On the use of the micro-macro decomposition to design multiscale numerical schemes for kinetic equations

Luc Mieussens

Institut de Mathématiques de Bordeaux,
Université de Bordeaux,
France

Workshop "Computational Kinetic Transport and Hybrid Methods", 2009 (IPAM, UCLA)

collaborators: P. Degond (Toulouse), G. Dimarco (Ferrara), M. Lemou (Rennes), J.-G. Liu (Duke)

Kinetic equations and asymptotic models

- scale factor: $\varepsilon = \frac{\text{mean free path}}{\text{macroscopic length}}$ (Knudsen number)
- when $\varepsilon \ll 1$: the kinetic equation is well approximated by an asymptotic "fluid" model
- Example 1: Rarefied gas dynamics (hydrodynamic limit)

$$\partial_t f + v \cdot \nabla f = \frac{1}{\varepsilon} Q(f)$$

$$\downarrow \varepsilon \ll 1$$

$$\partial_t U + \nabla \cdot F(U) + \varepsilon \nabla \cdot (D \nabla U) = 0$$

(compressible Navier-Stokes eq. for $U = \int (1, v, \frac{1}{2}|v|^2) f \, dv$)

$$\downarrow \varepsilon = 0$$

$$\partial_t U + \nabla \cdot F(U) = 0$$

(compressible Euler eq.)

other limits: low Mach number, boundary driven diffusion, ...

Kinetic equations and asymptotic models

- scale factor: $\varepsilon = \frac{\text{mean free path}}{\text{macroscopic length}}$ (Knudsen number)
- when $\varepsilon \ll 1$: the kinetic equation is well approximated by an asymptotic "fluid" model
- Example 2: Linear transport (diffusion limit)

$$\partial_t f + \frac{1}{\varepsilon} v \cdot \nabla f = \frac{1}{\varepsilon^2} L(f)$$

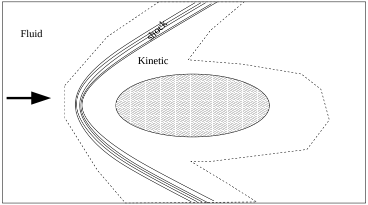
$$\downarrow \varepsilon = 0$$

$$\partial_t \rho - \nabla \cdot \left(\kappa \nabla \rho\right) = 0$$

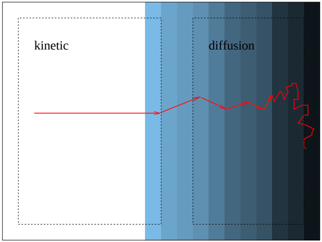
diffusion eq for $\rho = \langle f \rangle$

Multiscale kinetic problems

- multiscale: $\varepsilon = O(1)$ in some zones, $\varepsilon \ll 1$ in others



aerodynamics (Boltzmann)



linear transport

- usual numerical schemes: numerical constraint Δt and $\Delta x = O(\varepsilon)$

  cannot be used when $\varepsilon \ll 1$

- solutions:
  1. domain decomposition
  2. extended fluid models (higher order, moments, ...)
  3. AP schemes

this talk: a general method for (1), (2), (3)

Outline

Outline

Outline

Outline

Outline

1. AP schemes
   - Definition
   - Counter-example
   - References
2. AP scheme for the (linear) diffusion limit
   - Micro-macro decomposition
   - Discretization
   - Numerical tests
3. AP scheme for the compressible asymptotics of the Boltzmann equation
   - The Boltzmann equation
   - AP scheme based on the micro-macro decomposition
   - Numerical test
4. Fluid models with localized kinetic upscaling
   - Localization of the deviation part
   - Numerical tests

Asymptotic Preserving (AP) schemes.

