Mathematical Models for Charge Transport[1]

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Abstract

We review a few problems issued from the modeling of the transport of charged particles, subject to the influence of given or self-consistent electric fields. We describe some of the mathematical methods introduced to deal with these problems.

1 A review of the models

Let us start by presenting the different possible modeling for charge transport. The motion of charged particles is indeed governed by the surrounding electro-magnetic field. However, local concentration and movements of charges also create electric fields and currents which, in turn, lead to nonlinear PDEs. The topics is very active since the 80's, motivated by applications in plasma physics and the modeling of semiconductor devices, the reduction of the size of the devices demanding accurate description (see the classical treatise [72] or surveys in [76], [30]). Projects of energy production based on strong electromagnetic or inertial confinment is currently motivating intense research, it is definitely a source of challenging mathematical problems, see [56]. Interest for these models also comes from spacecraft engineering: a spacecraft interacts with the surrounding plasma and, depending on the environment, difference of potential might appear between some places of the spacecraft, or between the spacecraft and the plasma. These potential gradients can produce the formation of electrical arch which in turn produce devices failures and possibly the lost of the spacecraft. Understanding of these complex phenomena can help in preventing such violent consequences of the charge mechanisms. Such a problem is addressed for instance in [26, 25, 89, 90]. We mention also specific models for dealing with ionospheric plasmas [11]. It is worth pointing out that models having a similar structure than those of charge transport appear in different fields of physics and can lead to related interesting mathematical difficulties; we mention in particular the case of particles subject to gravitational forces in astrophysics [24], the recent development of kinetic or hydrodynamic models in biology [23]... These models are particularly challenging since

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singularities can appear in finite time. We also mention fluid/particles flows with application in combustion theory [94] or for describing pollutants transports [1, 2, 18, 38, 57, 52, 53, 20, 54]... We distinguish:

- quantum models, which are based on the Schrödinger equation,

- kinetic models, collisional or not, where a statistical viewpoint is adopted, particles being described by their particle distribution function in phase space,

- hydrodynamic models where the charge transport is described by classical models of continuum mechanics.

1.1 Quantum models

A fine description of electrons in a crystal takes into account quantum effects. Hence, electrons are described through their wave function $\psi(t,x) \in \mathbb{C}$, where t and x stand for the time and the space variables respectively. This quantity itself has no physical meaning, but instead $|\psi(t,x)|^2$ is interpreted as a probability density:

$$\int_\Omega |\psi(t,x)|^2 \, \mathrm{d}x$$

is the probability of finding electrons in the domain $\Omega \subset \mathbb{R}^N$ at time t. We thus associate to this quantity some physical observables, like the density of electrons at time t and position x defined by

$$n(t,x) = |\psi(t,x)|^2,$$

or the current density given by

$$J(t,x) = \mathrm{Im}\big(\overline{\psi(t,x)\nabla_x\psi(t,x)}\big)$$

The wave function obeys the Schrödinger equation

$$\mathrm{i}\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta_x\psi + qV\psi \tag{1.1}$$

where

- $\hbar > 0$ stands for the reduced Planck constant,
- $q > 0$ stands for the electron charge,
- $m > 0$ stands for the electron mass,

while V is the potential the charge particles are subject to. It splits into the given potential V_p of the crystal in which the electrons are evolving and the self-consistent potential V_s created by the charged particles themselves. The former takes into account the structure of the crystal and therefore presents naturally some oscillations which can be modeled either by periodic or random variations. The latter is defined through the following Poisson equation

$$-\frac{\varepsilon_0}{q}\Delta_x V_\mathrm{s} = n - B \tag{1.2}$$

where ε_0 is the permittivity and B some given background density of positive charges. Hence, it makes the problem non linear since the term $V_s\psi$ in (1.1) depends quadratically on the unknown.

Let us briefly comment on (1.2). When working on the whole space $\mathbb{R}^N$, the Poisson equation (1.2) should actually be understood as the convolution formula

$$\begin{aligned} \frac{\varepsilon_0}{q} V_s(t,x) &= \int_{\mathbb{R}^N} E_N(x-y)(n-B)(t,y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} E_N(x-y) f(t,y,\xi)\,\mathrm{d}\xi\,\mathrm{d}y - \int_{\mathbb{R}^N} E_N(x-y) B(t,y)\,\mathrm{d}y, \end{aligned} \tag{1.3}$$

where

$$E_N(x) = \begin{cases} -\dfrac{|x|}{2} & \text{if } N=1, \\ -\dfrac{1}{2\pi}\ln|x| & \text{if } N=2, \\ \dfrac{1}{4\pi\ |x|} & \text{if } N=3, \\ C_N \dfrac{1}{|x|^{N-2}} & \text{if } N>2 \text{ with } C_N>0 \end{cases} \tag{1.4}$$

stands for the elementary solution of the Laplacian $(-\Delta_x)$. Note that, when $B=0$, $n\geq 0$, we have $V_s \geq 0$ for $N \geq 3$.

1.2 Kinetic models

We can also adopt a statistical description of the charge transport by introducing the density of particles in phase space: $f(t,x,\xi)\geq 0$, where ξ is the so-called wave vector variable, is such that

$$f(t,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x$$

gives the number of particles at time t in the ball centred on (x,ξ) with volume $\mathrm{d}\xi\,\mathrm{d}x$ in phase space. This quantity obeys the following PDE

$$\partial_t f + v(\xi)\cdot\nabla_x f - \frac{q}{m}\nabla_x V\cdot\nabla_\xi f = \frac{1}{\tau}\,Q(f). \tag{1.5}$$

The left hand side describes the transport of particles, while the interaction mechanisms the particles are subject to (binary collisions, collisions with impurities or another species of particles...) are embodied into the "collision" operator $Q(f)$. The parameter $\tau > 0$ is a relaxation time; it characterizes how often such collision events occur. The function

$$\begin{aligned} v:\ \mathbb{R}^N &\longrightarrow \mathbb{R}^d \\ \xi &\longmapsto v(\xi) \end{aligned}$$

is the velocity of the particles. We associate to this statistical quantity some macroscopic quantities like the macroscopic density

$$n(t,x) = \int_{\mathbb{R}^N} f(t,x,\xi)\,\mathrm{d}\xi,$$

the macroscopic current

$$J(t,x) = \int_{\mathbb{R}^N} v(\xi)\ f(t,x,\xi)\,\mathrm{d}\xi = nu(t,x)$$

u being the bulk velocity, and the temperature

$$Nn\Theta(t,x) = \int_{\mathbb{R}^N} \left|v(\xi) - u\right|^2\ f(t,x,\xi)\,\mathrm{d}\xi.$$

Let us neglect temporarily the interaction mechanisms and assume $Q = 0$. Then (1.5) can be interpreted by reasoning on the characteristics equations: we define $X(t;s,x,\xi)$ and $\Xi(t;s,x,\xi)$ solution of the ODE system

$$\left\{\begin{array}{ll} \dfrac{\mathrm{d}}{\mathrm{d}t}X = v(\Xi), & \dfrac{\mathrm{d}}{\mathrm{d}t}\Xi = -\dfrac{q}{m}\nabla_x V(t,X) \\ X_{|t=s} = x, & \Xi_{|t=s} = \xi. \end{array}\right. \tag{1.6}$$

Then, (1.5), with $Q = 0$, reduces to the simple conservation ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} f\big(t, X(t;s,x,\xi), \Xi(t;s,x,\xi)\big) = 0.$$

