Asymptotic-Preserving Particle-In-Cell Method for the Vlasov-Poisson System Near Quasineutrality

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1. Introduction

General topic I

- Numerical modeling of a device such that
  - an **important physical scale**, λ, is:
    - very small in a part of the domain ($\lambda \ll 1$),
    - an order 1 parameter elsewhere ($\lambda = O(1)$),
  - you do not want to describe the scale λ.
- Starting from model M_λ :
  - Valid everywhere
  - Classical schemes stable and consistant

    iff λ is resolved by the mesh $\Rightarrow$ **very huge cost**.

General topic II

A possible solution

- Use M_λ where $\lambda = O(1)$.
- Use an asymptotic model where $\lambda \ll 1$:

$$M_0 = \lim_{\lambda \to 0} M_\lambda.$$

- Problems:
  - Position of the interface.
  - Reconnection of M_λ and M_0.
  - **Moving interface**: difficult numerical pb in 2D or 3D.

Another possible solution

- Use M_λ everywhere.
- Discretized it with a scheme such that:
  - it does not need to resolve the scale λ:

    Asymptotic stability,

  - it gives an approx. solution of M_0 when $\lambda \to 0$:

    Asymptotic consistency

Asymptotically stable and consistent

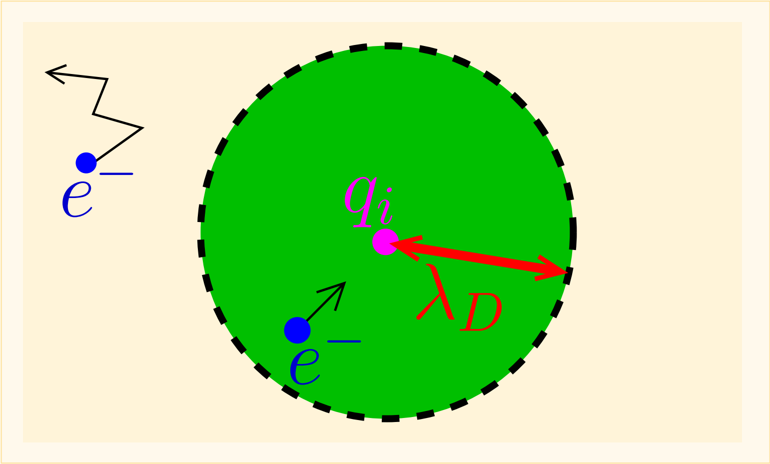
$\Rightarrow$ Asymptotic preserving scheme

([S.Jin] kinetic $\longrightarrow$ Hydro)

Caracteristic scales in plasmas

- **Debye length:**

$$\lambda_D = \left(\frac{\varepsilon_0 k_B T}{e^2 n} \right)^{1/2}$$



- Electrons are attracted by $q_i > 0$
- A cloud of < 0 charges around q_i
- Screening of q_i beyond the distance λ_D

$\Rightarrow$ Charge unbalances subsist only at scales $\leq \lambda_D$

Quasi-neutrality

- Quasi-neutral plasmas: (very frequent)

$$\lambda = \frac{\lambda_D}{L} \ll 1 \qquad \Rightarrow$$

Charge unbalances negligible

$$n_+(x,t) \approx n_-(x,t)$$

$L =$ caract. length of the problem

- Non quasi-neutral plasmas : (sheaths, beams, ...)

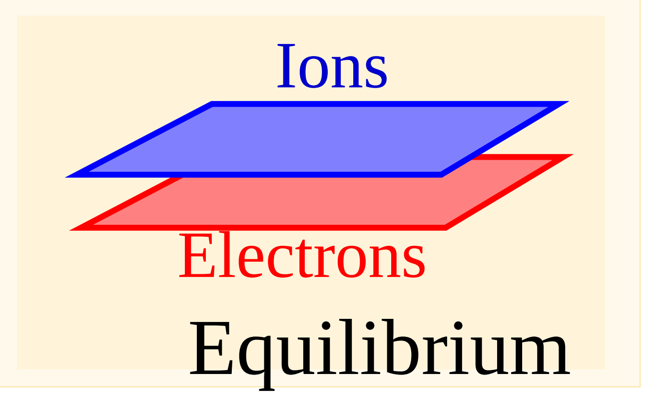
$$\lambda \sim 1 \qquad \Rightarrow$$

Charge unbalances of order 1

$$n_+(x,t) \neq n_-(x,t)$$

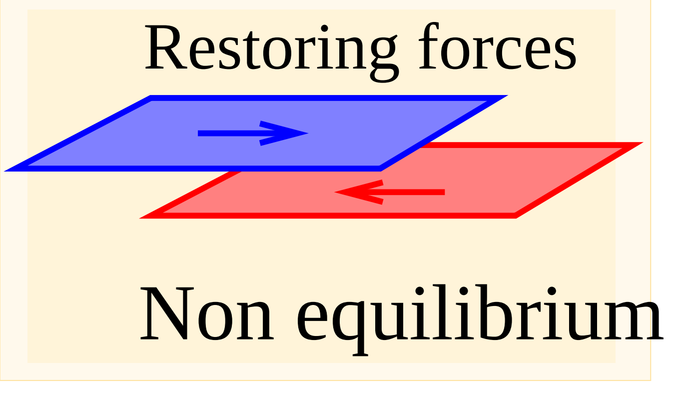
Time scale: plasma oscillations

- Plasma oscillations:
  - Charge unbalances
  - Restoring electric forces
  - Oscillations



