partial_t + v\cdot\nabla_x\Big)\Gamma(G) &= \frac{1}{\epsilon}\,\Gamma'(G)\mathcal{Q}(G,G)\,,\\ G(v,x,0) &= G^{in}(v,x)\geq 0\,. \end{aligned} \tag{26}$$

This is the so-called renormalized Boltzmann equation. A differentiable function $\Gamma : [0,\infty)\to\mathbb{R}$ is called *admissible* if for some constant $C_\Gamma<\infty$ it satisfies

$$\Big|\Gamma'(Z)\Big| \leq \frac{C_\Gamma}{\sqrt{1+Z}} \quad \text{for every } Z\geq 0\,.$$

The solutions lie in $C([0,\infty); w\text{-}L^1(M\,\mathrm{d}v\,\mathrm{d}x))$, where the prefix "$w$-" on a space indicates that the space is endowed with its weak topology.

DiPerna-Lions Solutions

We say that $G \geq 0$ is a weak solution of the renormalized Boltzmann equation provided that it is initially equal to G^{in}, and that for every $Y \in L^\infty(\mathrm{d}v; C^1(\mathbb{T}^D))$ and every $[t_1, t_2] \subset [0, \infty)$ it satisfies

$$\begin{aligned}
\epsilon \int_{\mathbb{T}^D} \langle \Gamma(G(t_2))\, Y \rangle \,\mathrm{d}x - \epsilon \int_{\mathbb{T}^D} \langle \Gamma(G(t_1))\, Y \rangle \,\mathrm{d}x& \\
- \int_{t_1}^{t_2} \int_{\mathbb{T}^D} \langle \Gamma(G)\, v \cdot \nabla_x Y \rangle \,\mathrm{d}x \,\mathrm{d}t& \\
= \frac{1}{\epsilon} \int_{t_1}^{t_2} \int_{\mathbb{T}^D} \Big\langle \Gamma'(G) \mathcal{Q}(G, G)\, Y \Big\rangle \, dx \, dt \,.&
\end{aligned}$$

If G is such a weak solution of for one such Γ with $\Gamma' > 0$, and if G satisfies certain bounds, then it is a weak solution for every admissible Γ. Such solutions are called *renormalized solutions* of the Boltzmann equation.

DiPerna-Lions Theorem - 1

Theorem. 1 (DiPerna-Lions Renormalized Solutions) *Let b satisfy*

$$\lim_{|v|\to\infty} \frac{1}{1+|v|^2} \int_{\mathbb{S}^{D-1}\times K} b(\omega, v_1 - v)\, \mathrm{d}\omega\, \mathrm{d}v_1 = 0$$

for every compact $K \subset \mathbb{R}^D$.

Given any initial data G^{in} in the entropy class

$$E(M\,\mathrm{d}v\,\mathrm{d}x) = \left\{ G^{in} \geq 0 \,:\, H(G^{in}) < \infty \right\}, \qquad (27)$$

there exists at least one $G \geq 0$ in $C([0,\infty); w\text{-}L^1(M\,\mathrm{d}v\,\mathrm{d}x))$ that is a renormalized solution of the Boltzmann equation.

This solution satisfies a weak form of the local conservation law of mass

$$\epsilon\, \partial_t \langle G \rangle + \nabla_x \cdot \langle v\, G \rangle = 0\,.$$

DiPerna-Lions Theorem - 2

Moreover, there exists a martix-valued distribution W *such that* $W\,\mathrm{d}x$ *is nonnegative definite measure and* G *and* W *satisfy a weak form of the local conservation law of momentum*

$$\epsilon\,\partial_t\langle v\,G\rangle + \nabla_x\cdot\langle v\otimes v\,G\rangle + \nabla_x\cdot W = 0\,,$$

and for every $t > 0$*, the global energy equality*

$$\int_{\mathbb{T}^D}\langle\tfrac{1}{2}|v|^2 G(t)\rangle\,\mathrm{d}x + \int_{\mathbb{T}^D}\tfrac{1}{2}\,\mathrm{tr}(W(t))\,\mathrm{d}x = \int_{\mathbb{T}^D}\langle\tfrac{1}{2}|v|^2 G^{in}\rangle\,\mathrm{d}x\,,$$

and the global entropy inequality

$$H(G(t)) + \int_{\mathbb{T}^D}\tfrac{1}{2}\,\mathrm{tr}(W(t))\,\mathrm{d}x + \frac{1}{\epsilon^2}\int_0^t R(G(s))\,\mathrm{d}s \le H(G^{in})\,.$$

DiPerna-Lions Theorem - 3

Remarks: DiPerna-Lions renormalized solutions are very weak — much weaker than standard weak solutions. They are not known to satisfy many properties that one would formally expect to be satisfied by solutions of the Boltzmann equation. In particular, the theory does not assert either the local conservation of momentum, the global conservation of energy, the global entropy equality, or even a local entropy inequality; nor does it assert the uniqueness of the solution.