- definition: a scheme uniformly stable and accurate w.r.t ε (Δt and Δx independent of ε)
- consequence: in the fluid regime ($\varepsilon \ll 1$): scheme consistent with the fluid equation
- requirements: special time (implicit) and space discretizations

Asymptotic Preserving (AP) schemes

- example of **non** AP scheme for the linear transport equation:

$$\partial_t f + \frac{1}{\varepsilon} v \partial_x f = \frac{1}{\varepsilon^2}([f] - f)$$

- explicit upwind scheme:

$$\frac{f^{n+1} - f^n}{\Delta t} + \frac{1}{\varepsilon}\left(v^+ \frac{f_i^n - f_{i-1}^n}{\Delta x} + v^- \frac{f_{i+1}^n - f_i^n}{\Delta x}\right) = \frac{1}{\varepsilon^2}([f_i^n] - f_i^n)$$

  - CFL constraint: $\Delta t \leq average(\varepsilon^2, \varepsilon \Delta x)$
  - numerical error $O(\Delta x / \varepsilon)$

Linear transport: contributions of Larsen, Morel, Miller, Adams, . . .

- linear transport equation (neutron transport) and its diffusion limit
- fully implicit scheme (collision and transport)
- main problem: AP space discretization for the steady equation
- study of different space discretizations
- main result: Discontinuous Galerkin (P1) approximation is AP

Unsteady equations: contributions of Klar, Jin, . . .

- main ingredients:
  - time splitting scheme, semi-implicit (collision)
  - decomposition of the solution (even-odd decomposition, zero-first order decomposition)
- contribution of A. Klar: linear transport and its diffusion limit, Boltzmann equation and the low Mach number limit
- contribution of S. Jin (with Pareschi, Toscani, Russo, Caflisch): linear transport, and some hyperbolic systems with relaxation (toy models for kinetic equations)
- other works: Goudon-Lafitte, Gosse-Toscani, ...

Our contribution

- a new kind of AP scheme
- no time splitting scheme
- use of the micro-macro decomposition
- can be used for diffusion <u>and</u> hydrodynamic limits
- can be used for coupling strategies

Outline

Diffusion limit of the linear transport equation

[M.Lemou-LM, SISC 2008]

- linear transport equation:

$$\partial_t f + \frac{1}{\varepsilon} v \partial_x f = \frac{1}{\varepsilon^2}([f] - f)$$

where $[f] = \dfrac{1}{2}\displaystyle\int_{-1}^{1} f(v)\,dv$

- limit $\varepsilon = 0$: diffusion equation for $\rho = [f]$

$$\partial_t \rho - \partial_x(\kappa \partial_x \rho) = 0$$

- collision and transport are stiff

Micro-macro decomposition.

- kinetic equation:

$$\partial_t f + \frac{1}{\varepsilon} v \partial_x f = \frac{1}{\varepsilon^2}([f] - f)$$

- micro-macro decomposition: $f = \rho + \varepsilon g$, where $\rho = [f]$
  property: $[g] = 0$
- decomposition of the kinetic equation:

$$\partial_t \rho + \varepsilon \partial_t g + \frac{1}{\varepsilon} v \partial_x \rho + v \partial_x g = -\frac{1}{\varepsilon} g$$

- scale separation: evolution equations for ρ and g

$$\partial_t \rho + \partial_x [vg] = 0,$$
$$\partial_t g + \frac{1}{\varepsilon} v \partial_x g = -\frac{1}{\varepsilon^2} g - (\frac{1}{\varepsilon} \partial_t \rho + \frac{1}{\varepsilon^2} v \partial_x \rho)$$

Micro-macro decomposition.

- kinetic equation:

$$\partial_t f + \frac{1}{\varepsilon} v \partial_x f = \frac{1}{\varepsilon^2}([f] - f)$$

- micro-macro decomposition: $f = \rho + \varepsilon g$, where $\rho = [f]$
  property: $[g] = 0$
- decomposition of the kinetic equation:

$$\partial_t \rho + \varepsilon \partial_t g + \frac{1}{\varepsilon} v \partial_x \rho + v \partial_x g = -\frac{1}{\varepsilon} g$$

- $\partial_t \rho$ is eliminated to get the micro-macro system:

$$\partial_t \rho + \partial_x [vg] = 0,$$

$$\partial_t g + \frac{1}{\varepsilon} v \partial_x g = -\frac{1}{\varepsilon^2} g - \left(\frac{1}{\varepsilon} \partial_t \rho + \frac{1}{\varepsilon^2} v \partial_x \rho\right)$$

Micro-macro decomposition.