Accordingly, given the initial data

$$f_{|t=0} = f_0$$

the solution is given by

$$f(t,x,\xi) = f_0\big(X(0;t,x,\xi), \Xi(0;t,x\xi)\big). \tag{1.7}$$

Then, the equation agrees with Newton's law: derivative of the position is the velocity, derivative of the momentum equals the applied force. As a matter of fact, in the absence of force, that is $V = 0$, particles are moving freely and keep constant velocity: we get $\Xi(t;s,x,\xi) = \xi$ and $X(t;s,x,\xi) = x + (t-s)\xi$ so that $f(t,x,\xi) = f_0(x - t\xi, \xi)$ in this simple case. However, this approach leads to a couple of mathematical difficulties, in particular when we consider non–linear models where the potential depends on the unknown by the coupling induced by (1.2). On the one hand, (1.6)–(1.7) is not a closed relation in this case since the potential and thus the characteristics depend on the unknown f itself. On the other hand, (1.6) makes sense under some regularity assumptions (applying the Cauchy-Lipshtiz Theorem requires $\nabla_x V$ to be locally Lipschitz with respect to the space variable...) and it is not clear at all that the necessary regularity is guaranteed when dealing with such nonlinear models. Anyway, this reasoning gives another (fruitful) way of thinking the motion of electrons: considering a finite set of M particles subject to a given (smooth) potential V_{p}, the trajectories obeys (1.6). Namely, for any $j \in \{1,...,M\}$, we have

$$\left\{\begin{array}{ll} \dfrac{\mathrm{d}}{\mathrm{d}t}X_j = v(\Xi_j), & \dfrac{\mathrm{d}}{\mathrm{d}t}\Xi_j = -\dfrac{q}{m}\nabla_x V_{\mathrm{p}}(t,X_j,) \\ X_{j,|t=s} = x_j, & \Xi_{j,|t=s} = \xi_j. \end{array}\right.$$

Then, we check that the distribution

$$f(t,x,\xi) = \sum_{i=1}^{M} \delta(x = X_j(t;0,x_j,\xi_j)) \otimes \delta(\xi = \Xi_j(t;0,x_j,\xi_j))$$

verifies

$$\partial_t f + v(\xi) \cdot \nabla_x f - \frac{q}{m} \nabla_x V_{\rm p} \cdot \nabla_\xi f = 0.$$

Self-interaction are described at the $M-$particles level by the coupled EDO system

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} X_j = v(\Xi_j), & \frac{\mathrm{d}}{\mathrm{d}t} \Xi_j = -\nabla_x V_{\rm p}(t, X_j) - \sum_{k \neq j} V_{\rm s}(|X_j - X_k|), \\ X_{j,|t=s} = x_j, & \Xi_{j,|t=s} = \xi_j. \end{cases}$$

Then a natural and delicate question consists in obtaining the corresponding Vlasov equation by passing to the limit $M \to \infty$.

In the right hand side of (1.5), the collision operator, which can be linear or not, has usually a specific structure. This structure is motivated by the physical modeling, but it also induces mathematical properties crucial to the analysis. Usually the operator Q acts only on the variable ξ and remains local with respect to time and space; it is for instance an integral or a differential operator with respect to ξ. In particular, a lot of models implies the following key features:

- The operator Q is orthogonal to some functions, usually of polynomial nature, of ξ, which induces local and conservation properties.
- The functions which make vanish the collision operator have a specific dependence with respect to the variable ξ, a typical example being

$$Q(f) = 0 \quad \text{iff } f(\xi) = nM(\xi), \quad M(\xi) = \frac{\mathrm{e}^{-E(\xi)}}{\displaystyle\int_{\mathbb{R}^N} \mathrm{e}^{-E(\xi')} \,\mathrm{d}\xi'}, \quad \nabla_\xi E(\xi) = v(\xi).$$

- The operator Q dissipates some quantities, which means that

$$\int_{\mathbb{R}^N} Q(f) \; \Psi(f) \,\mathrm{d}\xi \leq 0$$

  holds for some function $\Psi : \mathbb{R}^+ \to \mathbb{R}$. These dissipation properties are crucial for the mathematical analysis since they provide useful a priori estimates on the solution (in some sense they govern the functional spaces to be used). Moreover, the dissipation vanishes when $f = nM$ is a (local) equilibrium, indicating the relaxation effects of the collisions.

The simplest example is given by the relaxation (or linear BGK) operator

$$Q(f) = M(\xi) \int_{\mathbb{R}^N} f(\xi') \,\mathrm{d}\xi' - f(\xi). \tag{1.8}$$

It preserves the charge since

$$\int_{\mathbb{R}^N} Q(f) \,\mathrm{d}\xi = 0.$$

Accordingly, integrating (1.5) yields the local charge conservation equation relating the macroscopic charge and current

$$\partial_t n + \mathrm{div}_x J = 0,$$

and thus the global charge conservation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N} n\,\mathrm{d}x = 0.$$

A slightly more involved operator, the so–called linear Boltzmann operator, reads

$$Q(f) = \int_{\mathbb{R}^N} b(\xi,\xi')\big(M(\xi)f(\xi') - f(\xi)M(\xi')\big)\,\mathrm{d}\xi' \tag{1.9}$$

for some positive and symmetric kernel $b(\xi,\xi') = b(\xi',\xi) > 0$. It describes the binary collisions dynamics of the electrons with other particles which remain in an equilibrium state described by the Maxwellian $M(\xi)$. It also implies the charge conservation. Clearly the kernel of Q is spanned by functions proportional to M: $\mathrm{Ker}(Q) = \mathrm{Span}\{M\}$. We observe also that

$$\begin{aligned}
\int_{\mathbb{R}^N} Q(f)\Psi(f/M)\,\mathrm{d}\xi &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} b(\xi,\xi')M(\xi)M(\xi')\Big(\frac{f}{M}(\xi') - \frac{f}{M}(\xi)\Big)\Psi(f/M)(\xi)\,\mathrm{d}\xi'\,\mathrm{d}\xi \\
&= -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} b(\xi,\xi')M(\xi)M(\xi')\Big(\frac{f}{M}(\xi') - \frac{f}{M}(\xi)\Big) \\
&\qquad \times\Big(\Psi(f/M)(\xi') - \Psi(f/M)(\xi)\Big)\,\mathrm{d}\xi'\,\mathrm{d}\xi
\end{aligned}$$

is non positive for any non decreasing function Ψ. Specializing to $\Psi(z) = z$ and assuming $b(\xi,\xi') \geq \beta > 0$, it can be interpreted as a spectral gap inequality

$$\begin{aligned}
-\int_{\mathbb{R}^N} Q(f)\frac{f}{M}\,\mathrm{d}\xi &\geq \frac{\beta}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} M(\xi)M(\xi')\Big(\frac{f}{M}(\xi') - \frac{f}{M}(\xi)\Big)^2\,\mathrm{d}\xi'\,\mathrm{d}\xi \\
&\geq \frac{\beta}{2}\int_{\mathbb{R}^N}\Big(f - M(\xi)\int_{\mathbb{R}^N} f(\xi')\,\mathrm{d}\xi'\Big)^2\,\frac{1}{M(\xi)}\,\mathrm{d}\xi.
\end{aligned}$$

In turn, we deduce the following a priori estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^2\frac{\mathrm{d}\xi}{M}\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} b(\xi,\xi')M(\xi)M(\xi')\Big(\frac{f}{M}(\xi') - \frac{f}{M}(\xi)\Big)^2\,\mathrm{d}\xi'\,\mathrm{d}\xi \leq 0,$$

which tells us, at least, that the solution lies in $L^\infty((0,T);L^2(\mathbb{R}^N\times\mathbb{R}^N, M^{-1}\,\mathrm{d}\xi\,\mathrm{d}x))$.

We can also consider the Fokker-Planck operator

$$Q(f) = \nabla_\xi\cdot(v(\xi)f + \nabla_\xi f) = \mathrm{div}_\xi\Big[M\nabla_\xi\Big(\frac{f}{M}\Big)\Big] \tag{1.10}$$

which is of differential nature. It verifies

$$\int_{\mathbb{R}^N} Q(f)\frac{f}{M}\,\mathrm{d}\xi = -\int_{\mathbb{R}^N}\Big|\nabla_\xi\Big(\frac{f}{M}\Big)\Big|^2\,M\,\mathrm{d}\xi \leq 0.$$

A relevant example of non linear operator is given by

$$Q(f) = \int_{\mathbb{R}^N} b(\xi,\xi')\big(M(\xi)f(\xi')(1-f(\xi)) - f(\xi)(1-f(\xi'))M(\xi')\big)\,\mathrm{d}\xi', \tag{1.11}$$

which takes into account Fermi exclusion principles. It vanishes when

$$\frac{(1-f)M}{f}(\xi) = \mu \quad \text{is constant}$$

that is when f is the Fermi-Dirac distribution

$$f(\xi) = \frac{M(\xi)}{\mu + M(\xi)}.$$

It verifies the charge conservation too and the dissipative properties have been analyzed in [78].