Leray Theory

The DiPerna-Lions theory has many similarities with the Leray theory of global weak solutions of the initial-value problem for Navier-Stokes type systems. For the Navier-Stokes system with mean zero initial data, we set the Leray theory in the following Hilbert spaces of vector- and scalar-valued functions:

$$\begin{aligned}
\mathbb{H}_v &= \left\{ w \in L^2(\mathrm{d}x;\mathbb{R}^D) \,:\, \nabla_x \cdot w = 0\,, \int w \,\mathrm{d}x = 0 \right\}, \\
\mathbb{H}_s &= \left\{ \chi \in L^2(\mathrm{d}x;\mathbb{R}) \,:\, \int \chi \,\mathrm{d}x = 0 \right\}, \\
\mathbb{V}_v &= \left\{ w \in \mathbb{H}_v \,:\, \int |\nabla_x w|^2 \,\mathrm{d}x < \infty \right\}, \\
\mathbb{V}_s &= \left\{ \chi \in \mathbb{H}_s \,:\, \int |\nabla_x \chi|^2 \,\mathrm{d}x < \infty \right\}.
\end{aligned}$$

Let $\mathbb{H} = \mathbb{H}_v \oplus \mathbb{H}_s$ and $\mathbb{V} = \mathbb{V}_v \oplus \mathbb{V}_s$.

Leray Theorem

Theorem. 2 (Leray Solutions) *Given any initial data* $(u^{in}, \theta^{in}) \in \mathbb{H}$, *there exists at least one* $(u, \theta) \in C([0, \infty); w\text{-}\mathbb{H}) \cap L^2(dt; \mathbb{V})$ *that is a weak solution of the Navier-Stokes system and that for every* $t > 0$, *satisfies the dissipation inequalities*

$$\int \tfrac{1}{2}|u(t)|^2 \mathrm{d}x + \int_0^t \int \nu |\nabla_x u|^2 \mathrm{d}x \, \mathrm{d}s \leq \int \tfrac{1}{2}|u^{in}|^2 dx \,,$$
$$\int \tfrac{D+2}{4}|\theta(t)|^2 \mathrm{d}x + \int_0^t \int \kappa |\nabla_x \theta|^2 \mathrm{d}x \, \mathrm{d}s \leq \int \tfrac{D+2}{4}|\theta^{in}|^2 \mathrm{d}x \,.$$

Remarks: By arguing formally from the Navier-Stokes system, one would expect these inequalities to be equalities. However, that is not asserted by the Leray theory. Also, as was the case for the DiPerna-Lions theory, the Leray theory does not assert uniqueness of the solution.

A Variant of Leray Theory

Because the role of the above dissipation inequalities is to provide a-priori estimates, the existence theory also works if they are replaced by the single dissipation inequality

$$\begin{aligned} \int \tfrac{1}{2}|u(t)|^2 + \tfrac{D+2}{4}|\theta(t)|^2 dx + \int_0^t \int \nu |\nabla_x u|^2 + \kappa |\nabla_x \theta|^2 dx\, ds \\ \leq \int \tfrac{1}{2}|u^{in}|^2 + \tfrac{D+2}{4}|\theta^{in}|^2 dx\,. \end{aligned} \tag{28}$$

It is this version of the Leray theory that we will obtain in the limit.

VI. MAIN RESULTS

Our main theorem is the following.

