Title: The Structure of Radiative Shocks

Author(s): Robert B. Lowrie

Intended for: Presentation at the Computational Kinetic Transport and Hybrid Methods Workshop, Institute for Pure and Applied Mathematics, UCLA, Los Angeles, CA, March 30 - April 3, 2009
The Structure of Radiative Shocks

Robert B. Lowrie

Los Alamos National Laboratory
Computational Physics and Methods Group (CCS-2)
lowrie@lanl.gov

Collaborators

J. D. Edwards (Texas A&M)
R. M. Rauenzahn (T-3 LANL)
Non-trivial, analytic solutions do not exist for radiation hydrodynamics.

- Compromise: Generate *semi-analytic* solutions; solutions from numerically integrating nonlinear ODEs.
- Seek traveling wave solutions ⇒ *radiative shocks*.

In this talk, use solutions to

- verify code correctness
- test AMR (rad-shocks are multiscale problems)

Gives physics insight that would be *very* difficult to obtain by a series of runs from a computational physics code.
1. Overview of radiative shocks

2. Equations of radiation hydrodynamics
   - Grey nonequilibrium diffusion
   - Euler coupled with grey nonequilibrium diffusion
   - Equilibrium diffusion limit

3. Overview of Semi-analytic Approach

4. Sample Solutions

5. Code Comparison
Outline

1. Overview of radiative shocks

2. Equations of radiation hydrodynamics
   - Grey nonequilibrium diffusion
   - Euler coupled with grey nonequilibrium diffusion
   - Equilibrium diffusion limit

3. Overview of Semi-analytic Approach

4. Sample Solutions

5. Code Comparison
Physics Regime

As one example, regimes in astrophysics governed by

- Inviscid hydrodynamics (Euler equations; non-relativistic)
- Thermal radiation (X-ray) transport
- High-energy density:
  - Material temperatures $O(1 \text{ keV} = 11.6 \times 10^6 \text{ Kelvin})$
  - Radiation *pressure* affects hydrodynamics
Past Work on Radiative Shocks

- Most theory on “thick – thick” shocks: equilibrates on either side of shock.

- **Overview of Theory:**

- Grey VEF/AMR, ion/electron, calculations:

- **Theory and Solutions:**

- **Extended thermodynamics:**
Inviscid Hydrodynamic Shocks

Temperature

Shock Jump

T

x

T

T_0

T_1

T_0

T_1

lowrie@lanl.gov
LA-UR 09-01841
IPAM 2009
Subcritical Radiative Shocks \( (T_p < T_1) \)

Material and Radiation Temperatures

Hydro Shock

Relaxation Region

Precursor

Overall Shock Jump

\( T \) vs \( T_r \)
Supercritical Radiative Shocks \((T_p = T_1)\)

\[ T, T_r \]

Precursor \( \rightarrow \)

Hydro Shock (Zel’dovich Spike)

\[ T_p = T_1 \]

\[ T_s \]

\[ T_0 \]

\[ x \]

\[ T, T_r \]

\[ T_1 \]

\[ x \in [-0.035, 0.015] \]
Zel’dovich Spike

Hydro Shock (Zel’dovich Spike)

Relaxation Region

$T_p = T_1$

$T_s$

$T_1$

$T, T_r$

$x$

-5e-05 0 5e-05 0.0001 0.00015 0.0002

lowrie@lanl.gov

LA-UR 09-01841

IPAM 2009
Outline

1. Overview of radiative shocks

2. Equations of radiation hydrodynamics
   - Grey nonequilibrium diffusion
   - Euler coupled with grey nonequilibrium diffusion
   - Equilibrium diffusion limit

3. Overview of Semi-analytic Approach

4. Sample Solutions

5. Code Comparison
Grey nonequilibrium diffusion in 1 slide
1-D planar (slab) geometry

First two moments of radiative transfer equation:

$$\partial_t E + \partial_x F = c\sigma_a [B(T) - E] - \frac{\sigma_t}{c} \left( F - \frac{4}{3} vE \right) v$$

$$\frac{1}{c^2} \partial_t F + \partial_x P = -\frac{\sigma_t}{c} \left( F - \frac{4}{3} vE \right)$$