For considering further non linear operators, let us restrict to the simple case where

$$v(\xi) = \xi, \qquad E(\xi) = \xi^2/2.$$

A non linear version of (1.8) is then the BGK operator

$$Q(f) = \frac{n}{(2\pi\Theta)^{N/2}} \exp\Big(-\frac{|\xi-u|^2}{2\Theta}\Big) - f(\xi). \tag{1.12}$$

where n, u and Θ, considered as the charge, the bulk velocity and the temperature of the cloud of particles, are associated to f by the relations

$$n = \int_{\mathbb{R}^N} f\,\mathrm{d}\xi, \quad nu = \int_{\mathbb{R}^N} \xi\ f\,\mathrm{d}\xi, \quad nu^2 + N\Theta = \int_{\mathbb{R}^N} \xi^2\ f\,\mathrm{d}\xi.$$

Therefore, when using these definition into (1.5), the macroscopic quantities n, u, Θ depend on (t,x). Similarly, we can define the non linear Fokker-Planck operator

$$Q(f) = \nabla_\xi \cdot \big((\xi-u)f + \Theta\nabla_\xi f\big). \tag{1.13}$$

Both the BGK and the Fokker-Planck operator conserve charge, momentum and energy since, by construction

$$\int_{\mathbb{R}^N} \begin{pmatrix} 1 \\ \xi \\ \xi^2 \end{pmatrix} Q(f)\,\mathrm{d}\xi = 0$$

which yields the conservation laws

$$\partial_t \int_{\mathbb{R}^N} \begin{pmatrix} 1 \\ \xi \\ \xi^2 \end{pmatrix} f\,\mathrm{d}\xi + \nabla_x \int_{\mathbb{R}^N} \xi \begin{pmatrix} 1 \\ \xi \\ \xi^2 \end{pmatrix} f\,\mathrm{d}\xi + \frac{q}{m}\nabla_x V \cdot \int_{\mathbb{R}^N} \begin{pmatrix} 0 \\ \mathbb{I} \\ 2\xi \end{pmatrix} f\,\mathrm{d}\xi = 0. \tag{1.14}$$

Note however that (1.14) is not a closed system of equations since the equation for the kth moment involves the moment of order $k+1$. Equilibrium states for the BGK and the Fokker-Planck operators are the Maxwellian distributions

$$f(\xi) = \frac{n}{(2\pi\Theta)^{N/2}} \exp\Big(- \frac{|\xi - u|^2}{2\Theta}\Big) \tag{1.15}$$

which are parametrized by total charge, momentum and temperature. The dissipative properties of these operators are summarized in the so–called "H-Theorem" which consists in observing that

$$\begin{aligned}
\int_{\mathbb{R}^N} Q(f)\ln(f)\,\mathrm{d}\xi &= \int_{\mathbb{R}^N} Q(f)\ln\Big(\frac{f}{(2\pi\Theta)^{-N/2}\mathrm{e}^{-|\xi-u|^2/(2\Theta)}}\Big)\,\mathrm{d}\xi \\
&= \begin{cases}
\displaystyle\int_{\mathbb{R}^N} \Big(\frac{\mathrm{e}^{-|\xi-u|^2/(2\Theta)}}{(2\pi\Theta)^{N/2}} - f\Big)\ln\Big(\frac{f}{(2\pi\Theta)^{-N/2}\mathrm{e}^{-|\xi-u|^2/(2\Theta)}}\Big)\,\mathrm{d}\xi \\
\qquad\qquad \text{for the BGK operator} \\
\displaystyle\int_{\mathbb{R}^N} \Big|\nabla_\xi\Big(\frac{f}{(2\pi\Theta)^{-N/2}\mathrm{e}^{-|\xi-u|^2/(2\Theta)}}\Big)\Big|^2 \ \frac{(2\pi\Theta)^{-N/2}\mathrm{e}^{-|\xi-u|^2/(2\Theta)}}{f}\,\mathrm{d}\xi \\
\qquad\qquad \text{for the Fokker-Planck operator}
\end{cases} \\
&\leq 0.
\end{aligned}$$

1.3 Macroscopic models

Eventually, we can adopt a fully macroscopic or hydrodynamic picture of the electrons describing their motion by the evolution of their macroscopic density n, current J and temperature Θ. The simplest model relies on drift-diffusion PDE: starting from the charge conservation

$$\partial_t n + \mathrm{div}_x J = 0$$

we postulate the following relation between the current and the density

$$J = -qn\nabla_x\Phi - \kappa\nabla_x n.$$

The last term is reminiscent to the standard Fick law and induces diffusion of the particles. The first term means that particles are convected with $-q\nabla_x\Phi$ as velocity field. Taking into account the coupling with (1.2), the convection depends on the distribution of particles itself. We shall see below that such model can be derived from (1.5), considering suitable asymptotic regimes and conservative linear collision operator like (1.8) or (1.10). The interesting question in such derivation is the identification of the diffusion matrix κ. In the same spirit, non linear models can be obtained from (1.11). Dealing with more involved operators that conserve also the energy, we are led to the so-called Energy-Transport models where the unknown is the pair (n, Θ). We can also describe the plasma by the standard equations of fluid mechanics. For instance, we can use the Euler system

$$\begin{cases} \partial_t n + \mathrm{div}_x(nu) = 0, \\ \partial_t(nu) + \mathrm{Div}_x(nu \otimes u + p) = -\dfrac{q}{m} n\nabla_x V, \\ \partial_t(nE) + \mathrm{Div}_x\big((nE+p)u\big) = -\dfrac{q}{m} n\nabla_x V \cdot u \end{cases} \tag{1.16}$$

where $E = e + u^2/2$ is the total energy and $e \geq 0$ is the internal energy. The latter is related to the temperature Θ, while the pressure is determined by a state law. For instance, we can complete the problem by using the perfect gas law:

$$e = \frac{p}{(\gamma-1)\rho} \geq 0, \qquad p = R\rho\Theta.$$

The set of equations (1.16) can be obtained from the kinetic picture (1.5), endowed with, say the BGK operator (1.8). Let us assume that the relaxation time is small $0 < \tau \ll 1$: τ can be interpreted as the typical time between collision events. When these events become very frequent, it is natural to expect a description with a model of continuum mechanics type. Indeed, the penalization of the collision term forces $Q(f) = 0$ so that we guess in this regime that f relaxes to a Maxwellian distribution

$$f \simeq \frac{n}{(2\pi\Theta)^{N/2}} \exp\Big(-\frac{|\xi-u|^2}{2\Theta}\Big) = M_{n,u,\Theta}(\xi).$$

We obtain a closed system of equations for the macroscopic quantities $(n, u, \Theta)(t, x)$ by inserting this ansatz into (1.14). We get

$$\begin{cases} \partial_t \displaystyle\int_{\mathbb{R}^N} \begin{pmatrix} 1 \\ \xi \\ \xi^2/2 \end{pmatrix} M_{n,u,\Theta}\,\mathrm{d}\xi + \nabla_x \int_{\mathbb{R}^N} \xi \begin{pmatrix} 1 \\ \xi \\ \xi^2/2 \end{pmatrix} M_{n,u,\Theta}\,\mathrm{d}\xi + \frac{q}{m}\nabla_x V \cdot \int_{\mathbb{R}^N} \begin{pmatrix} 0 \\ \mathbb{I} \\ \xi \end{pmatrix} M_{n,u,\Theta}\,\mathrm{d}\xi = 0 \\ \partial_t \begin{pmatrix} n \\ nu \\ nu^2/2 + Nn\Theta/2 \end{pmatrix} + \nabla_x \begin{pmatrix} nu \\ nu\otimes u + n\Theta\mathbb{I} \\ (nu^2/2 + Nn\Theta/2 + n\Theta)u \end{pmatrix} + \dfrac{q}{m} \begin{pmatrix} 0 \\ n\nabla_x V \\ n\nabla_x V \cdot u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{cases}$$

This is an adaptation of the "hydrodynamic limit" in gas dynamics. Similarly, we can also derive models that include viscous corrections. We refer e. g. to [22, 45, 46, 92] and for specific aspects for plasmas to [30]. For details on the equations of compressible gas dynamics we refer [68].