Theorem. 3 *Let the collision kernel b satisfy the assumptions given above.*

Let G_ϵ^{in} be a family in the entropy class $E(M\,\mathrm{d}v\,\mathrm{d}x)$ given by (27) that satisfies the normalizations (6) and the bound

$$H(G_\epsilon^{in}) \leq C^{in}\epsilon^2\,, \tag{29}$$

for some positive constant C^{in}. Let g_ϵ^{in} be the associated family of fluctuations given by (22). Assume that for some $(u^{in},\theta^{in}) \in \mathbb{H}$ the family g_ϵ^{in} satisfies in the sense of distributions

$$\lim_{\epsilon\to 0}\Big(\Pi\Big\langle v\,g_\epsilon^{in}\Big\rangle, \Big\langle(\tfrac{1}{D+2}|v|^2-1)\,g_\epsilon^{in}\Big\rangle\Big) = (u^{in},\theta^{in})\,. \tag{30}$$

Let G_ϵ be any family of DiPerna-Lions renormalized solutions of the Boltzmann equation (5) that have G_ϵ^{in} as initial values. Let g_ϵ be the family of fluctuations given by (22).

Then the family g_ϵ is relatively compact in $w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\sigma M \,\mathrm{d}v\,\mathrm{d}x))$, where $\sigma = 1+|v|^2$. Every limit point g of g_ϵ has the infinitesimal Maxwellian form (23) where $(u,\theta) \in C([0,\infty); w\text{-}\mathbb{H}) \cap L^2(\mathrm{d}t; \mathbb{V})$ is a Leray solution with initial data (u^{in}, θ^{in}) of the Navier-Stokes system (20–21) with ν and κ given by (24). More specifically, (u,θ) satifies the weak form of the Navier-Stokes system and the dissipation inequality

$$\begin{aligned}\int_{\mathbb{T}^D} \tfrac{1}{2}|u(t)|^2 + \tfrac{D+2}{4}|\theta(t)|^2 \,\mathrm{d}x + \int_0^t\!\!\int_{\mathbb{T}^D} \nu|\nabla_x u|^2 + \kappa|\nabla_x \theta|^2 \,\mathrm{d}x\,\mathrm{d}s \\ \leq \liminf_{\epsilon\to 0} \frac{1}{\epsilon^2} H(G_\epsilon^{in}) \leq C^{in}\,. \end{aligned} \tag{31}$$

Moreover, every subsequence g_{ϵ_k} *of* g_ϵ *that converges to* g *as* $\epsilon_k \to 0$ *also satisfies*

$$\Pi\langle v\, g_{\epsilon_k}\rangle \to u \quad \textit{in } C([0,\infty);\mathcal{D}'(\mathbb{T}^D;\mathbb{R}^D))\,, \tag{32}$$

$$\langle(\tfrac{1}{D+2}|v|^2 - 1)\, g_{\epsilon_k}\rangle \to \theta \quad \textit{in } C([0,\infty); w\text{-}L^1(\mathrm{d}x;\mathbb{R}))\,. \tag{33}$$

where Π *is the orthogonal projection from* $L^2(\mathrm{d}x;\mathbb{R}^D)$ *onto divergence-free vector fields.*

Remark. The dissipation inequality (31) is just (28) with the right-hand side replaced by the `lim inf`. We can recover (28) in the limit by replacing (29) and (30) with the hypothesis

$$g_\epsilon^{in} \to v\cdot u^{in} + \left(\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2}\right)\theta^{in} \quad \text{entropically at order } \epsilon \text{ as } \epsilon \to 0\,. \tag{34}$$

The notion of *entropic convergence* is defined in [Bardos-Golse-L-93] as follows.

Definition 4 *Let G_ϵ be a family in the entropy class $E(M\,\mathrm{d}v\,\mathrm{d}x)$ and let g_ϵ be the associated family of fluctuations given by*

$$g_\epsilon = \frac{G_\epsilon - 1}{\epsilon}\,.$$

The family g_ϵ is said to converge entropically at order ϵ *to some $g \in L^2(M\,\mathrm{d}v\,\mathrm{d}x)$ if and only if*

$$\begin{gathered} g_\epsilon \to g \text{ in } w\text{-}L^1(M\,\mathrm{d}v\,\mathrm{d}x)\,, \\ \text{and} \\ \lim_{\epsilon\to 0} \frac{1}{\epsilon^2} H(G_\epsilon) = \int_{\mathbb{T}^D} \tfrac{1}{2}\langle g^2\rangle\,\mathrm{d}x\,. \end{gathered} \tag{35}$$

Remark. Entropic convergence is stronger than norm convergence in $L^1(\sigma M\,\mathrm{d}v\,\mathrm{d}x)$. It is thereby a natural tool for obtaining strong convergence results for fluctuations about an absolute Maxwellian.