- kinetic equation:

$$\partial_t f + \frac{1}{\varepsilon} v \partial_x f = \frac{1}{\varepsilon^2}([f] - f)$$

- micro-macro decomposition: $f = \rho + \varepsilon g$, where $\rho = [f]$
  property: $[g] = 0$
- decomposition of the kinetic equation:

$$\partial_t \rho + \varepsilon \partial_t g + \frac{1}{\varepsilon} v \partial_x \rho + v \partial_x g = -\frac{1}{\varepsilon} g$$

- $\partial_t \rho$ is eliminated to get the micro-macro system:

$$\partial_t \rho + \partial_x [vg] = 0,$$

$$\partial_t g + \frac{1}{\varepsilon} v \partial_x g = -\frac{1}{\varepsilon^2} g + \frac{1}{\varepsilon} \partial_x [vg] - \frac{1}{\varepsilon^2} v \partial_x \rho$$

Micro-macro decomposition.

- kinetic equation:

$$\partial_t f + \frac{1}{\varepsilon} v \partial_x f = \frac{1}{\varepsilon^2}([f] - f)$$

- micro-macro decomposition: $f = \rho + \varepsilon g$, where $\rho = [f]$
  property: $[g] = 0$
- decomposition of the kinetic equation:

$$\partial_t \rho + \varepsilon \partial_t g + \frac{1}{\varepsilon} v \partial_x \rho + v \partial_x g = -\frac{1}{\varepsilon} g$$

- $\partial_t \rho$ is eliminated to get the micro-macro system:

$$\partial_t \rho + \partial_x [vg] = 0,$$
$$\partial_t g + \frac{1}{\varepsilon}(I - [.])(v \partial_x g) = -\frac{1}{\varepsilon^2} g - \frac{1}{\varepsilon^2} v \partial_x \rho$$

(equivalent to the original equation (no approximation))

Micro-macro decomposition : another point of view.

- kinetic equation:

$$\partial_t f + \frac{1}{\varepsilon} v \partial_x f = \frac{1}{\varepsilon^2} L f \quad \text{where} \quad L = [.] - I$$

- micro-macro decomposition: $f = \rho + \varepsilon g$, where $\rho = [f]$
  property: $[g] = 0 \Leftrightarrow g \perp \mathcal{N}(L)$

- decomposed kinetic equation:

$$\partial_t \rho + \varepsilon \partial_t g + \frac{1}{\varepsilon} v \partial_x \rho + v \partial_x g = -\frac{1}{\varepsilon} g$$

- scale separation in the decomposed kinetic equation:
  - $\Pi := [\,.\,]$, orthogonal projection onto $\mathcal{N}(L)$
  - applying Π
  - applying $I - \Pi$ gives the evolution eq. for g

Micro-macro decomposition and the diffusion limit.

- this formulation is well suited to derive the diffusion limit:

$$\partial_t \rho + \partial_x [vg] = 0,$$
$$\partial_t g + \frac{1}{\varepsilon}(I - [.])(v\partial_x g) = -\frac{1}{\varepsilon^2} g - \frac{1}{\varepsilon^2} v\partial_x \rho.$$

- 2nd eq. gives: $g = -v\partial_x \rho + O(\varepsilon)$
- use this into 1st eq. to get:

$$\partial_t \rho - \partial_x \Big(\underbrace{\left[v^2\right]}_{\kappa} \partial_x \rho \Big) = O(\varepsilon),$$

- $\Rightarrow$ for $\varepsilon \ll 1$, we get the diffusion limit
- simple idea: numerical discretization of the micro-macro coupled system

Time discretization

- micro-macro coupled system:

$$\partial_t \rho + \partial_x [vg] = 0,$$
$$\partial_t g + \frac{1}{\varepsilon}(I - [.])(v\partial_x g) = -\frac{1}{\varepsilon^2} g - \frac{1}{\varepsilon^2} v\partial_x \rho.$$

- time semi-implicit scheme:

