About Renormalized Solutions and the Structure of Collision Kernels

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Renormalized solutions of an equation

Definition: Let (E) be an equation (EDP, or integrodifferential) for a nonnegative function f. We say that f is a renormalized solution of (E) when for some one-to-one (strictly increasing or strictly decreasing) smooth function ϕ from $\mathbb{R}_+$ to $\mathbb{R}_+$, the function $\phi\, o\, f$ satisfies in the sense of distributions the equation deduced form (E) for $\phi\, o\, f$ by applying at the formal level the chain rule.

Properties: All strong solutions are renormalized solutions. All renormalized solutions which are smooth are strong solutions.

Boltzmann Equation

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = Q(f,f)(t,x,v),$$

with

$$Q(f,f)(t,v) = \int_{\mathbb{R}^3} \int_{\sigma \in S^2} \Big\{ f(v') \, f(v'_*)$$

$$- f(v) \, f(v_*) \Big\} \, |v - v_*|^\gamma \, b\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma \right) d\sigma \, dv_*,$$

and

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \, \sigma,$$

$$v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \, \sigma.$$

Cross sections

$$B = |v - v_*|^\gamma \, b(\frac{v - v_*}{|v - v_*|} \cdot \sigma)$$

Angular cutoff of Grad for the angular part:

$$b \in L^1.$$

Classification for the kinetic part:

- ✓ $\gamma \in]0, 1],$ Hard Potentials or Hard Spheres,
- ✓ $\gamma = 0,$ Maxwellian Molecules,
- ✓ $\gamma \in]-2, 0[,$ Soft Potentials,
- ✓ $\gamma \in]-3, -2],$ Very Soft Potentials.

Remark : $\gamma = \frac{s-5}{s-1}$ for an intermolecular force in $1/r^s$.

Natural a priori estimates

Mass, energy and entropy:

$$\sup_{t\in[0,T]} \int\int f(t,x,v)\left(1+|x|^2+|v|^2+|\log f(t,x,v)|\right) dvdx$$

$$\leq Cst\left(T, \int\int f(0,x,v)\, dvdx, \int\int f(0,x,v)\,|x|^2\, dvdx,\right.$$

$$\left.\int\int f(0,x,v)\,|v|^2\, dvdx, \int\int f(0,x,v)\, \log f(0,x,v)\, dvdx\right).$$

Dissipation of entropy:

$$D(f) := \int_0^T \int .. \int \Big(f(t,x,v')\,f(t,x,v'_*) - f(t,x,v)\,f(t,x,v_*)\Big)$$

$$\times\Big(\log(f(t,x,v')\,f(t,x,v'_*)) - \log(f(t,x,v)\,f(t,x,v_*))\Big)\, B\, d\sigma dv_* dvdxdt \leq Cst(..).$$

Existence Theorem

Theorem (DiPerna, P.-L. Lions): Let B be a cross section with angular cutoff, for $\gamma \in]-3,1]$, and $f(0,\cdot,\cdot)$ be an initial datum with finite mass, $|x|^2$ moment, energy and entropy. Then, there exists a renormalized solution to the Boltzmann equation with this initial datum. More precisely, there exists $f \in C(\mathbb{R}_+; L^1)$ (nonnegative) such that $\frac{Q(f,f)}{\sqrt{1+f}}$ lies in $L^1_{loc}(\mathbb{R}_+; L^1)$ and

$$(\partial_t + v \cdot \nabla_x)\sqrt{1+f} = \frac{1}{2} \frac{Q(f,f)}{\sqrt{1+f}}.$$

Main estimate

$$\int_0^T \int \int \frac{Q(f,f)}{\sqrt{1+f}} dv dx dt = \int_0^T \int \int \frac{\int \int (f(v')\, f(v'_*) - f(v)\, f(v_*))\, B}{\sqrt{1+f(v)}} dv dx dt$$

$$= \int_0^T \int \int \frac{\int \int (\sqrt{f(v')\, f(v'_*)} + \sqrt{f(v)\, f(v_*)})\,(\sqrt{f(v')\, f(v'_*)} - \sqrt{f(v)\, f(v_*)})\, B}{\sqrt{1+f(v)}} dv dx dt$$

$$\leq Cst \left(\int .. \int \frac{(f(v')\, f(v'_*) + f(v)\, f(v_*))\, B}{1+f(v)} \right)^{1/2}$$

$$\times \left(\int .. \int (\sqrt{f(v')\, f(v'_*)} - \sqrt{f(v)\, f(v_*)})^2\, B \right)^{1/2}$$

$$\leq Cst \left(D(f) + \int_0^T \int \int f(t,x,v_*)\, B\, dv_* dx dt \right)^{1/2} (D(f))^{1/2}.$$