Nonequilibrium diffusion sets $P = E/3$ and drops $\partial_t F$ to yield

$$\partial_t E + \frac{4}{3} \partial_x (vE) - \partial_x \left( \frac{c}{3\sigma_t} \partial_x E \right) = c\sigma_a [B(T) - E] + \frac{1}{3} v \partial_x E$$

Underlined terms often dropped in “low-energy density” approximation.
Euler Coupled with Grey Nonequilibrium Diffusion

Nondimensional form, 1-D planar (slab) geometry

Most of this talk will be about solutions of

\[ \partial_t \rho + \partial_x (\rho v) = 0, \]

\[ \partial_t (\rho v) + \partial_x \left( \rho v^2 + p + \frac{1}{3} P_0 T_r^4 \right) = 0, \]

\[ \partial_t (\rho \varepsilon) + \partial_x [v(\rho \varepsilon + p)] = P_0 \sigma_a (T_r^4 - T^4) - \frac{1}{3} P_0 v \partial_x T_r^4, \]

\[ \partial_t (\rho \varepsilon + P_0 T_r^4) + \partial_x \left[ v \left( \rho \varepsilon + p + \frac{4}{3} P_0 T_r^4 \right) \right] = P_0 \partial_x (\kappa \partial_x T_r^4), \]

where \( \varepsilon = e + \frac{1}{2} v^2 \) and for a \( \gamma \)-law EOS

\[ p = \frac{\rho T}{\gamma}, \quad e = \frac{T}{\gamma(\gamma - 1)}, \quad P_0 = \frac{\tilde{a}_R \tilde{T}_0^4}{\gamma \tilde{\rho}_0} \approx \text{rad. pressure mat. pressure}. \]

Embedded discontinuities are hydrodynamic shocks.
Equilibrium Diffusion Limit

We’ll also discuss “1T” solutions. Optically thick limit, \( T_r \rightarrow T \), and our system reduces to

\[
\partial_t \rho + \partial_x (\rho v) = 0, \\
\partial_t (\rho v) + \partial_x \left( \rho v^2 + p + \frac{1}{3} \mathcal{P}_0 T^4 \right) = 0, \\
\partial_t (\rho E + \mathcal{P}_0 T^4) + \partial_x \left[ v \left( \rho E + p + \frac{4}{3} \mathcal{P}_0 T^4 \right) \right] = \mathcal{P}_0 \partial_x (\kappa \partial_x T^4).
\]

- Euler equations with nonlinear heat conduction and a modified equation-of-state.
- For small-\( \mathcal{P}_0 \), heat conduction may dominate.
- Embedded discontinuities are isothermal shocks.
Problem statement

- Assume that far from shock, \( T_r = T \) (“thick–thick”).
- Galilean invariant, so use steady frame.

**Given:** The value \( \gamma \) and

- Pre-shock state \((x \rightarrow -\infty)\): \( \rho_0, T_0, T_{r0} = T_0 \).
- Non-dimensional constants: \( P_0 \) and \( M_0 \) \((\equiv \tilde{\nu}_0/\tilde{a}_0, \text{Mach number})\).
- Functions \( \sigma_a(\rho, T) \) and \( \kappa(\rho, T) \) \((\tilde{\kappa} = \tilde{c}/3\tilde{\sigma}_t)\).