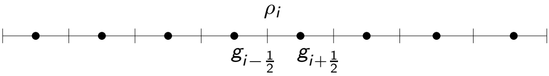
$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \partial_x \left[vg^{n+1}\right] = 0,$$
$$\frac{g^{n+1} - g^n}{\Delta t} + \frac{1}{\varepsilon}(I - [.])(v\partial_x g^n) = -\frac{1}{\varepsilon^2} g^{n+1} - \frac{1}{\varepsilon^2} v\partial_x \rho^n.$$

- diffusion limit:

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} - \partial_x \kappa \partial_x \rho^n = 0$$

Space discretization.

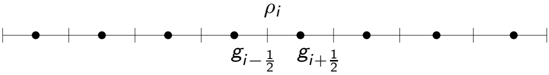
- staggered grid:



$$
\begin{aligned}
&\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} + \partial_x \left[v g^{n+1} \right]_i = 0, \\
&\frac{g_{i+\frac{1}{2}}^{n+1} - g_{i+\frac{1}{2}}^n}{\Delta t} + \frac{1}{\varepsilon}(I - [.]) v \partial_x g_{i+\frac{1}{2}}^n \\
&\qquad\qquad = -\frac{1}{\varepsilon^2} g_{i+\frac{1}{2}}^{n+1} - \frac{1}{\varepsilon^2} v \partial_x \rho_{i+\frac{1}{2}}^n .
\end{aligned}
$$

Space discretization.

- staggered grid:



$$\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t}+\partial_x\left[vg^{n+1}\right]_i=0,$$

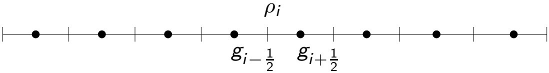
$$\frac{g_{i+\frac{1}{2}}^{n+1}-g_{i+\frac{1}{2}}^{n}}{\Delta t}+\frac{1}{\varepsilon}(I-[.])v\partial_x g_{i+\frac{1}{2}}^{n}$$

$$=-\frac{1}{\varepsilon^2}g_{i+\frac{1}{2}}^{n+1}-\frac{1}{\varepsilon^2}v\partial_x\rho_{i+\frac{1}{2}}^{n}.$$

upwind discretization of $v\partial_x g_{i+\frac{1}{2}}$ *(stability for kinetic regime)*

Space discretization.

- staggered grid:



$$\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t}+\partial_x\left[vg^{n+1}\right]_i=0,$$

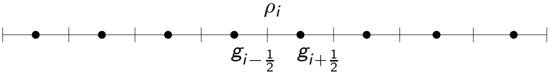
$$\frac{g_{i+\frac{1}{2}}^{n+1}-g_{i+\frac{1}{2}}^{n}}{\Delta t}+\frac{1}{\varepsilon}(I-[.])\left(v^{+}\frac{g_{i+\frac{1}{2}}^{n}-g_{i-\frac{1}{2}}^{n}}{\Delta x}+v^{-}\frac{g_{i+\frac{3}{2}}^{n}-g_{i+\frac{1}{2}}^{n}}{\Delta x}\right)$$

$$=-\frac{1}{\varepsilon^2}g_{i+\frac{1}{2}}^{n+1}-\frac{1}{\varepsilon^2}v\partial_x\rho_{i+\frac{1}{2}}^{n}.$$

upwind discretization of $v\partial_x g_{i+\frac{1}{2}}$ *(stability for kinetic regime)*

Space discretization.

- staggered grid:



$$\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t}+\partial_x\left[vg^{n+1}\right]_i=0,$$

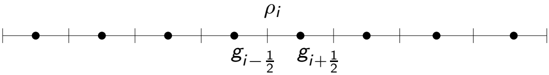
$$\frac{g_{i+\frac{1}{2}}^{n+1}-g_{i+\frac{1}{2}}^{n}}{\Delta t}+\frac{1}{\varepsilon}(I-[.])\left(v^+\frac{g_{i+\frac{1}{2}}^{n}-g_{i-\frac{1}{2}}^{n}}{\Delta x}+v^-\frac{g_{i+\frac{3}{2}}^{n}-g_{i+\frac{1}{2}}^{n}}{\Delta x}\right)$$

$$=-\frac{1}{\varepsilon^2}g_{i+\frac{1}{2}}^{n+1}-\frac{1}{\varepsilon^2}v\partial_x\rho_{i+\frac{1}{2}}^n.$$

central discretization of $\partial_x\left[vg\right]_i$ and $v\partial_x\rho_{i+\frac{1}{2}}$
(for accurate approx. of the diffusion terms)

Space discretization.