- (electronic) Plasma period

$$\tau_e = \left(\frac{\varepsilon_0 m_e}{e^2 n} \right)^{1/2}$$



Time scale and the quasi-neutrality

- In quasi-neutral regime

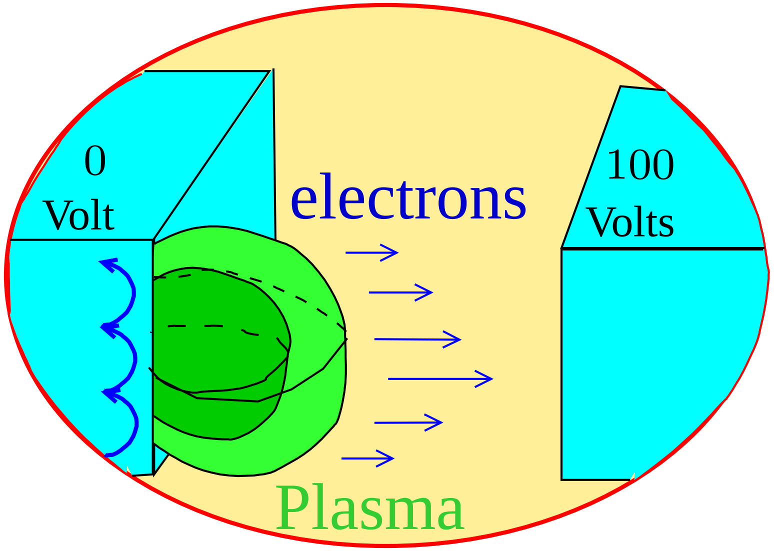
$$\tau := \frac{\tau_e}{t_0} \ll 1$$

t_0 = caracteristic time of the problem

- Quasi-neutral state = average over a very large number of plasma periods

Physical applications I

- Example 1: plasma expansion between 2 electrodes



- High current diodes,
  Spons. by CEA/DAM.
- Arcs on solar pannels,
  Sponsoring by CNES.

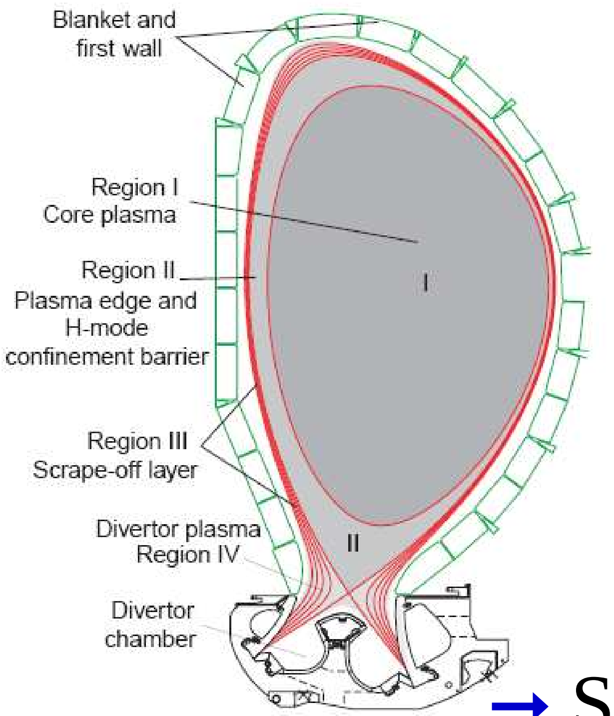
- AP scheme for Euler-Poisson in the quasineutral limit
  Crispel, Degond, MHV, 07 JCP.

- Asympt. stability : P. Degond, JG. Liu, MHV, SIAM08.

Physical applications II

- Example 2: ITER

  Deut.-Trit. fusion by magnetic confinement



- Sponsoring by CEA
- Collaboration with:
  - P. Degond, F. Deluzet, L. Navoret (Toulouse)
  - A.B. Sun (Xi'an, China)
  - A. Sangam (Nice)
  - S.Hirstoaga, E.Sonnendrücker (Strasbourg)
  - A. Ambroso, P. Omnès, J. Segré, X. Garbet, G. Falchetto, M. Ottaviani (CEA)

Different systems and limits

- AP schemes, quasineutral limit,
  - Euler-Maxwell,
  - Vlasov-Poisson
- Drift limits (Large magnetic field)
  - Euler-Lorentz
  - Vlasov-Lorentz

Outline

References

- Rigorous quasi-neutral limits
  - Cordier & Grenier, Wang, Alì & Jüngel
  - Brenier, Brenier & Grenier, Brenier & Corrias, Peng & Jüngel
- AP schemes in the quasi-neutral limit
  - Kinetic models
    - Cohen, Friedman, Langdon, Masson, . . .
    - Barnes, Brackbill, Forslund, Friedman, Hewett, Langdon, Masson, Wallace, . . .
  - Fluid models
    - [Fabre]
    - [Choe,Yoon,Kim,Choi], [Colella,Dorr,Wake],[Crispel,Degond,MHV]

2. An AP scheme in the quasineutral limit for Vlasov-Poisson

2.1. The quasineutral limit of Euler-Poisson

The Euler-Poisson model

- One species model for clarity

$$(EP) \begin{cases} \partial_t n + \nabla \cdot (n\, u) = 0, \\ \varepsilon\, \partial_t (n\, u) + \varepsilon\, \nabla\, (n\, u \otimes u) + \nabla p(n) = n \nabla \phi, \\ -\lambda^2 \Delta \phi = n_0 - n, \end{cases}$$

- $n_0 =$ constant ion density, $n =$ elec. density,
  $u =$ elec. velocity, $p(n) =$ elec. pressure,
  $\phi =$ potential, $\varepsilon = \dfrac{e^- \text{ mass}}{\text{ion mass}}$.