With the addition of hypothesis (34), it is clear from (31) and (35) that (28) is recovered. Moreover, one can prove in the style of Theorem 6.2 of [Golse-L-02] that if (28) is an equality for every $t \in [0,T]$ then as $\epsilon \to 0$ one obtains the strong convergences

$$g_\epsilon(t) \to v \cdot u(t) + \left(\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2}\right)\theta(t)$$

entropically at order ϵ for every $t \in [0,T]$, and

$$\frac{G'_{\epsilon 1} G'_\epsilon - G_{\epsilon 1} G_\epsilon}{\epsilon^2 (1 + \frac{1}{3}\epsilon\, g_\epsilon)^{\frac{1}{2}}} \to \Phi : \nabla_x u + \Psi \cdot \nabla_x \theta$$

in $L^1([0,T]; L^1((\sigma + \sigma_1)\mathrm{d}\mu\,\mathrm{d}x))$, where

$$\Phi = A + A_1 - A' + A'_1\,, \quad \text{and} \quad \Psi = B + B_1 - B' + B'_1\,.$$

In particular, one obtains these strong convergences for so long as (u, θ) is a classical solution of the Navier-Stokes system.

VII. PROOF OF MAIN THEOREM

Step 1: Compactness of Fluctuations. Because the family G_ϵ satisfies the entropy inequality

$$H\Big(G_\epsilon(t)\Big) + \frac{1}{\epsilon^2}\int_0^t R\Big(G_\epsilon(s)\Big)\,\mathrm{d}s \le H\Big(G_\epsilon^{in}\Big) \le C^{in}\epsilon^2\,, \tag{36}$$

we obtain compactness results for the family of fluctuations

$$g_\epsilon = \frac{G_\epsilon - 1}{\epsilon}\,,$$

and the associated family of scaled collision integrands defined by

$$q_\epsilon = \frac{G'_{\epsilon 1} G'_\epsilon - G_{\epsilon 1} G_\epsilon}{\epsilon^2}\,.$$

Proposition 3.1 (1) of [Bardos-Golse-L-93] implies that the family

$$\sigma g_\epsilon \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(M\,\mathrm{d}v\,\mathrm{d}x))\,,$$

where $\sigma = 1 + |v|^2$. Proposition 3.4 (1) of [BGL-93] imply that the family

$$\sigma \frac{q_\epsilon}{\sqrt{n_\epsilon}} \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}\mu\,\mathrm{d}x))\,,$$

where $n_\epsilon = 1 + \frac{1}{3}\epsilon\, g_\epsilon$, and $\mathrm{d}\mu = b(\omega, v_1 - v)\,\mathrm{d}\omega\, M_1 \mathrm{d}v_1\, M\mathrm{d}v$.

Consider any convergent subsequence of the family g_ϵ, abusively still denoted g_ϵ, such that the sequence $q_\epsilon/\sqrt{n_\epsilon}$ also converges.

Step 2: Form of the Limit Points. Let g be the limit point of the sequence g_ϵ, and q be the limit point of the sequence $q_\epsilon/\sqrt{n_\epsilon}$. Proposition 3.8 of [BGL-93] implies that g is an infinitesimal Maxwellian given by

$$g = \rho + v \cdot u + (\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\theta\,, \tag{37}$$

for some $(\rho, u, \theta) \in L^\infty(\mathrm{d}t; L^2(\mathrm{d}x; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}))$ that for every $t \geq 0$ satisfies

$$\begin{aligned}\int_{\mathbb{T}^D} \tfrac{1}{2}|\rho(t)|^2 + \tfrac{1}{2}|u(t)|^2 + \tfrac{D}{4}|\theta(t)|^2 \,\mathrm{d}x &= \int_{\mathbb{T}^D} \tfrac{1}{2}\Big\langle |g(t)|^2 \Big\rangle \,\mathrm{d}x \\ &\leq \liminf_{\epsilon \to 0} \frac{1}{\epsilon^2} H\Big(G_\epsilon(t)\Big)\,.\end{aligned} \tag{38}$$

Proposition 3.4 (2) of [BGL-93] implies that $q \in L^2(\mathrm{d}\mu\,\mathrm{d}x\,\mathrm{d}t)$. We will show that (ρ, u, θ) is a Leray solution of the Navier-Stokes system (20–21) with initial data (u^{in}, θ^{in}).