- staggered grid:



$$\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t} + \left[v\frac{g_{i+\frac{1}{2}}^{n+1} - g_{i-\frac{1}{2}}^{n+1}}{\Delta x} \right] = 0,$$

$$\frac{g_{i+\frac{1}{2}}^{n+1}-g_{i+\frac{1}{2}}^n}{\Delta t} + \frac{1}{\varepsilon}(I-[.])\left(v^+\frac{g_{i+\frac{1}{2}}^n - g_{i-\frac{1}{2}}^n}{\Delta x} + v^-\frac{g_{i+\frac{3}{2}}^n - g_{i+\frac{1}{2}}^n}{\Delta x}\right)$$

$$= -\frac{1}{\varepsilon^2} g_{i+\frac{1}{2}}^{n+1} - \frac{1}{\varepsilon^2} v\frac{\rho_{i+1}^n - \rho_i^n}{\Delta x}.$$

central discretization of $\partial_x [vg]_i$ and $v\partial_x \rho_{i+\frac{1}{2}}$
(for accurate approx. of the diffusion terms)

Numerical diffusion limit.

- the scheme:

$$\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t}+\left[v\frac{g_{i+\frac12}^{n+1}-g_{i-\frac12}^{n+1}}{\Delta x}\right]=0,$$

$$\frac{g_{i+\frac12}^{n+1}-g_{i+\frac12}^{n}}{\Delta t}+\frac{1}{\varepsilon}(I-[.])\left(v^+\frac{g_{i+\frac12}^{n}-g_{i-\frac12}^{n}}{\Delta x}+v^-\frac{g_{i+\frac32}^{n}-g_{i+\frac12}^{n}}{\Delta x}\right)$$

$$=-\frac{1}{\varepsilon^2}g_{i+\frac12}^{n+1}-\frac{1}{\varepsilon^2}v\frac{\rho_{i+1}^n-\rho_i^n}{\Delta x}$$

- limit $\varepsilon=0$: the scheme gives

$$\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t}-\kappa\frac{\rho_{i+1}^n-2\rho_i^n+\rho_{i-1}^n}{\Delta x^2}=0,$$

time explicit scheme with 3-point stencil

Analysis

[J.-G Liu-LM, 2009]

- the scheme is uniformly stable:

$$\|\rho^n\|^2 + \varepsilon^2 |||g^n|||^2 \leq C \left(\|\rho^0\|^2 + \varepsilon^2 |||g^0|||^2\right)$$

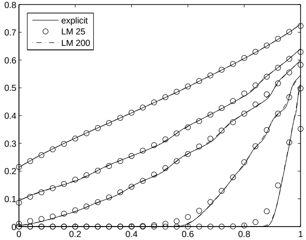
under the CFL constraint:

$$\Delta t \leq \frac{1}{2}\left(\frac{\Delta x^2}{4\kappa} + \varepsilon \Delta x\right)$$

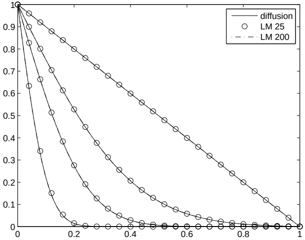
- small ε: $\Delta t \leq \frac{\Delta x^2}{8\kappa}$ (CFL for diffusion)
- large ε: $\Delta t \leq \frac{1}{2}\varepsilon \Delta x$ (CFL for convection)
- the scheme is uniformly accurate:

$$\|\rho(t_n) - \rho^n\| + \varepsilon |||g(t_n) - g^n||| \leq C(\Delta t + \Delta x^2 + \varepsilon \Delta x)$$