- $\lambda = \dfrac{\lambda_D}{L} = \dfrac{\text{Debye length}}{\text{caract. length}}$, $\tau = \lambda \sqrt{\varepsilon} = \dfrac{\text{plasma period}}{\text{caract. time}}$

$\lambda \to 0$: Quasi-neutral limit (I)

$$(QN) \begin{cases} \partial_t n + \nabla \cdot (n\, u) = 0 \\ \varepsilon\, \partial_t (n\, u) + \varepsilon\, \nabla\, (n\, u \otimes u) + \nabla p(n) = n \nabla \phi, \\ n = n_0. \end{cases}$$

➠ Equivalently: $\begin{cases} \nabla \cdot (n_0\, u) = 0, \\ \partial_t (n_0\, u) + \nabla\, (n_0\, u \otimes u) = \dfrac{n_0\, \nabla \phi}{\varepsilon}, \\ n = n_0. \end{cases}$

$n_0 = 1 \Rightarrow$ Incompressible Euler Eqs. (pressure$= -\phi$)

➠ $\phi =$ Lagrange multiplier of $\nabla \cdot (n_0\, u) = 0$

- Explicit eq. for the potential

$$\nabla \cdot \left(\partial_t (n_0\, u) + \nabla\, (n_0\, u \otimes u) = \frac{n_0\, \nabla \phi}{\varepsilon} \right)$$

$$\Downarrow \nabla \cdot (n_0\, u) = 0$$

QN elliptic eq. $- \nabla \cdot \left(\frac{n_0\, \nabla \phi}{\varepsilon} \right) = -\nabla^2 : (n_0\, u \otimes u)$

Reformulated systems

- Reformulated quasi-neutral model

$$
(RQN)\begin{cases}
n = n_0, \\
\varepsilon\,\partial_t(n_0\,u) + \varepsilon\,\nabla\,(n_0\,u\otimes u) + \nabla p(n) = n\nabla\phi, \\
-\nabla\cdot\left(\dfrac{n_0\,\nabla\phi}{\varepsilon}\right) = -\nabla^2 : (n_0\,u\otimes u)
\end{cases}
$$

- Is it possible to complete the diagram?

$$
\begin{array}{ccc}
\text{Euler-Poisson} & \xrightarrow{\lambda\to 0} & \text{Quasi-neutral} \\
\downarrow & & \downarrow \\
\begin{array}{c}\text{Reformulated}\\ \text{Euler-Poisson}\end{array} & \xrightarrow[\lambda\to 0]{} & \begin{array}{c}\text{Reformulated}\\ \text{quasi-neutral}\end{array}
\end{array}
$$

Reformulated Euler-Poisson system (I)

- Take the $\nabla\cdot$ of the momentum Eq.

$$\nabla \cdot (\partial_t (n\, u)) + \nabla^2 : S(n,u) = \nabla \cdot \left(\frac{n \nabla \phi}{\varepsilon} \right) \qquad (1)$$

with $\quad S(n,u) = n\, u \otimes u + p(n)\mathrm{Id}/\varepsilon$

- Take the ∂_t of the mass Eq.

$$\partial^2_{tt} n + \partial_t (\nabla \cdot (n\, u)) = 0 \qquad (2)$$

- Take the difference of (1) and (2)

$$-\partial^2_{tt} n + \nabla^2 : S(n,u) = \nabla \cdot \left(\frac{n \nabla \phi}{\varepsilon} \right)$$

Reformulated Euler-Poisson system (II)

- Use the Poisson Eq., $n = n_0 + \lambda^2 \Delta\phi$:

$$-\lambda^2 \Delta(\partial_{tt}^2 \phi) + \nabla^2 : S(n,u) = \nabla \cdot \left(\frac{n \nabla \phi}{\varepsilon} \right)$$

- The reformulated Poisson Eq.

$$\underbrace{\varepsilon\,\lambda^2}_{=\tau^2} \partial_{tt}^2(-\Delta\phi) - \nabla \cdot (n \nabla \phi) = -\varepsilon\, \nabla^2 : S(n,u)$$

Reformulated Euler-Poisson system (III)

$$(REP)\begin{cases}\partial_t n + \nabla\cdot(n\,u) = 0,\\ \varepsilon\,\partial_t(n\,u) + \varepsilon\,\nabla\,(n\,u\otimes u) + \nabla p(n) = n\nabla\phi,\\ \varepsilon\,\lambda^2\partial^2_{tt}(-\Delta\phi) - \nabla\cdot(n\nabla\phi) = -\varepsilon\,\nabla^2 : S(n,u),\end{cases}$$

➠ Reduces to (RQN) system when $\lambda = 0$

Properties of the reform. Poisson Eq. (I)

$$\varepsilon\,\lambda^2\partial_{tt}^2(-\Delta\phi) - \nabla\cdot(n\nabla\phi) = -\varepsilon\,\nabla^2 : S(n,u)$$