Step 3: Nonlinear Equi-Integrablity. A major breakthrough of Saint-Raymond was the development of the relative entropy cutoff method. Using this we show that the entropy inequality (36) implies the sequence

$$a\,\frac{g_\epsilon^2}{n_\epsilon} \quad \text{is bounded in } L^1_{loc}(\mathrm{d}t; L^1(M\,\mathrm{d}v\,\mathrm{d}x)),$$

and that for every $[0,T] \subset [0,\infty)$

$$\lim_{\eta\to 0} \int_0^T \int_{\mathbb{T}^D} \sup_{\langle \mathbf{1}_S \rangle < \eta} \left\langle \mathbf{1}_S\, a\, \frac{g_\epsilon^2}{n_\epsilon} \right\rangle \mathrm{d}x\,\mathrm{d}t = 0 \quad \text{uniformly in } \epsilon\,,$$

where the supremum is taken over all measurable $S \subset \mathbb{R}^D \times \mathbb{T}^D \times [0,T]$. Here $\mathbf{1}_S$ denotes the indicator function of S.

Step 4: Nonlinear Compactness by L^1-Velocity Averaging. Using the L^1-velocity averaging theory of [Golse and Saint-Raymond 02] and our equi-integrability result we show that the sequence

$$a\,\frac{g_\epsilon^2}{n_\epsilon} \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(M\,\mathrm{d}v\,\mathrm{d}x))\,. \tag{39}$$

They used this averaging theory to prove analogous compactness results while establishing Navier-Stokes-Fourier limits for collision kernels with a Grad cutoff that derive from hard potentials [GStR-04, GStR-08].

Step 5: Approximate Conservation Laws. In order to prove our main theorem we have to pass to the limit in approximate local and global conservation laws built from the renormalized Boltzmann equation (26). We choose to use the normalization of that equation given by

$$\Gamma(Z) = \frac{Z-1}{1+(Z-1)^2}\,.$$

After dividing by ϵ, equation (26) becomes

$$\epsilon\,\partial_t \tilde{g}_\epsilon + v\cdot\nabla_x \tilde{g}_\epsilon = \frac{1}{\epsilon^2}\Gamma'(G_\epsilon)\,\mathcal{Q}(G_\epsilon, G_\epsilon)\,, \tag{40}$$

where $\tilde{g}_\epsilon = \Gamma(G_\epsilon)/\epsilon$. By introducing $N_\epsilon = 1+\epsilon^2 g_\epsilon^2$, we can write

$$\tilde{g}_\epsilon = \frac{g_\epsilon}{N_\epsilon}\,, \qquad \Gamma'(G_\epsilon) = \frac{2}{N_\epsilon^2} - \frac{1}{N_\epsilon}\,.$$

When the moment of the renormalized Boltzmann equation (40) is formally taken with respect to any $\zeta \in \mathrm{span}\{1, v_1, \cdots, v_D, |v|^2\}$, one obtains

$$\partial_t \langle \zeta\, \tilde{g}_\epsilon \rangle + \frac{1}{\epsilon} \nabla_x \cdot \langle v\, \zeta\, \tilde{g}_\epsilon \rangle = \frac{1}{\epsilon} \left\langle\!\!\left\langle \zeta\, \Gamma'(G_\epsilon)\, q_\epsilon \right\rangle\!\!\right\rangle . \tag{41}$$

This fails to be a local conservation law because the so-called *conservation defect* on the right-hand side is generally nonzero.

The fact that this defect vanishes as $\epsilon \to 0$ follows from the fact ζ is a collision invariant and the compactness result (39). Specifically, following [Golse-L 02] we show that

$$\frac{1}{\epsilon} \left\langle\!\!\left\langle \zeta\, \Gamma'(G_\epsilon)\, q_\epsilon \right\rangle\!\!\right\rangle \to 0 \quad \text{in } L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}x)) \text{ as } \epsilon \to 0 . \tag{42}$$

Step 6: Establishing the Global Conservation Laws. By (42) the right-hand side of (41) vanishes with ϵ uniformly over all $[t_1, t_2]$ contained in any bounded interval of time. Letting $\epsilon \to 0$, and using the normalization (7) shows that for every $t \geq 0$ we have the limiting global conservation law

$$\int_{\mathbb{T}^D} \langle \zeta \, g(t) \rangle \, \mathrm{d}x = \int_{\mathbb{T}^D} \langle \zeta \, g(0) \rangle \, \mathrm{d}x = 0 \, .$$