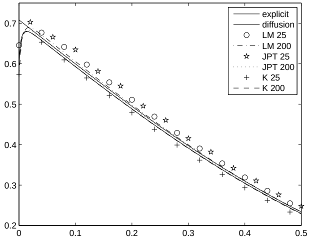
Numerical tests



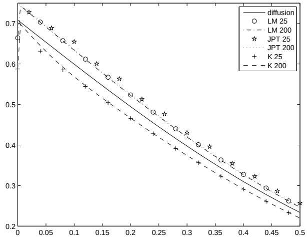
$\varepsilon = 1$



$\varepsilon = 10^{-8}$

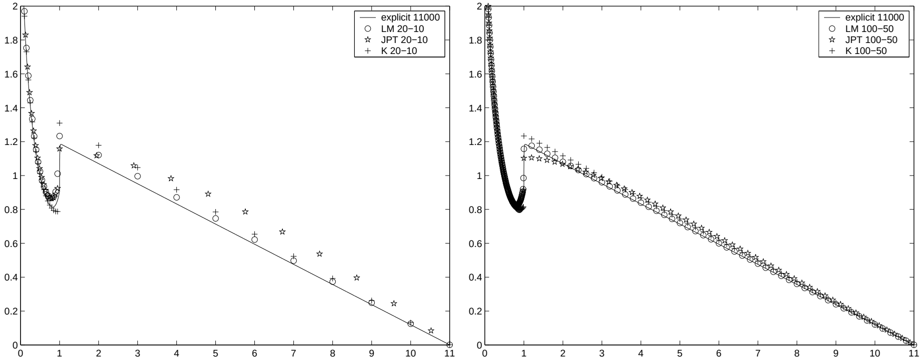


$\varepsilon = 10^{-2}$



$\varepsilon = 10^{-4}$

Numerical tests



Outline

The Boltzmann equation and its compressible limits.

- Boltzmann equation:

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f,f)$$

- $\varepsilon = 0$: $f = M(U) \Rightarrow$ Euler equations:

$$\partial_t U + \nabla_x \cdot F(U) = 0$$

where $U = (\rho, \rho u, E) = \langle mf \rangle$, $\qquad m = (1, v, \frac{1}{2}|v|^2)$

- $\varepsilon \ll 1$: $f = M(U) + O(\varepsilon) \Rightarrow$ compressible Navier-Stokes equations (CNS) :

$$\partial_t U + \nabla_x \cdot F(U) = -\varepsilon \begin{pmatrix} 0 \\ \nabla_x \cdot \sigma \\ \nabla_x \cdot (\sigma u + q) \end{pmatrix}$$

A simple non-AP scheme.

Example: time splitting scheme of Coron-Perthame for the BGK equation

- transport step: $\frac{f^{n+\frac{1}{2}} - f^n}{\Delta t} + v \cdot \nabla_x f^n = 0$
- relaxation step: $f^{n+1} = e^{-\Delta t/\varepsilon} f^{n+\frac{1}{2}} + (1 - e^{-\Delta t/\varepsilon}) M(f^{n+1})$
  (exact solution of: $\partial_t f = \frac{1}{\varepsilon}(M(f) - f)$)

Property: scheme consistent with Euler equations, but not with CNS equations:

$$f^{n+1} - M(f^{n+1}) = O(e^{-\Delta t/\varepsilon}) \ll O(\varepsilon)$$

AP scheme based on the micro-macro decomposition.

[Bennoune-Lemou-LM, JCP 08]

- micro-macro decomposition: $f = M[U] + \varepsilon g$
  property: $\langle mf \rangle = \langle mM[U] \rangle = U$ and $\langle mg \rangle = 0$
- collision operator:

$$
\begin{aligned}
Q(f,f) &= Q(M(U), M(U)) + \varepsilon 2Q(M(U), g) + \varepsilon^2 Q(g,g) \\
&= 0 + \varepsilon \mathcal{L}_{M(U)} g + \varepsilon^2 Q(g,g)
\end{aligned}
$$

- Boltzmann equation

$$
\partial_t M(U) + v \cdot \nabla_x M(U) + \varepsilon(\partial_t g + v \cdot \nabla_x g) = \mathcal{L}_{M(U)} g + \varepsilon Q(g,g)
$$

- property: $g \perp \mathcal{N}(\mathcal{L}_{M(U)})$
- define $\Pi_{M(U)}$: orthogonal projection onto $\mathcal{N}(\mathcal{L}_{M(U)})$
- scale separation: apply $\Pi_{M(U)}$ and $I - \Pi_{M(U)}$

AP scheme based on the micro-macro decomposition.