**Calculate:** \( \rho(x), v(x), T(x), \text{and } T_r(x) \).

**Optional:** Transform back to a frame where shock is moving.
Outline

1. Overview of radiative shocks

2. Equations of radiation hydrodynamics
   - Grey nonequilibrium diffusion
   - Euler coupled with grey nonequilibrium diffusion
   - Equilibrium diffusion limit

3. Overview of Semi-analytic Approach

4. Sample Solutions

5. Code Comparison
Euler Coupled with Grey Nonequilibrium Diffusion

Nondimensional form, 1-D planar (slab) geometry

Seek traveling-wave solutions of

\[ \partial_t \rho + \partial_x (\rho v) = 0, \]
\[ \partial_t (\rho v) + \partial_x \left( \rho v^2 + p + \frac{1}{3} \mathcal{P}_0 T_r^4 \right) = 0, \]
\[ \partial_t (\rho E) + \partial_x [v(\rho E + p)] = \mathcal{P}_0 \sigma_a (T_r^4 - T^4) - \frac{1}{3} \mathcal{P}_0 v \partial_x T_r^4, \]
\[ \partial_t (\rho E + \mathcal{P}_0 T_r^4) + \partial_x \left[ v \left( \rho E + p + \frac{4}{3} \mathcal{P}_0 T_r^4 \right) \right] = \mathcal{P}_0 \partial_x (\kappa \partial_x T_r^4), \]
Overall Jump Relation

Integrate conservation equations from $-\infty < x < \infty$ to give:

$$
\left( \begin{array}{c}
\rho v \\
\rho v^2 + p^* \\
(\rho E^* + p^*)v
\end{array} \right)_0 = \left( \begin{array}{c}
\rho v \\
\rho v^2 + p^* \\
(\rho E^* + p^*)v
\end{array} \right)_1,
$$

where

$$p^* = p + \frac{1}{3} P_0 T^4, \quad e^* = e + \frac{1}{\rho} P_0 T^4, \quad E^* = e^* + \frac{1}{2} v^2.$$

- Get ninth-order polynomial in $T_1$; see Bouquet et al (2000).
- Same procedure used for any radiation model.
Hydro Shock Relations

At a discontinuity separating state-$p$ and state-$s$:

$$\begin{pmatrix} \rho v \\ \rho v^2 + p \\ (\rho E + p)v \\ -\kappa \partial_x Tr^4 + \frac{4}{3} v Tr^4 \end{pmatrix}_p = \begin{pmatrix} \rho v \\ \rho v^2 + p \\ (\rho E + p)v \\ -\kappa \partial_x Tr^4 + \frac{4}{3} v Tr^4 \end{pmatrix}_s$$

- First 3 equations: Standard hydro jump conditions. Holds for any radiation model.
- Last equation: Continuity of Eulerian frame radiation flux.
Reduced Equations

4 PDEs reduce to 2 ODEs. We use Mach number ($\mathcal{M}$) as the independent variable:

$$\frac{dx}{d\mathcal{M}} = \frac{3\mathcal{M}_0 (\mathcal{M}^2 - 1) \rho \beta}{P_0},$$

$$\frac{dT}{d\mathcal{M}} = (\gamma - 1) \beta \left[ 4\mathcal{M}_0 T_r^3 T_r' + (\gamma \mathcal{M}^2 - 1) r \right],$$

where $\beta(T, \mathcal{M})$ and $r(T, \mathcal{M})$ are known functions, and

$$\rho(T, \mathcal{M}) = \frac{\mathcal{M}_0}{\mathcal{M} \sqrt{T}}, \quad T_r(T, \mathcal{M}) = \frac{1}{\gamma P_0} \left[ K_m - 3\gamma \frac{\mathcal{M}_0^2}{\rho(T, \mathcal{M})} - 3 T \rho(T, \mathcal{M}) \right].$$

If $T_r \approx$ const., then $\mathcal{M} = 1/\sqrt{\gamma}$ corresponds to maximum $T$ (isothermal sonic point – ISP).
Overview of Solution Procedure

1. Compute post-shock state \((x \rightarrow \infty)\): Find root of a ninth-order polynomial in \(T_1\).

2. End states are typically saddle points. Solve two IVPs:
   - Find precursor region: Integrate ODEs from \(M = M_0\) to \(M = 1\).
   - Find relaxation region: Integrate ODEs from \(M = M_1\) to \(M = 1\).