A further approximation consists in assuming that the gas is isentropic or isothermal. Then, we get rid of the energy equation and the flow is simply described by the coupled PDE

$$\begin{cases} \partial_t n + \mathrm{div}_x(nu) = 0, \\ \partial_t(nu) + \mathrm{Div}_x(nu \otimes u + p) = - -\dfrac{q}{m} n\nabla_x V \end{cases} \tag{1.17}$$

where now the pressure is determined by

$$p = \rho^\gamma, \qquad \gamma \geq 1$$

(the isothermal case corresponds to $\gamma = 1$). Even with such simplifications the analysis of the PDE system remains tough: it relies on the theory of hyperbolic systems, with additionaly the difficulty of dealing with the coupling that defines the potential. Such details are beyond the scope of these notes, we refer instead to [70, 80] or [27].

Remark 1 *Here, we present the equation set on the whole space. Discussion of the boundary conditions remains a tough questions, even for the modeling viewpoint. Furthermore, the influence of the boundary conditions on the dissipation properties and the underlying estimates needs some care, as pointed out in [28] for kinetic models.*

Remark 2 *A more detailed coupling involved, instead of the mere Poisson equation* (1.2) *for defining the electric field, the complete set of Maxwell equations. To be more specific, the kinetic equation reads*

$$\partial_t f + \xi \cdot \nabla_x f + \nabla_\xi \cdot (F f) = 0$$

where the force acting on the particles reads

$$F(t, x, \xi) = qE(t, x) + \xi \wedge B(tx)$$

where the electric field E and the magnetic field B verify

$$\begin{cases} \partial_t E - c^2 \mathrm{curl}_x B = -q\dfrac{J(t,x)}{\varepsilon_0}, & \partial_t B + \mathrm{curl}_x E = 0, \\ \mathrm{div}_x E = q\dfrac{\rho(t,x)}{\varepsilon_0}, & \mathrm{div}_x B = 0, \end{cases} \tag{1.18}$$

with c the speed of light, ε_0 the vaccuum permittivity. The system of PDEs which couples the kinetic equation (1.5) *to* (1.2) *can be derived from* (1.5) *coupled to* (1.18) *by investigating the limit $c \to \infty$ (up to a suitable choice of units...): we refer to [29, 84] or [14] for the specific case of time-periodic solutions. Similar problems can be addressed with magnéto-hydrodynamic equations.*

Remark 3 *The situation where we look at the evolution of the electrons only is already a simplified framework. Actually, electrons interact with positive charges, or with "holes" in semiconductors theory, so that we should deal with coupled system of PDEs like* (1.1), *or* (1.5) *or* (1.16) *or* (1.17), *with $q = \pm 1$; the coupling arises from the definition of the electric potential in* (1.2) *which then reads*

$$-\lambda \Delta_x V_{\mathrm{s}} = (n_{negative} - n_{positive})$$

instead of having a given background. In what follows we shall neglect this additional difficulty and deal with one species of charged particles only.

The mathematical questions concerning these PDEs can be summarized as follows:

- Mathematical analysis of the equations: well posedness of the Cauchy problem, qualitative properties of the solutions...
- Asymptotic analysis: identification of physical paramaters and asymptotic regimes which allow to derive relations between the different levels of modeling.
- Numerical analysis: design of performing numerical schemes.

2 Semi-classical limit: from quantum to classical mechanics

2.1 From Schrödinger to Liouville

The first goal consists in obtaining the models of classical mechanics from models describing the transport of charge at the quantum level. The derivation relies on asymptotic analysis arguments, letting (a dimensionless version of) the Planck constant $\hbar$ go to 0 in (1.1). The mathematical analysis of these questions has been performed by P.L. Lions-Th. Paul [65]. The main ingredient is the introduction of the Wigner transform [93] of a function $\psi : (0,T) \times \mathbb{R}^N \to \mathbb{C}$.

In what follows, we denote indifferently by $\mathscr{F}(f)$ or by $\widehat{f}$ the Fourier transform given by

$$\widehat{f}(y) = \int_{\mathbb{R}^N} \mathrm{e}^{-\mathrm{i}y\cdot\xi} f(\xi)\, d\xi,$$

and $\mathscr{F}^{-1}(f)(\xi) = \int_{\mathbb{R}^N} \mathrm{e}^{+\mathrm{i}y\cdot\xi} f(y)\, dy/(2\pi)^N$. We recall that these formulae make sense when both f and $\widehat{f}$ belong to $L^1(\mathbb{R}^N)$, but the Fourier transform and its inverse are extended to $L^2(\mathbb{R}^N)$ or to the Schwartz class $\mathscr{S}(\mathbb{R}^N)$ by density, and then by duality to $\mathscr{S}'(\mathbb{R}^N)$.

Definition 1 *Let $\psi \in L^2(\mathbb{R}^N)$. The Wigner transform of ψ is the real valued $L^2(\mathbb{R}^N \times \mathbb{R}^N)$ function defined by*

$$W[\psi](x,\xi) = \int_{\mathbb{R}^N} \psi(x - y/2)\ \overline{\psi(x + y/2)}\, \mathrm{e}^{\mathrm{i}y\cdot\xi} \frac{\mathrm{d}y}{(2\pi)^N}.$$

The change of variable $y \to -y$ yields

$$W[\psi](x,\xi) = \int_{\mathbb{R}^N} \psi(x + y/2)\ \overline{\psi(x - y/2)}\, \mathrm{e}^{-\mathrm{i}y\cdot\xi} \frac{\mathrm{d}y}{(2\pi)^N} = \overline{W[\psi](x,\xi)}$$

so that $W[\psi](x,\xi) \in \mathbb{R}$. Next, it is convenient to rewrite

$$W[\psi](x,\cdot) = \mathscr{F}^{-1}_{y\to\xi}\Big[\psi(x - y/2)\ \overline{\psi(x + y/2)}\Big].$$

The change of variables $(x', y') = (x - y/2, x + y/2)$ shows that

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} |\psi(x - y/2)\ \overline{\psi(x + y/2)}|^2\, \mathrm{d}y\, \mathrm{d}x = 2^{2N} \int_{\mathbb{R}^N\times\mathbb{R}^N} |\psi(x')\ \overline{\psi(y')}|^2\, \mathrm{d}y'\, \mathrm{d}x' = 2^{2N}\|\psi\|^2_{L^2}$$

and the Plancherel formula implies that $W[\psi](x,\xi)$ belongs to $L^2(\mathbb{R}^N \times \mathbb{R}^N)$. We shall describe below further functional spaces adapted for studying the properties of the Wigner transform.

The point is that the Wigner transform changes the Schrödinger equation into a transport equation, where the Fourier variable ξ plays the role of velocity. The potential term is transformed into a pseudo-differential operator.

Proposition 1 *Let ψ be the solution of the Schrödinger equation*

$$\begin{cases} \mathrm{i}\partial_t \psi = -\dfrac{1}{2}\Delta\Psi + V\psi, \\ \psi_{|t=0} = \psi^{\text{Init}} \in L^2(\mathbb{R}^N), \end{cases} \tag{2.19}$$

with a smooth and real valued potential, say $V \in C_c^\infty(\mathbb{R}^N)$. Then, $W[\psi]$ verifies the following Wigner (or "quantum Liouville") equation

$$\partial_t f + \xi \cdot \nabla_x f = \Theta(V)(f) \qquad \text{in } \mathbb{R}_t^+ \times \mathbb{R}_x^N \times \mathbb{R}_\xi^N, \tag{2.20}$$

with initial data $f_{|t=0} = W[\psi^{\text{Init}}]$ and where the right hand side is defined by the pseudo-differential operator

$$\Theta(V)(f) = \mathrm{i}\mathscr{F}^{-1}_{y\to\xi}\big((V(x+y/2) - V(x-y/2))\widehat{f}(t,x,y)\big). \tag{2.21}$$