- coupled system:

$$
\begin{aligned}
&\partial_t U + \nabla_x \cdot F(U) + \varepsilon \nabla_x \cdot \langle vmg \rangle = 0, \\
&\partial_t g + (I - \Pi_{M(U)})(v \cdot \nabla_x g) = \frac{1}{\varepsilon} \mathcal{L}_{M(U)} g + Q(g, g) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \frac{1}{\varepsilon}(I - \Pi_{M(U)})(v \cdot \nabla_x M(U)).
\end{aligned}
$$

- equivalent to the Boltzmann equation (no approximation!)
- can be viewed as a fluid model (Euler) plus an *upscaling* term which takes kinetic effects into account

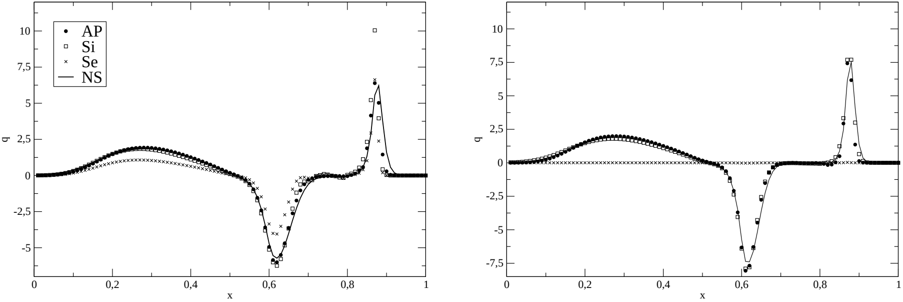
AP scheme based on the micro-macro decomposition.

- fully discrete scheme

$$\frac{U_i^{n+1}-U_i^n}{\Delta t}+\frac{F_{i+\frac12}(U^n)-F_{i-\frac12}(U^n)}{\Delta x}+\varepsilon\left\langle vm\frac{g_{i+\frac12}^{n+1}-g_{i-\frac12}^{n+1}}{\Delta x}\right\rangle=0,$$

$$\frac{g_{i+\frac12}^{n+1}-g_{i+\frac12}^{n}}{\Delta t}+(I-\Pi_{i+\frac12}^n)\left(v^+\frac{g_{i+\frac12}^n-g_{i-\frac12}^n}{\Delta x}+v^-\frac{g_{i+\frac32}^n-g_{i+\frac12}^n}{\Delta x}\right)$$
$$=\frac{1}{\varepsilon}\mathcal{L}_{M_{i+\frac12}^n}g_{i+\frac12}^{n+1}+Q(g_{i+\frac12}^n,g_{i+\frac12}^n)-\frac{1}{\varepsilon}(I-\Pi_{i+\frac12}^n)(v\frac{M_{i+1}^n-M_i^n}{\Delta x}),$$

- property: **asymptotically equivalent** (up to $O(\varepsilon^2)$) to a scheme which is:
  - consistent with **CNS equations**.
  - second order accurate (in space) for the diffusive fluxes



Sod test case: heat flux (scaled)
($\varepsilon = 2 \times 10^{-3}$ (left) and $\varepsilon = 2 \times 10^{-4}$ (right))
$\Delta t = 2 \times 10^{-3}$.

- AP and Si schemes: CNS regime is captured
- Se scheme: CNS regime is not captured

Outline

Localization of the deviation part.