The infinitesimal Maxwellian form (37) then implies that, as stated by the Main Theorem,

$$\int_{\mathbb{T}^D} \rho \, \mathrm{d}x = 0 \, , \qquad \int_{\mathbb{T}^D} u \, \mathrm{d}x = 0 \, , \qquad \int_{\mathbb{T}^D} \theta \, \mathrm{d}x = 0 \, . \tag{43}$$

Step 7: Establishing the Incompressibility and Boussinesq Relations. By proceeding as in the proof of Proposition 4.2 of [BGL93], multiply (41) by ϵ, pass to the limit, and use the infinitesimal Maxwellian of the form to see that

$$\nabla_x \cdot u = 0\,, \qquad \nabla_x(\rho + \theta) = 0\,.$$

The first of these is the incompressibility relation while the second is a weak form of the Boussinesq relation. By (43) we see that

$$\int_{\mathbb{T}^D} \rho + \theta \, \mathrm{d}x = 0\,.$$

This then implies the Boussinesq relation

$$\rho + \theta = 0 \quad \text{for almost every } (x,t) \in \mathbb{T}^D \times [0,\infty)\,.$$

Step 8: Establishing the Dissipation Inequality. By passing to the limit in the weak form of (40) we show that for every $\widehat{\xi} \in L^2(aM\,\mathrm{d}v)$

$$\left\langle\!\!\left\langle \widehat{\xi}\, q \right\rangle\!\!\right\rangle = \left\langle \widehat{\xi}\, A \right\rangle : \nabla_{\!x} u + \left\langle \widehat{\xi}\, B \right\rangle \cdot \nabla_{\!x} \theta\,. \tag{44}$$

Then by arguing as in the proof of Proposition 4.6 of [BGL 93] we obtain

$$\begin{aligned} \int_0^t \!\! \int_{\mathbb{T}^D} \nu |\nabla_{\!x} u|^2 + \kappa |\nabla_{\!x} \theta|^2 \,\mathrm{d}x\,\mathrm{d}s &\leq \int_0^t \!\! \int_{\mathbb{T}^D} \tfrac{1}{4} \left\langle\!\!\left\langle q^2 \right\rangle\!\!\right\rangle \mathrm{d}x\,\mathrm{d}s \\ &\leq \liminf_{\epsilon \to 0} \frac{1}{\epsilon^4} \int_0^t R\Big(G_\epsilon(s)\Big)\,\mathrm{d}s\,, \end{aligned} \tag{45}$$

where ν and κ are given by (24). The dissipation inequality (31) asserted by the Main Theorem follows by combining (36), (38), and (45).

Step 9: Approximate Dynamical Equations. The approximate motion and heat equations are

$$\partial_t \langle v\, \tilde{g}_\epsilon \rangle + \frac{1}{\epsilon} \nabla_x \cdot \langle A\, \tilde{g}_\epsilon \rangle + \frac{1}{\epsilon} \nabla_x \langle \tfrac{1}{D} |v|^2 \tilde{g}_\epsilon \rangle = \frac{1}{\epsilon} \left\langle\!\!\left\langle v\, \Gamma'(G_\epsilon)\, q_\epsilon \right\rangle\!\!\right\rangle , \tag{46}$$

$$\partial_t \langle (\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2})\, \tilde{g}_\epsilon \rangle + \frac{1}{\epsilon} \nabla_x \cdot \langle B\, \tilde{g}_\epsilon \rangle = \frac{1}{\epsilon} \left\langle\!\!\left\langle (\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2})\, \Gamma'(G_\epsilon)\, q_\epsilon \right\rangle\!\!\right\rangle . \tag{47}$$

Second, the approximate momentum equation (46) will be integrated against divergence-free test functions. The last term in its flux will thereby be eliminated, and we only have to pass to the limit in the flux terms above that involve A and B — namely, in the terms

$$\frac{1}{\epsilon} \langle A\, \tilde{g}_\epsilon \rangle , \qquad \frac{1}{\epsilon} \langle B\, \tilde{g}_\epsilon \rangle . \tag{48}$$

Recall that $A = \mathcal{L}\widehat{A}$ and $B = \mathcal{L}\widehat{B}$ and that each entry of $\widehat{A}$ and $\widehat{B}$ is in $L^p(aM\,\mathrm{d}v)$ for every $p \in [1, \infty)$.