Proof. We compute the time derivative of the Wigner transform

$$\begin{aligned}
\partial_t W[\psi](t,x,\xi) &= \int_{\mathbb{R}^N} \Big(\partial_t\psi(x-y/2)\ \overline{\psi(x+y/2)} + \psi(x-y/2)\ \partial_t\overline{\psi(x+y/2)}\Big)\mathrm{e}^{\mathrm{i}y\cdot\xi}\frac{\mathrm{d}y}{(2\pi)^N} \\
&= \int_{\mathbb{R}^N} \Big(\frac{1}{\mathrm{i}}\big(-\frac{1}{2}\Delta_x + V(x-y/2)\big)\psi(x-y/2)\ \overline{\psi(x+y/2)} \\
&\quad + \psi(x-y/2)\overline{\Big(\frac{1}{\mathrm{i}}\Big)}\ \big(-\frac{1}{2}\Delta_x + V(x+y/2)\big)\overline{\psi(x+y/2)}\Big)\mathrm{e}^{\mathrm{i}y\cdot\xi}\frac{\mathrm{d}y}{(2\pi)^N} \\
&= \mathrm{i}\int_{\mathbb{R}^N} \big(V(x+y/2) - V(x-y/2)\big)\ \psi(x-y/2)\ \overline{\psi(x+y/2)}\ \mathrm{e}^{\mathrm{i}y\cdot\xi}\frac{\mathrm{d}y}{(2\pi)^N} \\
&\quad + \frac{\mathrm{i}}{2}\int_{\mathbb{R}^N} \Big(\Delta_x\psi(x-y/2)\ \overline{\psi(x+y/2)} - \psi(x-y/2)\ \Delta_x\overline{\psi(x+y/2)}\Big)\mathrm{e}^{\mathrm{i}y\cdot\xi}\frac{\mathrm{d}y}{(2\pi)^N}.
\end{aligned}$$

The first integral is nothing but $\Theta(V)(W[\psi])$. Let us now integrate by part in the second term, remarking that

$$\nabla_y\Big[\psi(x\pm y/2)\Big] = \pm\frac{1}{2}(\nabla\psi)(x\pm y/2), \qquad \Delta_x\psi(x\pm y/2) = 4\Delta_y\Big[\psi(x\pm y/2)\Big].$$

We get

$$\begin{aligned}
&-2\mathrm{i}\int_{\mathbb{R}^N} \Big(\nabla_y\Big[\psi(x-y/2)\Big]\cdot\big(\nabla_y + \mathrm{i}\xi\big)\Big[\overline{\psi(x+y/2)}\Big] \\
&\qquad\qquad -\big(\nabla_y + \mathrm{i}\xi\big)\Big[\psi(x-y/2)\Big]\cdot\nabla_y\Big[\overline{\psi(x+y/2)}\Big]\ \mathrm{e}^{\mathrm{i}y\cdot\xi}\frac{\mathrm{d}y}{(2\pi)^N} \\
&= -\xi\cdot\int_{\mathbb{R}^N} \Big(\nabla\psi(x-y/2)\overline{\psi(x+y/2)} + \psi(x-y/2)\nabla\overline{\psi(x+y/2)}\Big)\ \mathrm{e}^{\mathrm{i}y\cdot\xi}\frac{\mathrm{d}y}{(2\pi)^N} \\
&= --\xi\cdot\nabla_x W[\psi](t,x\xi),
\end{aligned}$$

which ends the proof. ∎

In fact, for our purposes, we need a definition that takes into account the Planck scale $\hbar$. To this end, we adapt Definition 1.

Definition 2 *Let $\big(\psi_\hbar\big)_{\hbar>0}$ be a sequence bounded in $L^2(\mathbb{R}^N)$. We associate the following sequence of Wigner transform*

$$W_\hbar[\psi_\hbar](x,\xi) = \int_{\mathbb{R}^N} \psi_\hbar(x-\hbar y/2)\ \overline{\psi_\hbar(x+\hbar y/2)}\,\mathrm{e}^{\mathrm{i}\hbar y\cdot\xi}\,\frac{\mathrm{d}y}{(2\pi)^N}.$$

We notice that the L^2 norm of $W_\hbar[\psi_\hbar]$ is now of order $\mathcal{O}(\hbar^{-N})$. Since we aim at extracting convergent subsequences, this is not a useful information. To go further, we introduce the space

$$\mathscr{A} = \big\{\phi\in C^0(\mathbb{R}^N\times\mathbb{R}^N),\ \mathscr{F}_{\xi\to y}\phi(x,y)\in L^1(\mathbb{R}^N_y; C^0\cap L^\infty(\mathbb{R}^N))\big\},$$

endowed with the norm

$$\|\phi\|_{\mathscr{A}} = \int_{\mathbb{R}^N}\|\mathscr{F}_{\xi\to y}\phi(\cdot,y)\|_{L^\infty(\mathbb{R}^N)}\,\mathrm{d}y = \int_{\mathbb{R}^N}\sup_{x\in\mathbb{R}^N}\Big(\mathscr{F}_{\xi\to y}|\phi(x,y)|\Big)\,\mathrm{d}y.$$

Clearly, $\mathscr{A}$ contains the space $\mathscr{S}$ of Schwartz functions on $\mathbb{R}^N\times\mathbb{R}^N$. Hence, the dual $\mathscr{A}'$ embeds into $\mathscr{S}'$. The starting point of the analysis relies in the following claim.

Proposition 2 *Let $\big(\psi_\hbar\big)_{\hbar>0}$ be a sequence bounded in $L^2(\mathbb{R}^N)$. Then $\big(W_\hbar[\psi_\hbar]\big)_{\hbar>0}$ is bounded in $\mathscr{A}'$. Accordingly, we can suppose that there exists a subsequence which converges (weakly-$\star$ in $\mathscr{A}'$) in $\mathscr{S}'$*

Proof. By definition we have

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} W_\hbar[\psi_\hbar]\ \varphi(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N\times\mathbb{R}^N}\mathscr{F}_{\xi\to y}\varphi(x,-y)\ \psi_\hbar(x-\hbar y/2)\overline{\psi_\hbar(x+\hbar y/2)}\,\mathrm{d}y\,\mathrm{d}x$$

Hence, we get

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N\times\mathbb{R}^N} W_\hbar[\psi_\hbar]\ \varphi(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\Big|\\
&\qquad\le \int_{\mathbb{R}^N}\|\mathscr{F}_{\xi\to y}\varphi(\cdot,y)\|_{L^\infty(\mathbb{R}^N)}\,\mathrm{d}y\times\sup_{y\in\mathbb{R}^N}\int_{\mathbb{R}^N}|\psi_\hbar(x-\hbar y/2)\overline{\psi_\hbar(x+\hbar y/2)}|\,\mathrm{d}x\\
&\qquad\le \|\varphi\|_{\mathscr{A}}\sup_{y\in\mathbb{R}^N}\Big\{\Big(\int_{\mathbb{R}^N}|\psi_\hbar(x-\hbar y/2)|^2\,\mathrm{d}x\Big)^{1/2}\Big(\int_{\mathbb{R}^N}|\psi_\hbar(x+\hbar y/2)|^2\,\mathrm{d}x\Big)^{1/2}\Big\}\\
&\qquad\le \|\varphi\|_{\mathscr{A}}\|\psi_\hbar\|^2_{L^2(\mathbb{R}^N)}.
\end{aligned}$$

∎

Corollary 1 *Let $\psi_\hbar$ be the solution of* (1.1). *Then, its Wigner transform $f_\hbar = W_\hbar[\psi_\hbar]$ verifies*

$$\partial_t f_\hbar + \xi\cdot\nabla_x f_\hbar = \Theta_\hbar(V)(f_\hbar)\qquad \text{in } \mathbb{R}^+_t\times\mathbb{R}^N_x\times\mathbb{R}^N_\xi, \tag{2.22}$$

with initial data $f_{\hbar,|t=0} = W_\hbar[\psi_\hbar^{\mathrm{Init}}]$ and where the right hand side is defined by the pseudo-differential operator

$$\Theta_\hbar(V)(f) = \frac{\mathrm{i}}{\hbar}\mathscr{F}^{-1}_{y\to\xi}\big((V(x+\hbar y/2) - V(x-\hbar y/2))\widehat{f}(t,x,y)\big). \tag{2.23}$$

The connection with the classical Liouville equation can be understood with the following formal ansatz

$$\frac{V(x+\hbar y/2)-V(x-\hbar y/2)}{\hbar} \simeq \nabla_x V(x)\cdot y$$

so that as $\hbar \to 0$

$$\begin{aligned}\Theta_\hbar(V)(f) &\simeq \nabla_x V(x)\cdot \mathrm{i}\int_{\mathbb{R}^N} y\ \widehat{f}(y)\mathrm{e}^{\mathrm{i}\hbar y\cdot\xi}\frac{\mathrm{d}y}{(2\pi)^N}\\ &\simeq \nabla_x V(x)\cdot \mathscr{F}^{-1}_{y\to\xi}(\mathrm{i}y\widehat{f}) = \nabla_x V(x)\cdot\nabla_\xi f(\xi).\end{aligned}$$

Making the argument rigorous leads to the following claim.