- New elliptic eq. replaces Poisson eq.
  - Equivalent to Poisson eq. under initial cond.

$$(\lambda^2\Delta\phi = n - n_0)|_{t=0} \quad \text{and} \quad \frac{d}{dt}(\lambda^2\Delta\phi = n - n_0)|_{t=0}.$$

- Does not degenerate when $\lambda \to 0$

Properties of the reform. Poisson Eq. (II)

- $n = constant$

$$\tau^2 \, \partial^2_{tt}\rho + n\,\rho = -\varepsilon\,\nabla^2 : S(n,u) \qquad (3)$$

- Harmonic oscillator Eq. on $\rho = -\Delta\phi$
- Explicit scheme $\Rightarrow$ conditional stab. $\Delta t \le \tau$
- Implicit scheme $\Rightarrow$ unconditional stability

2.2. The quasineutral limit of Vlasov-Poisson

The Vlasov-Poisson model

- One species model for clarity
  - Distribution function $f(x, v, t)$

$$
(VP) \begin{cases} \partial_t f + v \cdot \nabla_x f + \dfrac{\nabla_x \phi}{\varepsilon} \cdot \nabla_v f = 0, & (3) \\ -\lambda^2 \Delta \phi = n_0 - n, \quad n = \displaystyle\int f \, dv \, . \end{cases}
$$

- What is the quasi-neutral limit?

The reformulated Vlasov-Poisson model

➠ Taking the velocity moments of Vlasov (eq. (3))

$$\begin{cases} \partial_t n + \nabla_x \cdot (n\,u) = 0, & (4) \\ \partial_t (n\,u) + \nabla_x S = \dfrac{n \nabla_x \phi}{\varepsilon}, & (5) \\ n\,u = \displaystyle\int f\,v\,dv, \qquad S = \int f\,v \otimes v\,dv. \end{cases}$$

➠ $\nabla_x \cdot (5) - \partial_t (4)$ and $n = n_0 + \lambda^2 \Delta \phi$

$$\lambda^2 \partial_{tt}^2 (-\Delta \phi) - \nabla_x \cdot \left(\frac{n}{\varepsilon} \nabla_x \phi \right) = -\nabla_x^2 : S$$

- Reformulated Vlasov-Poisson model

$$(RVP)\begin{cases} \partial_t f + v\cdot\nabla_x f + \nabla_x\phi\cdot\nabla_v f = 0, \\ \varepsilon\,\lambda^2\partial_{tt}^2(-\Delta\phi) - \nabla_x\cdot(n\nabla_x\phi) = -\varepsilon\,\nabla_x^2 : S \\ n = \int f\,dv, \qquad S = \int f\,v\otimes v\,dv. \end{cases}$$

- Quasi-neutral limit of VP: $\lambda \to 0$

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + \nabla_x\phi\cdot\nabla_v f = 0, \\ -\nabla_x\cdot(n\nabla_x\phi) = -\varepsilon\,\nabla_x^2 : S \\ n = \int f\,dv\,, \qquad S = \int f\,v\otimes v\,dv. \end{cases}$$

2.3. Classical and Asymptotic Preserving PIC schemes

Principle of PIC method (I)

Particles In Cells

- Initially, (X_j^0, V_j^0) given numerical particles

$$f(x,v,0) \approx \sum_j \omega_j \;\delta\Big(x - X_j^0\Big)\,\delta\Big(v - V_j^0\Big)\,,$$

- $\phi \approx \phi_h =$ constant function on a grid of size h
  - Finite Difference approximation
  - A cell contains several numerical particles

Principle of PIC method (II)

➠ We follow the numerical particles

$$\begin{cases} \dfrac{dX_j(t)}{dt} = V_j(t)\,, & \dfrac{dV_j(t)}{dt} = \dfrac{(\nabla_x \phi)_h(X_j(t),t)}{\varepsilon}\,, \\[2ex] X_j(0) = X_j^0, & V_j(0) = V_j^0, \end{cases}$$

$$f(x,v,t) \approx \sum_j \omega_j \;\delta\Big(x - X_j(t)\Big)\,\delta\Big(v - V_j(t)\Big)\,.$$

Classical PIC scheme (I)

- Leapfrog scheme, $(X_j^m, V_j^{m+1/2})$ given:

$$\frac{X_j^{m+1} - X_j^m}{\Delta t} = V_j^{m+1/2},$$

$$\frac{V_j^{m+3/2} - V_j^{m+1/2}}{\Delta t} = \frac{(\nabla_x \phi)_h^{m+1}(X_j^{m+1})}{\varepsilon},$$

- ϕ_h finite difference approx. of Poisson eq.

$$-\lambda^2 \left(\Delta \phi\right)_h^{m+1} = \left(n_0 - n\right)_h^{m+1}$$

Classical PIC scheme (I)

- uncoupled scheme
  - Calculate separately X_j^{m+1}, ϕ_h^{m+1}, $V_j^{m+3/2}$
- Stable and consistant iff

$$\Delta t, h = \mathcal{O}(\lambda)$$

Asymptotic Preserving PIC scheme (I)