Consequence of $(x-y)\,(\log x - \log y) \geq Cst\, (\sqrt{x} - \sqrt{y})^2$.

Remarks

Restriction to spatially homogeneous data: The theory of renormalized solutions also holds for the spatially homogeneous case with a very soft (cutoff) potential.

Simplifications in the proof: It is possible to use an averaging lemma (Cf. Golse, P.-L. Lions, Perthame, Sentis) with a r.h.s. bounded in L^1 rather than weakly compact in L^1 (Cf. Gérard, Golse). It is possible to remove the use of subsolutions and supersolutions (Cf. P.-L. Lions).

Still lacking: the combined use of the strong compactness in v of Q^+ (Cf. Bouchut-LD, Lu, P.-L. Lions, Wennberg) and the averaging lemma in order to produce strong compactness in t, x, v for f.

Extensions (equations with first order derivatives only),

Cf. DiPerna, P.-L. Lions
Vlasov-Maxwell equation:

$$(\partial_t + v \cdot \nabla_x) f + (E + v \times B) \cdot \nabla_v f = 0,$$

Renormalized form:

$$(\partial_t + v \cdot \nabla_x) \frac{1}{1+f} + (E + v \times B) \cdot \nabla_v \frac{1}{1+f} = 0.$$

Useful when $f(0, \cdot, \cdot)$ is not bounded (E, B naturally lie in L^2).

Transport equation with divergence free $W^{1,1}$ coefficients.

Renormalized solutions with defect measure

Definition: Let (E) be a time-dependent equation (EDP, or integrodifferential) for a nonnegative function f, such that the mass $\int f$ is conserved at the formal level. We say that f is a renormalized solution with defect measure of (E) when for some one-to-one (strictly increasing or strictly decreasing) smooth function ϕ from $\mathbb{R}_+$ to $\mathbb{R}_+$, the function $\phi\, o\, f$ satisfies in the sense of distributions an inequality corresponding to the equation deduced from (E) for $\phi\, o\, f$ by applying at the formal level the chain rule.

Properties: All strong solutions are renormalized solutions with defect measures. To be verified and discussed on each case: all renormalized solutions which are smooth are strong solutions.

The example of the Landau equation (Cf. P.-L. Lions, Villani)

Landau equation:

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot \Big((a * f)\, \nabla_v f - (b * f)\, f \Big),$$

where

$$a(x) = \Phi(|x|^2) \Big\{ |x|^2\, Id - x \otimes x \Big\},$$

$$b(x) = \nabla \cdot a(x) = -\,(N-1)\, \Phi(|x|^2)\, x,$$

and Φ is a cross section.

According to P.-L. Lions, Villani, and under suitable assumptions on Φ, if $f(0,\cdot,\cdot)$ has finite mass, energy and entropy, then for β a function of class C^2 on $\mathbb{R}_+$, concave, and γ, ζ defined in such a way that

$$\forall x \in \mathbb{R}_+, \qquad \gamma'(x)^2 = -\beta''(x), \quad \zeta(x) = \beta(x) - x\,\beta'(x).$$

there exists a renormalized solution f with defect measure to the Landau equation, that is a solution of

$$(\partial_t + v \cdot \nabla_x)\beta(f) \geq \nabla_v \nabla_v : \Big((a * f)\,\beta(f) \Big) + (a * f) : \nabla_v \gamma(f) \nabla_v \gamma(f)$$

$$-\,2\,\nabla_v \cdot \Big((b * f)\,\beta(f) \Big) + ((\nabla \cdot b) * f)\,\zeta(f).$$

Extension to the Boltzmann equation without angular cutoff: Alexandre, Villani.