Theorem 1 *Let $\psi_\hbar$ be the solution of (1.1) with a smooth given potential $V\in C^1\cap W^{1,\infty}(\mathbb{R}^N)$. Then, up to a subsequence, as $\hbar\to 0$, the Wigner transform $W_\hbar[\psi_\hbar]=f_\hbar$ tends to f in $C^0([0,T],\mathscr{A}'-$ weak $-\star)$, solution of the Liouville equation (1.5) with initial data f^{Init} the limit, in the weak-$\star$ sense in $\mathscr{A}'$ of $W_\hbar[\psi_\hbar^{\mathrm{Init}}]$.*

Obtaining (1.5) by using the Wigner transform might sound strange since when introducing (1.5), we said that the unknown f is interpreted as a density and is therefore non negative while there is no reason that give a sign to $W_\hbar[\psi_\hbar]$. Nevertheless, the limit when $\hbar\to 0$ has indeed the required sign (which motivates the terminology of "Wigner measure").

Proposition 3 *If the sequence of Wigner transform $W_\hbar[\psi_\hbar]$ associated to a sequence $\psi_\hbar\in L^2(\mathbb{R}^N)$ converges in $\mathscr{S}'$ to some W, then $W\geq 0$.*

Proof. We aim at proving that

$$\lim_{\hbar\to 0}\int_{\mathbb{R}^N\times\mathbb{R}^N} W_\hbar[\psi_\hbar]\ \phi(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x \geq 0$$

holds for any non negative $\phi\in\mathscr{S}$. It suffices to establish the result for $\phi=|\widehat{\varphi}|^2=\widehat{\varphi}\overline{\widehat{\varphi}}$. We start by remarking

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} W_\hbar[\psi_\hbar]\ |\widehat{\varphi}|^2(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x = \int_{\mathbb{R}^N\times\mathbb{R}^N}\psi_\hbar(x-\hbar y/2)\overline{\psi_\hbar(x+\hbar y/2)}\mathscr{F}^{-1}_{\xi\to y}|\widehat{\varphi}|^2(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

Then, the standard rules of the Fourier transform yield

$$\begin{aligned}\mathscr{F}^{-1}_{\xi\to y}a\overline{b}(y) &= \frac{1}{(2\pi)^N}\widehat{a}\star\widehat{\overline{b}}(-y)\\ &= \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}\widehat{a}(z)\widehat{\overline{b}}(-y-z)\,\mathrm{d}z = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}\widehat{a}(-z)\widehat{\overline{b}}(z-y)\,\mathrm{d}z\\ &= (2\pi)^N\int_{\mathbb{R}^N}\mathscr{F}^{-1}a(z)\mathscr{F}^{-1}\overline{b}(y-z)\,\mathrm{d}z.\end{aligned}$$

Furthermore, we have

$$\mathscr{F}^{-1}\overline{b}(y) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} \mathrm{e}^{\mathrm{i}y\cdot\xi}\overline{b}(\xi)\,\mathrm{d}\xi = \frac{1}{(2\pi)^N}\overline{\int_{\mathbb{R}^N} \mathrm{e}^{-\mathrm{i}y\cdot\xi}b(\xi)\,\mathrm{d}\xi} = \overline{\mathscr{F}^{-1}b(-y)}.$$

It follows that

$$\mathscr{F}^{-1}_{\xi\to y}|\widehat{\varphi}|^2(x,y) = \mathscr{F}^{-1}_{\xi\to y}\widehat{\varphi}\overline{\widehat{\varphi}}(x,y) - (2\pi)^N\int_{\mathbb{R}^N}\varphi(x,z)\overline{\varphi(x,z-y)}\,\mathrm{d}z.$$

Thus, we arrive at

$$\begin{aligned}
&\int_{\mathbb{R}^N\times\mathbb{R}^N} W_\hbar[\psi_\hbar]\ |\widehat{\varphi}|^2(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\\
&= (2\pi)^N\int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N}\psi_\hbar(x-\hbar y/2)\overline{\psi_\hbar(x+\hbar y/2)}\varphi(x,z)\overline{\varphi(x,z-y)}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\\
&= (2\pi)^N\int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N}\psi_\hbar(x'-\hbar z)\overline{\psi_\hbar(x-\hbar y')}\varphi(x'-\hbar(z+y')/2,z)\overline{\varphi(x'-\hbar(z+y')/2,y')}\,\mathrm{d}z\,\mathrm{d}y'\,\mathrm{d}x'
\end{aligned}$$

with the change of variables $(x,y) = (x'-\hbar(z+y')/2, z-y')$ (having jacobian 1). The leading term is obtained by replacing $\varphi(x-\hbar(z+y')/2,\cdot)$ by $\varphi(x,\cdot)$; that is

$$\begin{aligned}
&\int_{\mathbb{R}^N\times\mathbb{R}^N} W_\hbar[\psi_\hbar]\ |\widehat{\varphi}|^2(x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x\\
&\qquad\simeq (2\pi)^N\int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N}\psi_\hbar(x'-\hbar z)\overline{\psi_\hbar(x'-\hbar y')}\varphi(x',z)\overline{\varphi(x',y')}\,\mathrm{d}z\,\mathrm{d}y'\,\mathrm{d}x'\\
&\qquad\simeq \int_{\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\overline{\psi_\hbar(x'-\hbar y')}\ \overline{\varphi(x',y')}\,\mathrm{d}y'\Big)\Big(\int_{\mathbb{R}^N}\psi_\hbar(x'-\hbar z)\ \varphi(x',z)\,\mathrm{d}z\Big)\,\mathrm{d}x'\\
&\qquad\simeq \int_{\mathbb{R}^N}\Big|\int_{\mathbb{R}^N}\psi_\hbar(x'-\hbar y')\ \varphi(x',y')\,\mathrm{d}y'\Big|^2\,\mathrm{d}x' \geq 0.
\end{aligned}$$

The remainder terms which have been neglected can been shown to be of order $\mathcal{O}(\hbar)$. ∎

Proof of Theorem 1. We multiply (2.22) by a test function $\phi\in\mathscr{S}$ such that $\widehat{\phi}\in C^\infty_c$. Set $D_V(x,y) = V(x+y/2) - V(x-y/2)$. Note that

$$\begin{aligned}
\int_{\mathbb{R}^N}\Theta[V](f)\ \phi\,\mathrm{d}\xi &= \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\mathrm{e}^{\mathrm{i}y\cdot\xi}D_V(x,y)\widehat{f}(y)\ \phi(\xi)\,\mathrm{d}y\,\mathrm{d}\xi\\
&= -\frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}D_V(x,-y)\widehat{f}(y)\widehat{\phi}(-y)\,\mathrm{d}y = -\frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}D_V(x,y)\widehat{f}(-y)\widehat{\phi}(y)\,\mathrm{d}y\\
&= -\int_{\mathbb{R}^N}\mathscr{F}^{-1}_{\xi\to y}f(y)D_V(x,y)\widehat{\phi}(y)\,\mathrm{d}y = -\int_{\mathbb{R}^N}f\ \Theta[V](\phi)\,\mathrm{d}\xi.
\end{aligned}$$

Hence, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N\times\mathbb{R}^N}f_\hbar\ \phi\,\mathrm{d}\xi\,\mathrm{d}x = \int_{\mathbb{R}^N\times\mathbb{R}^N}f_\hbar\ v\cdot\nabla_x\phi\,\mathrm{d}\xi\,\mathrm{d}x - \int_{\mathbb{R}^N\times\mathbb{R}^N}f_\hbar\Theta_\hbar[V]\phi\,\mathrm{d}\xi\,\mathrm{d}x$$

since the operator (2.23) is skew-adjoint. Then, we first show that $\big(\Theta_\hbar[V]\phi\big)_{\hbar>0}$ remains in a bounded set of $\mathscr{A}$. This is guaranteed for $V \in C^1 \cap W^{1,\infty}(\mathbb{R}^N)$ and $\phi \in \mathscr{S}$. This already proves, by applying the Arzela-Ascoli theorem that $\big\{ \int_{\mathbb{R}^N\times\mathbb{R}^N} f_\hbar\ \phi\,\mathrm{d}\xi\,\mathrm{d}x,\ \hbar > 0\big\}$ lies in a compact set of $C^0([0,T])$. Second, we establish the convergence

$$\Theta_\hbar[V]\phi \xrightarrow[\hbar\to 0]{} \nabla_x V \cdot \nabla_v \phi \qquad \text{in } \mathscr{A},$$

as a consequence of the fact

$$\mathscr{F}_{\xi\to y}\Theta_\hbar[V]\phi = \mathrm{i}\widehat{\phi}(x,y)\ \frac{V(x+\hbar y/2) - V(x-\hbar y/2)}{\hbar} \xrightarrow[\hbar\to 0]{} \mathrm{i}y\cdot\nabla_x V \qquad \text{in } L^1(\mathbb{R}^N_x; C^0_b(\mathbb{R}^N)).$$