- Semi-implicit scheme

$$\frac{X_j^{m+1} - X_j^m}{\Delta t} = V_j^{m+1}, \qquad \frac{V_j^{m+1} - V_j^m}{\Delta t} = \frac{\nabla_x \phi^{m+1}(X_j^m)}{\varepsilon},$$

- ϕ_h finite diff. approx. of the reformulated Poisson eq.

$$\lambda^2 \, \varepsilon \, \frac{-\Delta_h \phi^{m+1} + 2\, \Delta_h \phi^m - \Delta_h \phi^{m-1}}{\Delta t^2}$$

$$-(\nabla_x)_h \cdot \left(n_h^m \, (\nabla_x \phi)_h^{m+1} \right) = -\varepsilon \, \nabla_x^2 : S_h^m$$

Asymptotic Preserving PIC scheme (II)

- $(\Delta\phi)_h^{m-1,m} \quad \Rightarrow \quad$ large truncation error if ϕ fluctuates
- Two different strategies
  - First strategy: PICAP-1
    - Eliminate $\Delta\phi^{m,m-1}$ using Poisson eq.

$$-(\nabla_x)_h \cdot \left[\left(\lambda^2 \varepsilon + \Delta t^2 \, n_h^m \right) (\nabla_x \phi)_h^{m+1} \right]$$

$$= -\varepsilon \, \Delta t^2 \nabla_x^2 : S_h^m - 2\, n_h^m + n_h^{m-1} + n_0,$$

    - steps m, $m-1 \Rightarrow$ step $m+1$

- Second strategy: PICAP-2

→ Eliminate $n_h^m - n_h^{m-1} = \Delta t\, \nabla_x \cdot (n\, u)_h^m$ using mass eq.

$$-(\nabla_x)_h \cdot \left[\left(\lambda^2 \varepsilon \, + \Delta t^2 \, n_h^m \right) (\nabla_x \phi)_h^{m+1} \right]$$

$$= -\varepsilon \, \Delta t^2 \nabla_x^2 : S_h^m - n_h^m + n_0 - \Delta t \, \nabla_x \cdot (n\, u)_h^m ,$$

→ steps $m \Rightarrow$ step $m + 1$

Asymptotic Preserving PIC scheme (III)

➠ Summary

$$\frac{X_j^{m+1} - X_j^m}{\Delta t} = V_j^{m+1}, \qquad \frac{V_j^{m+1} - V_j^m}{\Delta t} = \frac{\nabla_x \phi^{m+1}(X_j^m)}{\varepsilon},$$

➠ PICAP-1

$$-(\nabla_x)_h \cdot \left[\left(\lambda^2 \varepsilon + \Delta t^2 \, n_h^m \right) (\nabla_x \phi)_h^{m+1} \right] = G^m + H^{m-1},$$

➠ PICAP-2

$$-(\nabla_x)_h \cdot \left[\left(\lambda^2 \varepsilon + \Delta t^2 \, n_h^m \right) (\nabla_x \phi)_h^{m+1} \right] = I^m,$$

➠ uncoupled ➠ $\Delta t, h = \mathcal{O}(1)$ ➠ consistant $\lambda \to 0$

2.4. Numerical results

Test case (I)

Grismayer, Mora, Adam, Héron, Phys. Rev. 2008

- One dimensional two-species plasma expansion test case

$$\partial_t f_i + v\,\partial_x f_i - \partial_x\phi\,\partial_v f_i = 0,$$
$$\partial_t f_e + v\,\partial_x f_e + \frac{1}{\varepsilon}\partial_x\phi\,\partial_v f_e = 0,$$
$$-\lambda^2\partial_{xx}^2\phi = n_i - n_e, \quad n_{i,e} = \int f_{i,e}\,dv.$$

- Initially, ions and electrons are Maxwellian

$$f_{e0} = n_{e0}\sqrt{\frac{\varepsilon}{2\pi}}\,e^{-\varepsilon v^2/2}, \quad f_{i0} = n_{i0}\sqrt{\frac{1}{2\pi T_{i0}/T_{e0}}}\,e^{-v^2/(2T_{i0}/T_{e0})},$$