Reaction-diffusion systems for reversible chemistry

$$\mathcal{A}_1 + \mathcal{A}_3 \rightleftharpoons \mathcal{A}_2 + \mathcal{A}_4$$

Unknown : $a_i \equiv a_i(t,x) \geq 0$ concentration of $\mathcal{A}_i$ at $t \geq 0$ and $x \in \Omega$ (bounded regular open set of $\mathbb{R}^N$).

Equations (after rescaling):

$$\begin{cases} \partial_t a_1 - d_1\, \Delta_x a_1 = a_2\, a_4 - a_1\, a_3, \\ \partial_t a_2 - d_2\, \Delta_x a_2 = -(a_2\, a_4 - a_1\, a_3), \\ \partial_t a_3 - d_3\, \Delta_x a_3 = a_2\, a_4 - a_1\, a_3, \\ \partial_t a_4 - d_4\, \Delta_x a_4 = -(a_2\, a_4 - a_1\, a_3). \end{cases}$$

Equivalently

$$\partial_t a_i - d_i \Delta_x a_i = (-1)^{i+1}\,(a_2\, a_4 - a_1\, a_3).$$

Boundary and initial condition:

$$\nabla_x a_i(t,x) \cdot n(x) = 0 \text{ for } x \in \partial\Omega, \qquad a_i(0,x) \geq 0.$$

A priori estimates (1)

Conservation of numbers of molecules:

$$\int_\Omega \Big(a_1(t,x) + a_2(t,x)\Big)\,dx = \int_\Omega \Big(a_{10}(x) + a_{20}(x)\Big)\,dx := M_{12}$$

$$\int_\Omega \Big(a_1(t,x) + a_4(t,x)\Big)\,dx = \int_\Omega \Big(a_{10}(x) + a_{40}(x)\Big)\,dx := M_{14}$$

$$\int_\Omega \Big(a_3(t,x) + a_4(t,x)\Big)\,dx = \int_\Omega \Big(a_{30}(x) + a_{40}(x)\Big)\,dx := M_{34}$$

Consequence: estimate in $L^\infty(\mathbb{R}_+; L^1(\Omega))$.

Remark: enough to conclude in the case of ODEs or when all d_i are equal.

A priori estimates (2)

Entropy inequality:

$$-\frac{d}{dt}H(a_1, a_2, a_3, a_4) = D(a_1, a_2, a_3, a_4) \geq 0,$$

with the entropy

$$H(a_1, a_2, a_3, a_4) := \sum_{i=1}^{4} \int_\Omega (a_i \, \log a_i - a_i + 1) \, dx,$$

and the entropy dissipation

$$D(a_1, a_2, a_3, a_4) := \sum_{i=1}^{4} d_i \int_\Omega \frac{|\nabla_x \, a_i|^2}{a_i} \, dx + \int_\Omega (a_1 \, a_3 - a_2 \, a_4) \log \left(\frac{a_1 \, a_3}{a_2 \, a_4} \right) dx.$$

Kinetic equations for reactive rarefied gases

One dominant specie of density

$$M(v) = (2\pi)^{-3/2}\, e^{-\frac{|v|^2}{2}}$$

and 4 species of (relatively) "small" density.

Unknown: density $f^i(t,x,v) \geq 0$ of molecules of species i ($i = 1, .., 4$) which at time t and point x have velocity v.

(Rescaled) equation:

$$\epsilon \frac{\partial f^i}{\partial t} + v \cdot \nabla_x f^i = \frac{1}{\epsilon}\, Q_{EL}(f^i, M) + \sum_{j=1}^{4} Q_{EL}(f^i, f^j) + \epsilon\, Q^i_{CH}(f^1, .., f^4)\,.$$

where Q_{EL} is a kernel for elastic (non-reactive) collisions and Q^i_{CH} is a kernel for reactive collisions.

Link with the reaction-diffusion

(Formal) limit when $\varepsilon \longrightarrow 0$ (LD, M. Bisi):

$$f_\varepsilon^i(t,x,v) \to a^i(t,x)\,\frac{e^{-\frac{|v|^2}{2}}}{(2\pi)^{3/2}},$$

where

$$\frac{\partial a_i}{\partial t} - d_i\,\Delta_{\mathbf{x}} a_i = (-1)^{i+1}\,(a_2\,a_4 - a_1\,a_3).$$

Cf. also R. Spigler, D. Zanette for asymptotics starting from BGK.