■

Let us comment this result:

- With the regularity assumptions made here, namely $V \in C^1 \cap W^{1,\infty}(\mathbb{R}^N)$, we can show the uniqueness of the solution of (1.5) in $C^0(\mathbb{R}^+; \mathcal{M}^1(\mathbb{R}^N \times \mathbb{R}^N) - \text{weak} - \star)$ by appealing to the characteristics, which are indeed well defined under such conditions, see e. g. [13, 65]. Accordingly, assuming the convergence of the initial data, we can conclude that the statement hold for the entire sequence $f_\hbar$.
- The regularity assumptions on V can be slightly relaxed, but in such a case the uniqueness of the limit equation might not be guaranteed. Conversely, assuming more regularity on V and the initial data, we can obtain sharp ansatz, with estimates on the solutions like $\mathcal{O}(\hbar^m)$.
- Non linear situations where the potential is defined by a convolution formula $V_\mathrm{s} = U \star n(t,x)$ can be dealt with, including the singular kernels corresponding to (1.2).
- For some hints on the numerical treatment of the semi–classical scale, we mention [87], [48] and references therein.

2.2 Interaction between the semi-classical scale and the periodicity of the medium

As said above, the potential due to the environment has naturally a periodic structure, according to the structure of the crystal in which the electrons are embodied. New difficulties appear when the typical length scale of the crystal lattice coincides with the Planck scale. Keeping $\hbar$ as notation for the common value of the parameter, we are interested in the limit $\hbar \to 0$ in the Schrödinger equation

$$\mathrm{i}\hbar\partial_t\psi_\hbar = -\frac{\hbar^2}{2}\Delta_x\psi_\hbar + V_\mathrm{p}(x/\hbar)\psi_\hbar \tag{2.24}$$

where V_p is periodic:

$$V_\mathrm{p}(x+\gamma) = V_\mathrm{p}(x)$$

for any $\gamma \in \Gamma$, the lattice of the crystal. The asymptotic analysis of (2.24) combines the semi-classical limit with homogenization aspects and the framework designed above is not well designed to take into account the specific periodicity of the medium embodied into the potential $V_{\rm p}$ and the scales interactions. Following [71], the analysis relies on the combination of two ingredients: a suitable spectral decomposition of the solutions, the so–called Bloch decomposition, and an adaptation of the Wigner formalism to the periodic setting.

The starting point relies in the following remark: changing x into $x/\hbar = y$, one changes $H_\hbar \psi(x) = -\frac{\hbar^2}{2}\Delta_x \psi + V_{\rm p}(x/\hbar)\psi$ into $H\psi(y) = -\frac{1}{2}\Delta_y \psi + V_{\rm p}(y)\psi$. Hence, we shall concentrate on the spectral properties of the operator H where we get rid of the small scale. We shall see also below that passing from $H_\hbar$ to H, we can also reduce the space domain, exchanging the whole space $\mathbb{R}^N$ to a bounded domain C, with suitable boundary conditions. This considerably simplifies the spectral structure of the operator. Let us introduce a few notation:

- we denote by $C \subset \mathbb{R}^N$ the period cell of Γ,
- we define the dual lattice

$$\Gamma^\star = \big\{\gamma^\star \in \mathbb{R}^N, \text{for any } \gamma \in \Gamma, \text{ one has } \gamma \cdot \gamma^\star \in 2\pi\mathbb{Z}\big\},$$

- finally B stands for the period cell of $\Gamma^\star$: this bounded set of $\mathbb{R}^N$ is the so–called Brillouin zone.

Lemma 1 *Let $\psi \in L^2(\mathbb{R}^N)$; we set*

$$u(x,k) = \sum_{\gamma\in\Gamma} \psi(x+\gamma)\ \mathrm{e}^{-\mathrm{i}k\cdot\gamma}$$

for $x \in \mathbb{R}^N$ and $k \in B$. Clearly, $k \mapsto u(x,k)$ is $B-$periodic: for any $\gamma^\star \in \Gamma^\star$ we have $u(x, k+\gamma^\star) = u(x,k)$. Furthermore, $x \mapsto u(x,k)$ is $k-$quasiperiodic, that is for any $\gamma' \in \Gamma$, we have

$$u(x+\gamma', k) = \mathrm{e}^{\mathrm{i}k\cdot\gamma'} u(x,k).$$

Eventually we have

$$\psi(x) = \int_B u(x,k)\,\mathrm{d}k$$

where $\mathrm{d}k$ stands for the normalized Lebesgue measure on B.

Now, consider ψ, solution of the Schrödinger equation

$$\mathrm{i}\partial_t \psi = -\frac{1}{2}\Delta_x \psi + V_{\rm p}\psi$$

with $V_{\rm p}$ a $C-$periodic potential. Then,

$$u(t,x,k) = \sum_{\gamma\in\Gamma} \psi(t, x+\gamma)\mathrm{e}^{-\mathrm{i}k\cdot\gamma}$$

verifies

$$\mathrm{i}\partial_t u = -\frac{1}{2}\Delta_x u + V_{\mathrm{p}} u$$

endowed with the $k-$quasiperiodic boundary condition:

$$u(x+\gamma', k) = \mathrm{e}^{\mathrm{i}k\cdot\gamma'} u(x,k).$$

This remark permits to consider the problem in a bounded domain, and thus to use the spectral decomposition associated to the operator

$$-\frac{1}{2}\Delta_x + V_{\mathrm{p}}$$

with $k-$quasiperiodic boundary condition (see [86]).

Lemma 2 *The operator $\mathscr{H} = -\frac{1}{2}\Delta_x + V_{\mathrm{p}}$ on C with $k-$quasiperiodic boundary condition is symmetric and admits a compact resolvant; hence, for any fixed $k \in B$, there exists a sequence of real eigenvalues $\big\{E_n(k),\ n \in \mathbb{N}\big\}$ and associated eigenfunctions $\big\{x \mapsto u_n(x,k),\ n \in \mathbb{N}\big\}$ which define a orthonormal basis of $L^2(C)$. The set $\{E_n(k),\ k \in B\}$ is called the $n-$th energy band of $\mathscr{H}$.*

We now reintroduce the parameter $\hbar$ by using the following claim, see [86].

Proposition 4 *For $\psi \in L^2(\mathbb{R}^N)$, we set*

$$\tilde{\psi}_\hbar(n,k) = \int_{\mathbb{R}^N} \psi(x)\ \frac{1}{\hbar^{N/2}} u_n(x/\hbar, k)\,\mathrm{d}x.$$

Then

- $\psi(x) = \displaystyle\sum_{n\in\mathbb{N}} \int_B \tilde{\psi}_\hbar(n,k) \frac{1}{\hbar^{N/2}} u_n(x/\hbar, k)\,\mathrm{d}k,$
- $\widetilde{H_\hbar\psi}(n,k) = E_n(k)\tilde{\psi}_\hbar(n,k).$

Proof. We only sketch the manipulations that lead to the second item. Indeed, for $\hbar = 1$ and $\psi(x) = \int_B u(x,k)\,\mathrm{d}k$ with $u(x,k) = \sigma(k)u_n(x,k)$ we have

$$\begin{aligned} H\psi(x) &= -\frac{1}{2}\Delta_x\psi + V_{\mathrm{p}}(x)\psi(x) = \Big(-\frac{1}{2}\Delta_x + V_{\mathrm{p}}(x)\Big)\int_B u(x,k)\,\mathrm{d}k \\ &= \int_B \mathscr{H} u(x,k)\,\mathrm{d}k = \int_B E_n(k)u(x,k)\,\mathrm{d}k \end{aligned}$$