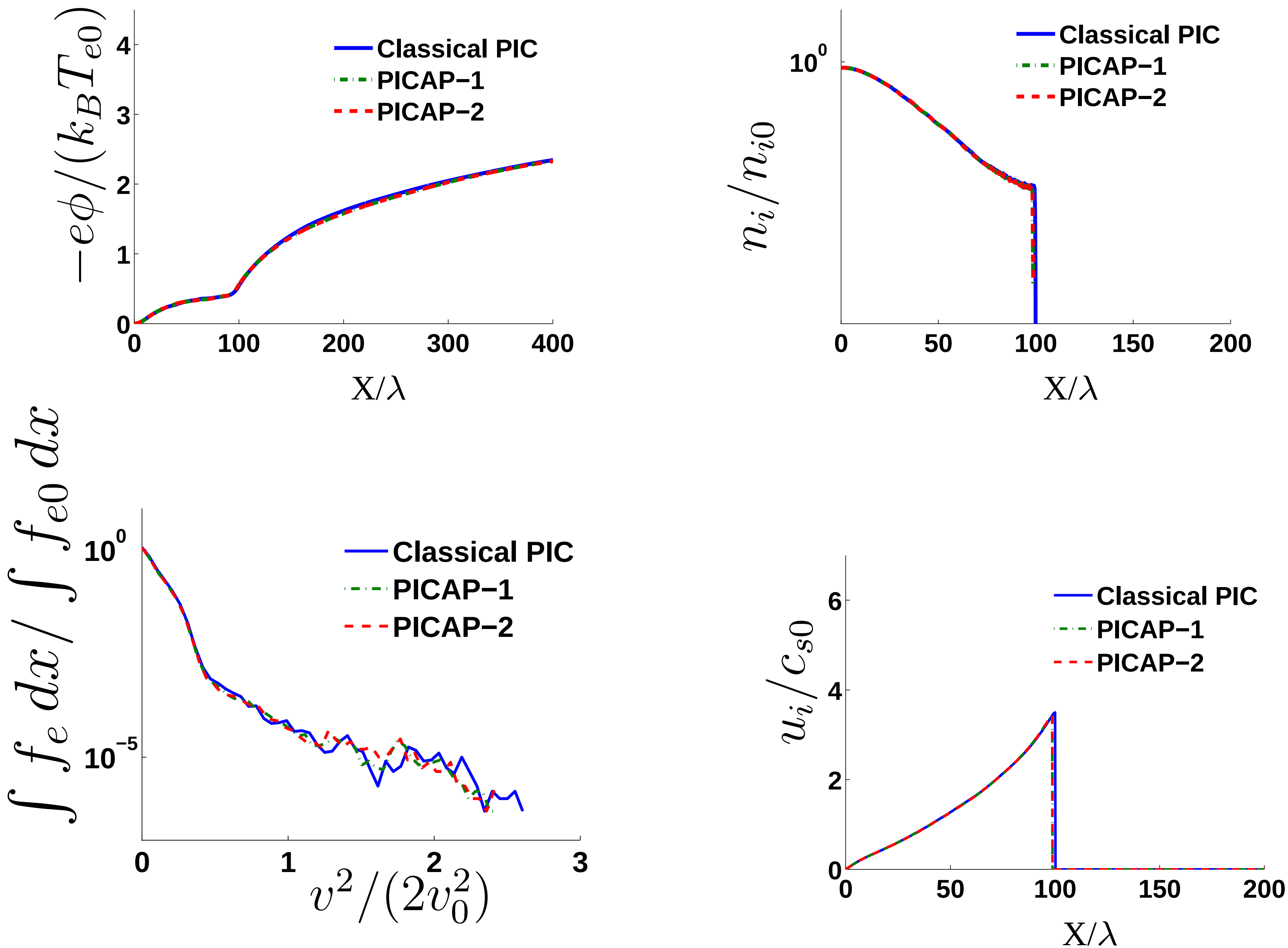
Test case (II)

- Domain: $x \in [0, 3.10^4\,\lambda]$.
- Initially

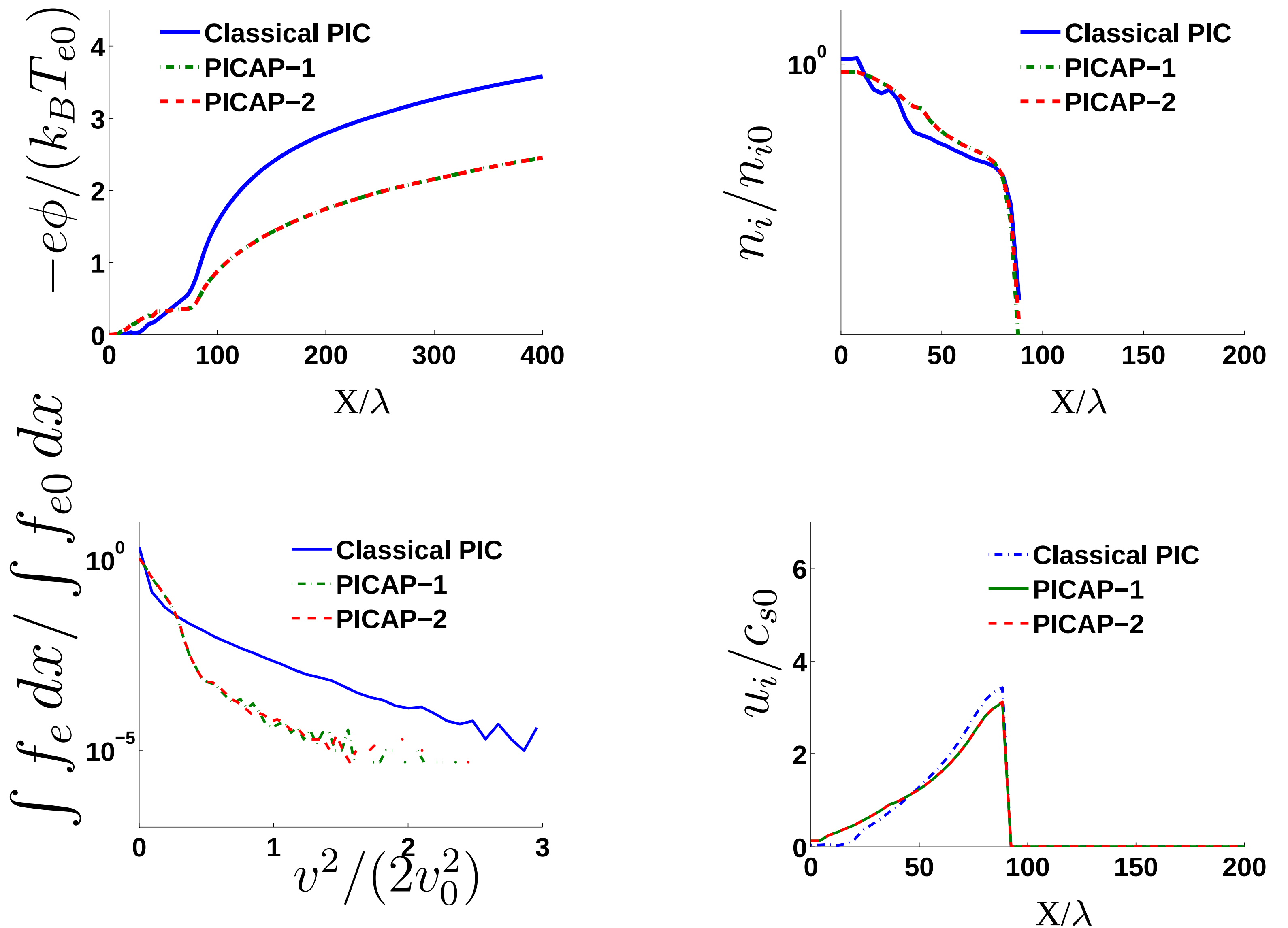
$$n_{i0} = \begin{cases} 1, & 0 \le x \le 20\,\lambda, \\ 0, & 20\,\lambda \le x \le 3.10^4. \end{cases} \qquad \begin{cases} n_{e0} = \exp(\phi_0), \\ -\partial^2_{xx}\phi_0 = n_{i0} - \exp(\phi_0). \end{cases}$$