Step 10: Compactness of the Flux Terms. We claim that the sequences

$$\frac{1}{\epsilon}\,\langle A\,\tilde{g}_\epsilon\rangle \text{ and } \frac{1}{\epsilon}\,\langle B\,\tilde{g}_\epsilon\rangle \text{ are relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x))\,.$$

More generally, we show the following. Let $s \in (1, \infty]$ be from the assumed bound (14) on b. Let $p = 2 + \frac{1}{s-1}$, so that $p = 2$ when $s = \infty$. Let $\widehat{\xi} \in L^p(aM\mathrm{d}v)$ and set $\xi = \mathcal{L}\widehat{\xi}$. We show that the sequence

$$\frac{1}{\epsilon}\,\langle \xi\,\tilde{g}_\epsilon\rangle \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x))\,. \tag{49}$$

Because each entry of $\widehat{A}$ and $\widehat{B}$ is in $L^p(aM\mathrm{d}v)$, the claim follows. The proof uses the compactness result (39), a decomposition and new quadratic estimates.

If we define $\tilde{q}_\epsilon$ and T_ϵ by

$$\tilde{q}_\epsilon = \frac{q_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} = \frac{1}{\epsilon^2} \frac{G'_{\epsilon 1} G'_\epsilon - G_{\epsilon 1} G_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} \,,$$

$$\frac{1}{\epsilon} \left(\tilde{g}_\epsilon + \tilde{g}_{\epsilon 1} - \tilde{g}'_\epsilon - \tilde{g}'_{\epsilon 1} \right) = \tilde{g}'_{\epsilon 1} \tilde{g}'_\epsilon - \tilde{g}_{\epsilon 1} \tilde{g}_\epsilon - \tilde{q}_\epsilon + T_\epsilon \,,$$

the flux terms decompose as

$$\begin{aligned} \frac{1}{\epsilon} \langle \xi \, \tilde{g}_\epsilon \rangle &= \frac{1}{\epsilon} \left\langle \left(\mathcal{L} \hat{\xi} \right) \tilde{g}_\epsilon \right\rangle = \frac{1}{\epsilon} \left\langle \hat{\xi} \, \mathcal{L} \tilde{g}_\epsilon \right\rangle = \frac{1}{\epsilon} \left\langle\!\!\left\langle \hat{\xi} \left(\tilde{g}_\epsilon + \tilde{g}_{\epsilon 1} - \tilde{g}'_\epsilon - \tilde{g}'_{\epsilon 1} \right) \right\rangle\!\!\right\rangle \\ &= \left\langle\!\!\left\langle \hat{\xi} \left(\tilde{g}'_{\epsilon 1} \tilde{g}'_\epsilon - \tilde{g}_{\epsilon 1} \tilde{g}_\epsilon \right) \right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle \hat{\xi} \, \tilde{q}_\epsilon \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \hat{\xi} \, T_\epsilon \right\rangle\!\!\right\rangle \\ &= \left\langle \hat{\xi} \, \mathcal{Q}(\tilde{g}_\epsilon, \tilde{g}_\epsilon) \right\rangle - \left\langle\!\!\left\langle \hat{\xi} \, \tilde{q}_\epsilon \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \hat{\xi} \, T_\epsilon \right\rangle\!\!\right\rangle . \end{aligned}$$

The first term in this decomposition is quadratic in $\tilde{g}_\epsilon$, the second is linear in $\tilde{q}_\epsilon$, while the last is a remainder that vanishes as $\epsilon \to 0$.

We control the quadratic term with the following facts. Let $\Xi = \Xi(\omega, v_1, v)$ be in $L^p(\mathrm{d}\mu)$ and let $\tilde{g}$ and $\tilde{h}$ be in $L^2(aM\,\mathrm{d}v)$. Then $\Xi\,\tilde{g}_1\,\tilde{h}$ is in $L^1(\mathrm{d}\mu)$ and satisfies the L^1 bound

$$\left\langle\!\!\left\langle |\Xi\,\tilde{g}_1\,\tilde{h}| \right\rangle\!\!\right\rangle \le C_b^{\frac{1}{p^*}} \left\langle\!\!\left\langle |\Xi|^p \right\rangle\!\!\right\rangle^{\frac{1}{p}} \langle a\,\tilde{g}^2\rangle^{\frac{1}{2}}\,\langle a\,\tilde{h}^2\rangle^{\frac{1}{2}}\,,$$

where $\frac{1}{p} + \frac{1}{p^*} = 1$ and $\tilde{g}_1$ denotes $\tilde{g}(v_1)$. Moreover, if the family

$$\langle a\tilde{g}_\epsilon^2\rangle \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x))\,,$$

while the family

$$\langle a\tilde{h}_\epsilon^2\rangle \quad \text{is bounded in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x))\,,$$

then the family

$$\Xi\,\tilde{g}_{\epsilon 1}\,\tilde{h}_\epsilon \quad \text{is relatively compact in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}\mu\,\mathrm{d}x))\,.$$