[Degond-Liu-LM, MMS 06]

- the micro-macro model (for Boltzmann):

$$\partial_t U + \nabla_x \cdot F(U) + \nabla_x \cdot \langle vmg \rangle = 0,$$
$$\partial_t g + (I - \Pi_{M(U)})(v \cdot \nabla_x g) = \mathcal{L}_{M(U)} g + Q(g,g) - (I - \Pi_{M(U)})(v \cdot \nabla_x M(U))$$

- simple idea: localization of g
- use the cutoff function h:
  - determine "Fluid zones": set $h = 0$
  - determine "Kinetic zones": set $h = 1$
  - define "buffer zones" between (F) and (K) zones: set $0 \leq h \leq 1$
- define $g = g_K + g_F$ with $g_K = hg$ and $g_F = (1-h)g$
- note that g_F is zero in (K) zones
- localization: g_F is set to 0 in (F) zones (*approximation*)

Localization of the deviation part.

[Degond-Liu-LM, MMS 06]

- the localized micro-macro model

$$\partial_t U + \nabla_x \cdot F(U) + \nabla_x \cdot \langle vmg_K \rangle = 0,$$
$$\partial_t g_K + h(I - \Pi_{M(U)})(v \cdot \nabla_x g_K) = h\mathcal{L}_{M(U)} g_K + hQ(g_K, g_K) - h(I - \Pi_{M(U)})(v \cdot \nabla_x M(U))$$

- simple idea: localization of g
- use the cutoff function h:
  - determine "Fluid zones": set $h = 0$
  - determine "Kinetic zones": set $h = 1$
  - define "buffer zones" between (F) and (K) zones: set $0 \leq h \leq 1$
- define $g = g_K + g_F$ with $g_K = hg$ and $g_F = (1 - h)g$
- note that g_F is zero in (K) zones
- localization: g_F is set to 0 in (F) zones (*approximation*)

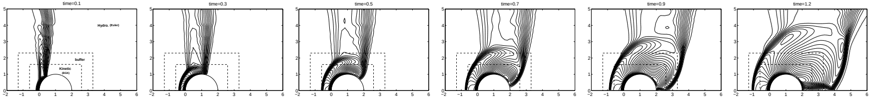
Fluid model with localized kinetic upscaling.

- the localized micro-macro model:

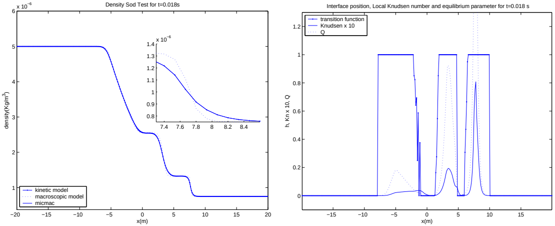
$$\partial_t U + \nabla_x \cdot F(U) + \nabla_x \cdot \langle vmg_K \rangle = 0,$$
$$\partial_t g_K + h(I - \Pi_{M(U)})(v \cdot \nabla_x g_K) = h\mathcal{L}_{M(U)} g_K + hQ(g_K, g_K)$$
$$- h(I - \Pi_{M(U)})(v \cdot \nabla_x M(U))$$

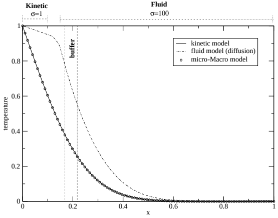
- smooth transition between Euler in (F) zones to Boltzmann in (K) zones
- dynamic transition is possible: $h = h(t)$
- no domain decomposition
- easy to implement

Plane shock wave for the BGK equation, static coupling.



Sod test (density and *h* function) for the BGK equation with dynamic localization.





Temperature for the radiative-heat transfer model, static localization.

Summary

- the micro-macro decomposition is well suited for AP schemes and multiscale methods

- Perspectives
  - Boundary conditions for our AP schemes?
  - 2D (3D) AP schemes
  - AP schemes for other asymptotics

Shi Jin's remark.

Navier−Sokes

$\Downarrow$

boundary layers

$\Downarrow$

$\Delta x < \varepsilon$

$\Downarrow$

$\Delta t < \varepsilon$

$\Downarrow$

this scheme is not AP!!!