3. If the two ODE solutions do not match at \(M = 1\), then shift the solutions such that they are connected with a hydro shock.

See Lowrie & Edwards (Shock Waves, 2008) for the details.
Outline

1. Overview of radiative shocks

2. Equations of radiation hydrodynamics
   - Grey nonequilibrium diffusion
   - Euler coupled with grey nonequilibrium diffusion
   - Equilibrium diffusion limit

3. Overview of Semi-analytic Approach

4. Sample Solutions

5. Code Comparison
Radiative shock solutions may be characterized by the following:

- Is there an embedded hydro shock?
  - If no, all variables continuous.
  - If yes,
    - Subcritical ($T_p < T_1$).
    - Supercritical ($T_p \approx T_1$).

- Is there an isothermal sonic point (ISP; $M = 1/\sqrt{\gamma}$)?
  - If yes, then there is a temperature (Zel’dovich) spike $\implies T_{\text{max}} > T_1$.
  - Lowrie & Edwards (2008) derive an algebraic expression for $T_{\text{max}}$ that is very accurate.

- May have a hydro shock without a spike.
- May have a spike without a hydro shock.

lowrie@lanl.gov

LA-UR 09-01841

IPAM 2009 23 / 52
Isothermal Shock (Equilib.) ⇔ Spike (Non-equilib.)

$M_0 = 1.05$ solution

$\mathcal{P}_0 = 10^{-4}$, $\sigma_a = 10^6$, $\kappa = 1$, $\gamma = 5/3$; subcritical; no embedded shock or isothermal sonic point (ISP).
\( M_0 = 1.2 \) solution

Subcritical; shock, but no ISP.
$\mathcal{M}_0 = 2$ solution
Subcritical; ISP coincident with shock.
\( M_0 = 3 \) solution

Subcritical; ISP downstream of shock.
$M_0 = 3$ solution

Spike region.
\( M_0 = 5 \) solution

Supercritical; ISP downstream of shock.
$M_0 = 5$ solution (continued)

Zel’dovich spike region

![Temperature Graph]

![Density Graph]
Isothermal Shock (Equilib.) \iff Spike (Non-equlib.)

\( \gamma = 5/3. \) See Lowrie & Rauenzahn (2007).

\[ P_0 \sim \left( \frac{\text{radiation pressure}}{\text{material pressure}} \right)_0 \]
\( \mathcal{M}_0 = 27 \) solution

Supercritical; ISP downstream of shock. NOTE OVER COMPRESSION.
\( M_0 = 27 \) solution (continued)

Zel’dovich spike region

![Graphs of Temperature and Density](image-url)
$\mathcal{M}_0 = 30$ solution

No shock, but still an ISP!
$\mathcal{M}_0 = 30$ solution (continued)

Spike region

Temperature

Density

\begin{align*}
\text{Temperature} \\
\text{Density}
\end{align*}
$M_0 = 50$ solution

No shock or ISP
Outline

1. Overview of radiative shocks

2. Equations of radiation hydrodynamics
   - Grey nonequilibrium diffusion
   - Euler coupled with grey nonequilibrium diffusion
   - Equilibrium diffusion limit

3. Overview of Semi-analytic Approach

4. Sample Solutions

5. Code Comparison
Code Comparison Test Cases

- Hydrogen gas, \( \gamma = \frac{5}{3} \)
- Bremsstrahlung absorption model (Zel’dovich & Razier; assume fully ionized):
  \[
  \sigma_a(\rho, T) = \sigma_{a,0}\rho^2 T^{-7/2}
  \]
- Thomson scattering:
  \[
  \sigma_s(\rho) = \sigma_{s,0}\rho
  \]
- \( \rho_0 = 1 \text{ g/cc} \)
- Cases:
  1. \( T_0 = 10 \text{ eV}, \mathcal{M}_0 = 10 \), subcritical
  2. \( T_0 = 100 \text{ eV}, \mathcal{M}_0 = 5 \), supercritical
  3. \( T_0 = 100 \text{ eV}, \mathcal{M}_0 = 45 \), no embedded hydro shock
- Comparison of relaxation rates shows that 3T effects should be small.
- Compare with a finite-volume, Godunov-based AMR code (RAGE).
Two Initialization Methods