- Number of numer. particles (ions+electrons) $\approx 5.10^6$.
  - Resolved case: $\Delta t = 0.05\tau$, $h = 0.2\,\lambda$
  - Half-resolved case: $\Delta t = 0.05\tau$, $h = 4\,\lambda$
  - Unresolved case: $\Delta t = 3\,\tau$, $h = 4\,\lambda$

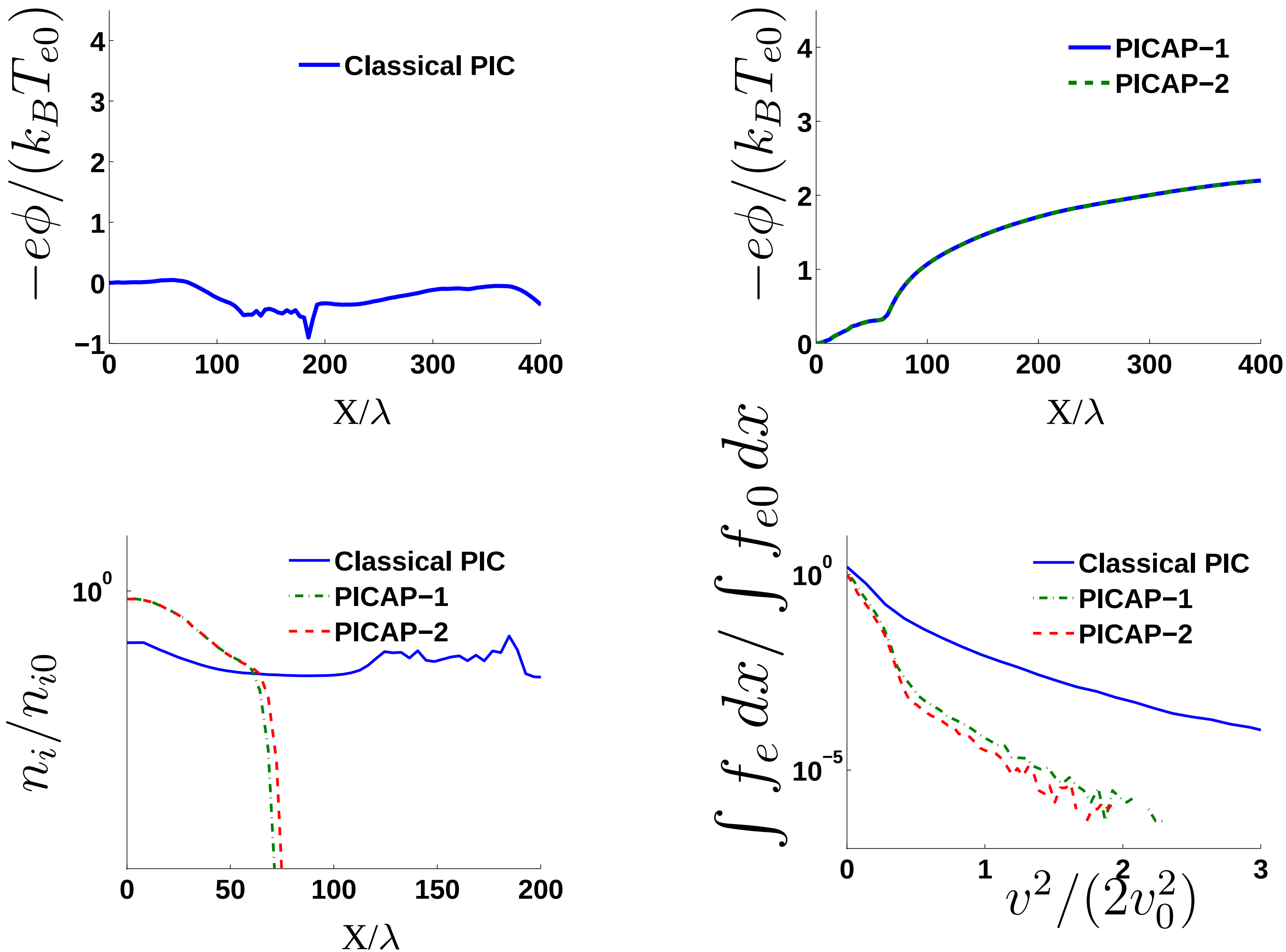
Resolved case: $\Delta t = 0.05\tau$, $h = 0.2\,\lambda$