Entropy inequality at the kinetic level

$$-\frac{d}{dt} H_{kin}(f_\varepsilon^1, f_\varepsilon^2, f_\varepsilon^3, f_\varepsilon^4) = D_{kin}(f_\varepsilon^1, f_\varepsilon^2, f_\varepsilon^3, f_\varepsilon^4),$$

with the entropy

$$H_{kin}(f_1, f_2, f_3, f_4) := \sum_{i=1}^{4} \int_\Omega \int_{\mathbb{R}^3} f_i(t,x,v) \, \log f_i(t,x,v) \, dv dx \,,$$

and the entropy dissipation $D_{kin}(f_1, f_2, f_3, f_4) \geq 0$.

The entropy estimate passes to the limit when $\varepsilon \to 0$.

Consequence of the entropy estimate

Entropy and entropy dissipation:

$$\sup_{t\in\mathbb{R}_+} \left(\int_\Omega \Big(a_i(t,x)\,\log(a_i(t,x)) - a_i(t,x) + 1 \Big)\, dx \right) \le C,$$

$$\int_0^\infty \int_\Omega |\nabla_x \sqrt{a_i}|^2 \, dx dt \le C.$$

Estimates in Sobolev and Orlicz spaces:

$$a_i \in L^\infty(\mathbb{R}_+; L \log L(\Omega)),$$

$$\sqrt{a_i} \in L^2(\mathbb{R}_+; H^1(\Omega)).$$

Use of Sobolev inequality

$$\int_0^T \left(\int_\Omega |a_i(t,x)|^{N/(N-2)}\, dx \right)^{(N-2)/N} dt \le C\,(1+T).$$

Interpolation with the $L^\infty(\mathbb{R}_+; L^1(\Omega))$ estimate:

$$||a_i||_{L^{1+2/N}([0,T]\times\Omega)} \le C\,(1+T)^{N/(N+2)}.$$

When $N = 2$, this estimate is replaced by

$$\int_0^T \left(\int_\Omega |a_i(t,x)|^p\, dx \right)^{1/p} dt \le C\,(1+T),$$

for all $p \in [1, +\infty[$, and after interpolation,

$$||a_i||_{L^p([0,T]\times\Omega)} \le C\,(1+T)^{1/p}$$

for all $p \in [1, 2[$.

Two-dimensional case: use of Trudinger's inequality

For some $s > 0$,

$$\int_0^T \int_\Omega a_i(t,x)\, \exp\left(\frac{s\, a_i(t,x)}{\|\sqrt{a_i}(t,\cdot)\|^2_{H^1(\Omega)}} \right) dxdt \le C\,(1+T).$$

Thanks to the interpolation with

$$a_i \in L^\infty(\mathbb{R}_+; L\log L(\Omega)),$$

and Young's inequality (valid for any $\gamma > 0$)

$$x\,y \le e^{\gamma x} + \frac{y}{\gamma}(\log(\frac{y}{\gamma}) - 1),$$

one gets

$$||a_i||_{L^2([0,T]\times\Omega)} \le C\,(1+T)^{1/2}.$$

One-dimensional case : Smoothness (1)

When $N = 1$, thanks to Sobolev inequality,

$$\int_0^T \left(\sup_{x\in\Omega} a_i(t,x) \right) dt \leq C\,(1+T),$$

and after interpolation,

$$||a_i||_{L^2([0,T]\times\Omega)} \leq C\,(1+T)^{1/2}.$$

Then (Cf. Rothe), if the initial datum is smooth,

$$a_i \in L^2([0,T]\times\Omega) \qquad \Rightarrow \qquad a_2\,a_4; a_1\,a_3 \in L^1([0,T]\times\Omega)$$

$$\Rightarrow \qquad \partial_t a_i - d_i\,\Delta_x a_i \in L^1([0,T]\times\Omega) \Rightarrow \qquad a_i \in L^{3-0}([0,T]\times\Omega)$$