Here $\tilde{g}_{\epsilon 1}$ denotes $\tilde{g}_\epsilon(v_1, x, t)$.

Step 11: Convergence of the Denisty Terms. We show that as $\epsilon \to 0$ the densities terms from (46) and (47) satisfy

$$\begin{aligned} \Pi\langle v\, \tilde{g}_\epsilon\rangle &\to u && \text{in } C([0,\infty); w\text{-}L^2(\mathrm{d}x;\mathbb{R}^D))\,, \\ \langle(\tfrac{1}{2}|v|^2 - \tfrac{D+2}{2})\, \tilde{g}_\epsilon\rangle &\to \tfrac{D+2}{2}\theta && \text{in } C([0,\infty); w\text{-}L^2(\mathrm{d}x))\,. \end{aligned} \tag{50}$$

Here Π is the Leray projection in $L^2(\mathrm{d}x;\mathbb{R}^D)$. The limits asserted in (32) and (33) of the Main Theorem then follow. Moreover, by combining these results with (36), (38), and (45), we conclude that $(u,\theta) \in C([0,\infty); w\text{-}\mathbb{H}) \cap L^2(\mathrm{d}t;\mathbb{V})$. By hypothesis (30) of the Main Theorem we also can argue that

$$u(x,0) = u^{in}(x)\,, \qquad \theta(x,0) = \theta^{in}(x)\,, \qquad \text{for almost every } x\,. \tag{51}$$

Step 12: Convergence of the Flux Terms. By using (44) we see as in [BGL-93] that as $\epsilon \to 0$ one has

$$\begin{aligned} &\left\langle\!\left\langle \widehat{A}\,\tilde{q}_\epsilon \right\rangle\!\right\rangle \to \nu\left[\nabla_x u + (\nabla_x u)^T\right] && \text{in } w\text{-}L^2_{loc}(\mathrm{d}t; w\text{-}L^2(\mathrm{d}x; \mathbb{R}^{D\vee D}))\,, \\ &\left\langle\!\left\langle \widehat{B}\,\tilde{q}_\epsilon \right\rangle\!\right\rangle \to \kappa\,\nabla_x\theta && \text{in } w\text{-}L^2_{loc}(\mathrm{d}t; w\text{-}L^2(\mathrm{d}x; \mathbb{R}^{D}))\,, \end{aligned}$$

where ν and κ are given by (24). Following [Lions and Masmoudi 2001] we pass the limit in the quadratic terms as

$$\left.\begin{aligned} \lim_{\epsilon\to 0} \Pi\nabla_x\cdot \left\langle \widehat{A}\,\mathcal{Q}(\tilde{g}_\epsilon, \tilde{g}_\epsilon) \right\rangle &= \Pi\nabla_x\cdot (u\otimes u) \\ \lim_{\epsilon\to 0} \nabla_x\cdot \left\langle \widehat{B}\,\mathcal{Q}(\tilde{g}_\epsilon, \tilde{g}_\epsilon) \right\rangle &= \tfrac{D+2}{2}\nabla_x\cdot (\theta\, u) \end{aligned}\right\} \quad \text{in } w\text{-}L^1_{loc}(\mathrm{d}t; \mathcal{D}'(\mathbb{T}^D))\,.$$

We thereby obtain the limiting fluxes for the Navier-Stokes motion and heat equations, thereby completing the proof of the Main Theorem. □

VIII. CONCLUSION

The DiPerna-Lions theory has been a great starting point for theories of fluid dynamical limits. However, much remains to be done. Major open problems in the program include:

- the acoustic limit with optimal scaling;
- limits for domains with boundaries;
- limits for non-cutoff collision kernels;
- dominant-balance Stokes, Navier-Stokes, and Euler limits;
- uniform in time results (compressible Stokes system).

THANK YOU!

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