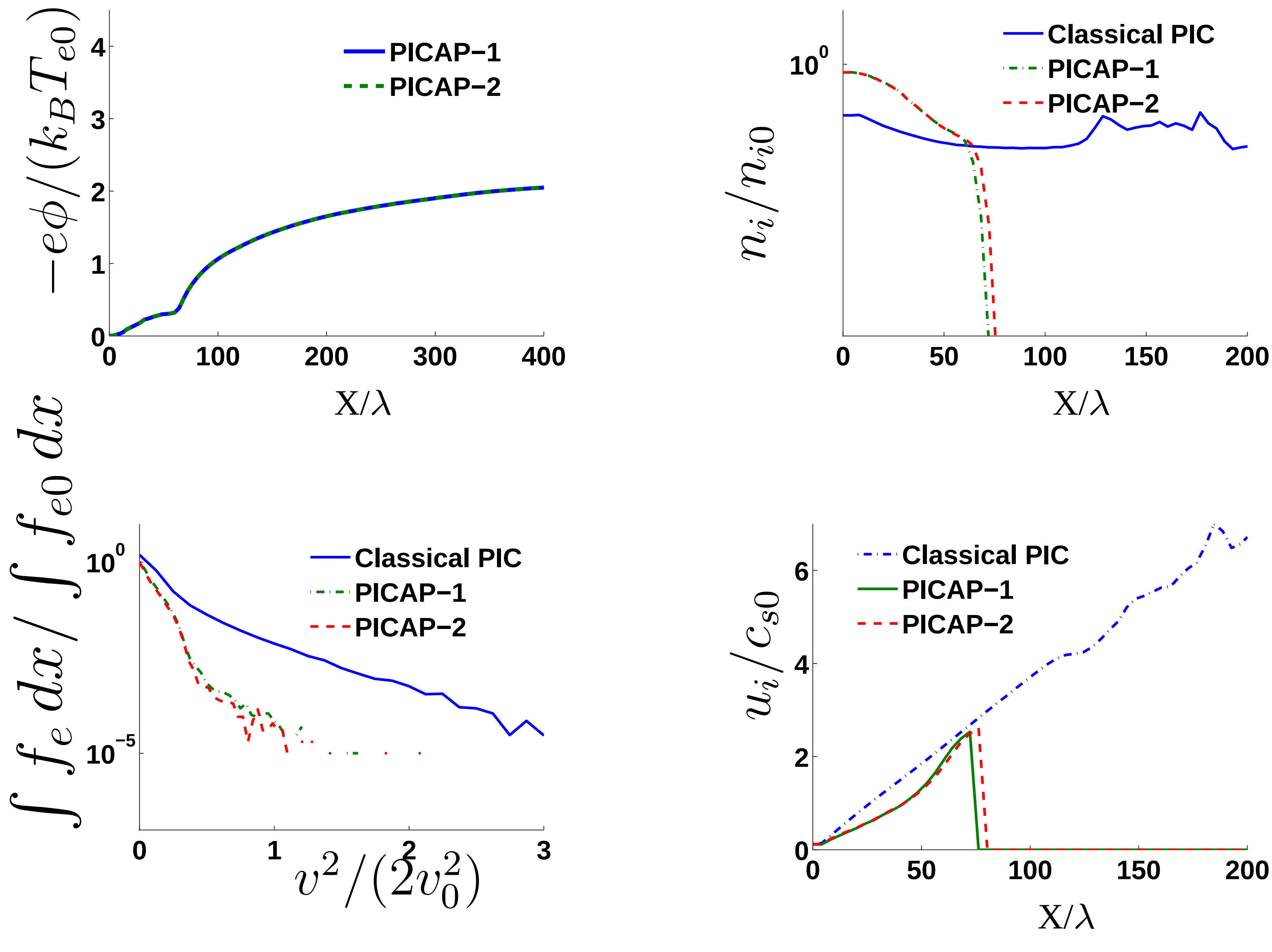
Half-resolved case: $\Delta t = 0.05\tau$, $h = 4\,\lambda$



Unresolved case: $\Delta t = 3\,\tau$, $h = 4\,\lambda$



Unresolved case, less particles /20



Gain of CPU time

- Ratios

$$\frac{\text{CPU Resolved case}}{\text{CPU Unresolved case}} = 48,$$

$$\frac{\text{CPU Resolved case}}{\text{CPU Unresolved case less particles}} = 960.$$

- About 1000 times faster in one dimension.

3. Works in progress

Works in progress

➠ In periodic perturbation of a quasi-neutral plasma equilibrium

$$\Downarrow$$

Problem of energy dissipation

➠ Can be reduced with reduction of noise

➠ Order two discretization (Leap-frog) for the particles trajectories

➠ Coupling with Maxwell equations.