One-dimensional case : Smoothness (2)

$$\Rightarrow \quad a_2\, a_4; a_1\, a_3 \in L^{3/2-0}([0,T]\times\Omega) \quad \Rightarrow \quad \partial_t a_i - d_i\, \Delta_x a_i \in L^{3/2-0}([0,T]\times\Omega)$$

$$\Rightarrow \quad a_i \in L^{\infty-0}([0,T]\times\Omega) \quad \Rightarrow \quad a_2\, a_4; a_1\, a_3 \in L^{\infty-0}([0,T]\times\Omega)$$

$$\Rightarrow \quad \partial_t a_i - d_i\, \Delta_x a_i \in L^{\infty-0}([0,T]\times\Omega) \quad \Rightarrow \quad a_i \in C^{0,1-0}([0,T]\times\overline{\Omega})$$

$$\Rightarrow \quad a_i \in C^{1,1-0}([0,T]\times\overline{\Omega}) \quad \Rightarrow \quad a_i \in C^{1,2-0}([0,T]\times\overline{\Omega})$$

$$\Rightarrow \quad \ldots \quad \Rightarrow \quad a_i \in C^{\infty}([0,T]\times\overline{\Omega}).$$

Remark:

$$||a_i||_{C^p([0,T]\times\overline{\Omega})} \leq Pol(T).$$

The method of duality applied to the entropy

$$\partial_t \left(\sum_{i=1}^{4} (a_i \log a_i - a_i + 1) \right) - \sum_{i=1}^{4} d_i \, \Delta_x \Big(a_i \log a_i - a_i + 1 \Big) \leq 0,$$

so that

$$\partial_t \left(\sum_{i=1}^{4} (a_i \log a_i - a_i + 1) \right) - \Delta_x \left(A \sum_{i=1}^{4} (a_i \log a_i - a_i + 1) \right) \leq 0,$$

where

$$A = \frac{\sum_{i=1}^{4} d_i \, (a_i \log a_i - a_i + 1)}{\sum_{i=1}^{4} (a_i \log a_i - a_i + 1)}.$$

Estimates for singular parabolic equations

Lemma (R.-H. Martin, M. Pierre): Let $T > 0$ and Ω be a bounded regular open set of $\mathbb{R}^N$. We suppose that $A_0 \leq A(t,x) \leq A_1$, for some constants A_0, $A_1 > 0$.
We assume that $z : [0,T] \times \Omega \to \mathbb{R}$ is a nonnegative solution to the parabolic inequation :

$$\partial_t z - \Delta_x (A\, z) \leq 0, \qquad \nabla_x (A\, z) \cdot n = 0 \quad \text{on} \quad \partial\Omega.$$

Then, there exists a constant C (depending on T, A_0, A_1, Ω) such that

$$||z||_{L^2([0,T]\times\Omega)} \leq C\, ||z(0,\cdot)||_{L^2(\Omega)}.$$

Existence for all dimensions

Using the approximate equation

$$\partial_t a_i^n - d_i \, \Delta_x a_i^n = (-1)^{i+1} \, \frac{a_2^n \, a_4^n - a_1^n \, a_3^n}{1 + \frac{1}{n} \, \sum_i (a_i^n)^2},$$

(with Neumann boundary condition), one can prove the

Theorem (LD, K. Fellner, M. Pierre, J. Vovelle): Let Ω be a bounded regular open set of $\mathbb{R}^N$, let $d_i > 0$ be diffusivity constants, and let $a_i(0, \cdot) \geq 0$ be initial data in $L^2(\log L)^2(\Omega)$.
Then, there exists a weak solution $a_i \in L^2_{loc}(\mathbb{R}_+; L^2(\Omega))$ to the system

$$\partial_t a_i - d_i \, \Delta_x a_i = (-1)^{i+1} \, (a_2 \, a_4 - a_1 \, a_3),$$

with Neumann boundary condition.
In dimension 2, Goudon, Vasseur have proven that those solutions are strong.