1. Initialize with exact shock profile.
   - How well can the code propagate the shock and maintain the profile?
   - Pros: Boundary conditions not an issue.
   - Cons: Requires code initialize from exact solution.
   - This method was used for all error-norm calculations.

   - Black-body piston moving at $v_1$, radiating at $T_r = T_1$.
   - Given enough time, the resulting shock should match the semi-analytic solution.
   - Pros: Easy setup; better test.
   - Cons: Requires “large” solution domain; boundary conditions become an issue.

With either method, you should propagate the shock long enough to establish a self-similar profile.
$T_0 = 10$ eV, $M_0 = 10$ Sample Results

$\Delta x = 50 \mu m$, propagated 1 cm, $T_1 \approx 321$ eV, $T_{\text{max}} \approx 419$ eV
$T_0 = 10 \text{ eV, } M_0 = 10$ Error Convergence

Exact solution computed without flux limiter.
$T_0 = 10 \text{ eV}, \mathcal{M}_0 = 10$, Piston Problem Initialization

Legend indicates distance exact profile has moved. Solutions shifted to align. $\Delta x = 50 \mu\text{m}$. 

$0.1 \quad 0.2 \quad 0.3 \quad 0.4 \quad 0.5$  
$x (\text{cm})$ 

$0 \quad 5 \quad 10 \quad 15 \quad 20 \quad 25 \quad 30 \quad 35 \quad 40 \quad 45 \quad 45$  
$T / T_0$ 

Exact  
0 cm  
1 cm  
2 cm  
5 cm  
10 cm  

lowrie@lanl.gov  
LA-UR 09-01841  
IPAM 2009 43 / 52
\( T_0 = 100 \text{ eV}, \mathcal{M}_0 = 5 \) Sample Results

AMR (8 levels): \( \Delta x_{\text{max}} = 5 \text{ cm}, \Delta x_{\text{min}} \approx 391 \mu\text{m} \), propagated 230 cm, \( T_1 \approx 857 \text{ eV}, T_{\text{max}} \approx 1.08 \text{ keV} \)
$T_0 = 100 \text{ eV}, \ M_0 = 5 \text{ Spike Region}$
$T_0 = 100 \text{ eV}, \mathcal{M}_0 = 5$ Refinement Distribution

8 levels $\Rightarrow$ 814 cells; 10 levels $\Rightarrow$ 528 cells (10-level mesh derefined in precursor).
Refinement and Error Distribution

Even though 10-level mesh derefined in $27 \lesssim x \lesssim 49$, it is more accurate.
$T_0 = 100\;\text{eV}, \; M_0 = 45$ Sample Results

$\Delta x = 1.25\;\text{cm},$ propagated 20 m, $T_1 \approx 8.36\;\text{keV}, \; T_{\text{max}} \approx 8.48\;\text{keV}$
$T_0 = 100 \text{ eV}, \ M_0 = 45 \text{ Spike Region}$

RAGE results with $\Delta x = 1.25 \text{ cm}/2^N$, with $N = 0, 1, 2, 3$
$T_0 = 100 \text{ eV}, \quad M_0 = 45$ Convergence

A hard-coded tolerance was limiting the convergence rate.
The radiative shock solutions test several issues:
- Fully-coupled radiation hydrodynamics
- Embedded shock problems $\Rightarrow$ ability to capture hydro shocks
- Smooth problems $\Rightarrow$ ability to attain theoretical convergence rate
- Refinement criteria for AMR (solutions are multiscale)
- Other radiation models

Future work
- Separate ion/electron temperatures
- More advanced radiation models