Extension to higher order chemistry

Proposition (LD, K. Fellner, M. Pierre, J. Vovelle): Assume $a_i(0,\cdot) \in L^2\,(\log L)^2(\Omega)$. If $p_i \leq 2$ for all i, then equation

$$\partial_t a_i - d_i\,\Delta_x a_i = (-1)^i\,(a_1^{p_1}\,a_3^{p_3} - a_2^{p_2}\,a_4^{p_4})$$

(with Neumann boundary conditions) has a renormalized solution with defect measure in any dimension. That is, $a_i \in L^2_{loc}(\mathbb{R}_+; L^2(\Omega))$ and

$$\partial_t (1+a_i)^{-2} - d_i\,\Delta_x (1+a_i)^{-2} \leq \frac{2\,(-1)^i}{(1+a_i)^3}\,(a_1^{p_1}\,a_3^{p_3} - a_2^{p_2}\,a_4^{p_4}) - 6\,d_i\,\frac{|\nabla_x a_i|^2}{(1+a_i)^4}.$$

In dimension 1, it is also the case as soon as $p_i \leq 3$ for all i. It is moreover a weak solution if in addition $p_1 + p_3 \leq 3$ and $p_2 + p_4 \leq 3$.

Sketch of the proof ($p_i = 2$)

Thanks to the duality method, we obtain that $a_i \log |a_i|$ is bounded in $L^2([0,T] \times \Omega)$.
Morover the entropy estimate writes

$$\sup_{t\in[0,T]} \int_\Omega \sum a_i(t)\, \log a_i(t) + 2\int_0^T \int_\Omega \sum d_i\, \frac{|\nabla_x a_i|^2}{a_i}$$

$$+ \int_0^T \int_\Omega (a_1^2\, a_3^2 - a_2^2\, a_4^2) \log \left(\frac{a_1^2\, a_3^2}{a_2^2\, a_4^2} \right) \le C_T.$$

So

$$\frac{a_1^2\, a_3^2 - a_2^2\, a_4^2}{(1+a_i)^3}$$

is well defined.
Then, using an approximate sequence a_i^n of solutions, strong compactness is obtained thanks to the heat kernel, and one can pass to the limit in each term, except $\frac{|\nabla_x a_i^n|^2}{(1+a_i^n)^4}$, in which one uses Fatou's lemma to recover an inequality.

Coagulation equations and the problem of gelation

Discrete coagulation equation for diffusive polymers in dimension 1: ($c_i := c_i(t,x)$ density of polymers of size $i \in \mathbb{N}^*$):

$$\partial_t c_i - d_i\, \partial_{xx} c_i = -\sum_{j=1}^{\infty} a_{ij}\, c_i\, c_j + \sum_{j=1}^{i} a_{ji-j}\, c_j\, c_{i-j}\,,$$

where $0 < d_{min} \leq d_i \leq d_{max} < +\infty$, and $a_{ij} = a_{ji} \geq 0$.

At the formal level, the conservation of mass holds:

$$\sum_{i=1}^{\infty} \int i\, c_i(t,x)\, dx = \sum_{i=1}^{\infty} \int i\, c_i(0,x)\, dx.$$

A priori estimates

According to Laurençot-Mischler and Canizo-LD-Fellner, the quantities $\sum_{j=1}^{i} a_{j\,i-j}\, c_j\, c_{i-j}$ and $\sum_{j=1}^{\infty} a_{ij}\, c_i\, c_j$ lie in L^1, so that thanks to the heat kernel properties (in dimension 1): $c_i \in L^{3-0}$.
Passing to the limit in an approximate equation,

$$\partial_t c_i - d_i\, \partial_{xx} c_i \leq - \sum_{j=1}^{\infty} a_{ij}\, c_i\, c_j + \sum_{j=1}^{i} a_{j\,i-j}\, c_j\, c_{i-j}.$$

Problem: interpretation of this result when gelation may occur (that is, for example when $a_{ij} \geq Cst\,(i+j)^{1+\delta}$, with $\delta > 0$).

A few open problems

1. Direct proof of strong stability for the Boltzmann equation with cutoff
2. (Polynomially growing) L^∞ estimates in all dimensions for quadratic or superquadratic reactions?
3. Existence of (renormalized with defect measure) solutions for very high-order reactions?
4. Rigorous passage from reactive Boltzmann equations to quadratic reaction-diffusion equations?