We introduce the Fourier coefficients

$$\varepsilon_n(\gamma) = \int_B E_n(k)\mathrm{e}^{-\mathrm{i}\gamma\cdot k}\,\mathrm{d}k, \qquad E_n(k) = \sum_{\gamma\in\Gamma} \varepsilon_n(\gamma)\mathrm{e}^{\mathrm{i}\gamma\cdot k}.$$

The previous equality becomes

$$\begin{aligned} H\psi(x) &= \sum_{\gamma\in\Gamma} \varepsilon_n(\gamma) \int_B e^{ik\cdot\gamma} u(x,k)\,dk = \sum_{\gamma,\gamma'\in\Gamma} \varepsilon_n(\gamma) \int_B e^{ik\cdot\gamma} e^{-ik\cdot\gamma'} \psi(x+\gamma')\,dk \\ &= \sum_{\gamma,\gamma'\in\Gamma} \varepsilon_n(\gamma)\psi(x+\gamma')\delta(\gamma=\gamma') = \sum_{\gamma\in\Gamma} \varepsilon_n(\gamma)\psi(x+\gamma). \end{aligned}$$

Then, we multiply by u_n to obtain

$$\begin{aligned} \int_{\mathbb{R}^N} H\psi(x) u_n(x,k)\,dx &= \sum_{\gamma\in\Gamma} \varepsilon_n(\gamma) \int_{\mathbb{R}^N} \psi(x+\gamma) u_n(x,k)\,dx \\ &= \sum_{\gamma\in\Gamma} \varepsilon_n(\gamma) \int_{\mathbb{R}^N} \psi(x) u_n(x-\gamma,k)\,dx \\ &= \sum_{\gamma\in\Gamma} \varepsilon_n(\gamma) \int_{\mathbb{R}^N} \psi(x)\ e^{ik\cdot\gamma} u_n(x,k)\,dx \\ &= \sum_{\gamma\in\Gamma} \varepsilon_n(\gamma)) e^{ik\cdot\gamma}\ \tilde{\psi}(n,k) = E_n(k)\tilde{\psi}(n,k). \end{aligned}$$

∎

The next step consists in defining the Wigner series associated to the corresponding solution of (2.24):

$$W^S_\hbar[\psi_\hbar](x,k) = \sum_{\gamma\in\Gamma} \psi_\hbar(x-\hbar\gamma/2)\overline{\psi_\hbar(x+\hbar\gamma/2)}\ e^{ik\cdot\gamma}$$

Again, we observe that $W^S_\hbar[\psi_\hbar]$ is real valued; we can show boundedness in appropriate functional space and establish that cluster points (defined in $\mathscr{S}'$) are non negative. Let us detail the basic asymptotic result.

Proposition 5 *Assume that initially $\psi_\hbar^{\mathrm{Init}}$ belongs to the $n-$th band space that is*

$$\psi_\hbar^{\mathrm{Init}}(x) = \int u(k)\ \frac{1}{\hbar^{N/2}} u_n(x/\hbar,k)\,dk$$

for some $u\in L^2(B)$. Then, the solution $\psi_\hbar$ of (2.24) has the same property and we get

$$i\hbar\partial_t\psi_\hbar = \sum_{\gamma\in\Gamma} \varepsilon_n(\gamma)\ \psi_\hbar(x+\hbar\gamma)$$

where the $\varepsilon_n(\gamma)$'s stand for the Fourier coefficients of $E_n(k)$. Furthermore, the associated Wigner series satisfies

$$\partial_t W^S_\hbar[\psi_\hbar](t,x,k) + \frac{i}{\hbar}\sum_{\gamma\in\Gamma} \varepsilon_n(\gamma) e^{i\gamma\cdot k}\big(W^S_\hbar[\psi_\hbar](t,x+\hbar\gamma/2,k) - W^S_\hbar[\psi_\hbar](t,x-\hbar\gamma/2,k)\big) = 0.$$

Proof. We only detail the equation for the Wigner series. Of course, we have

$$\partial_t W^S_\hbar[\psi_\hbar] = -\frac{\mathrm{i}}{\hbar}\sum_{\gamma\in\Gamma} \mathrm{e}^{\mathrm{i}k\cdot\gamma}\Big(H_\hbar\psi_\hbar(x-\hbar\gamma/2)\overline{\psi_\hbar(x+\hbar\gamma/2)} - \psi_\hbar(x-\hbar\gamma/2)\overline{H_\hbar\psi_\hbar(x+\hbar\gamma/2)}\Big).$$

We make use of the following identities:

- since $\psi_\hbar$ remains in the $n-$band space, we get

$$H_\hbar\psi_\hbar(x) = \sum_{\gamma\in\Gamma}\varepsilon_n(\gamma)\psi_\hbar(x+\hbar\gamma),$$

- and we have clearly $\overline{\varepsilon_n(\gamma)} = \varepsilon_n(-\gamma)$.

Hence, we can write

$$\begin{aligned}
\partial_t W^S_\hbar[\psi_\hbar] &= -\frac{\mathrm{i}}{\hbar}\sum_{\gamma,\gamma'\in\Gamma} \mathrm{e}^{\mathrm{i}k\cdot\gamma}\Big(\varepsilon_n(\gamma')\psi_\hbar(x-\hbar\gamma/2+\hbar\gamma')\overline{\psi_\hbar(x+\hbar\gamma/2)} \\
&\qquad -\varepsilon_n(-\gamma')\psi_\hbar(x-\hbar\gamma/2)\overline{\psi_\hbar(x+\hbar\gamma/2+\hbar\gamma')}\Big) \\
&= -\frac{\mathrm{i}}{\hbar}\sum_{\gamma'\in\Gamma}\varepsilon_n(\gamma')\sum_{\gamma\in\Gamma}\mathrm{e}^{\mathrm{i}k\cdot\gamma}\Big(\psi_\hbar(x-\hbar\gamma/2+\hbar\gamma')\overline{\psi_\hbar(x+\hbar\gamma/2)} - \psi_\hbar(x-\hbar\gamma/2)\overline{\psi_\hbar(x+\hbar\gamma/2-\hbar\gamma')}\Big) \\
&= -\frac{\mathrm{i}}{\hbar}\sum_{\gamma'\in\Gamma}\varepsilon_n(\gamma')\mathrm{e}^{\mathrm{i}k\dot{\gamma}'}\sum_{\gamma\in\Gamma}\mathrm{e}^{\mathrm{i}k\cdot\gamma}\Big(\psi_\hbar(x-\hbar(\gamma-\gamma')/2)\overline{\psi_\hbar(x+\hbar(\gamma+\gamma')/2)} \\
&\qquad -\psi_\hbar(x-\hbar(\gamma+\gamma')/2)\overline{\psi_\hbar(x+\hbar(\gamma-\gamma')/2)}\Big)
\end{aligned}$$

by using the change of variables $\gamma\to\gamma+\gamma'$. We recognize the asserted formula. ∎

When $\hbar\to 0$ we guess that

$$\begin{aligned}
\frac{\mathrm{i}}{\hbar}\sum_{\gamma\in\Gamma}\varepsilon_n(\gamma)\mathrm{e}^{\mathrm{i}\gamma\cdot k}\big(f(t,x+\hbar\gamma/2,k)-f(t,x-\hbar\gamma/2,k)\big) &\simeq \mathrm{i}\sum_{\gamma\in\Gamma}\varepsilon_n(\gamma)\mathrm{e}^{\mathrm{i}\gamma\cdot k}\nabla_x f(t,x,k)\cdot\gamma \\
&\simeq \nabla_k\Big(\sum_{\gamma\in\Gamma}\varepsilon_n(\gamma)\mathrm{e}^{\mathrm{i}\gamma\cdot k}\Big)\cdot\nabla_x f(t,x,k) \\
&\simeq \nabla_k E_n(k)\cdot\nabla_x f(t,x,k).
\end{aligned}$$

Theorem 2 *Assume that the initial data for* (2.24) *belongs to the $n-$th band space. Then, up to a subsequence, the Wigner series converges (in a appropriate weak sense...) to $f\geq 0$ solution of the transport equation*

$$\partial_t f + v_n(k)\cdot\nabla_x f = 0,$$

with velocity $v_n(k) = \nabla_k E_n(k)$.

Of course, by using orthogonality properties, we can consider sums of functions belonging to different band spaces. It works as soon as there is no band crossing. The situation is definitely much more intricate when we consider mixed states and possible band crossing: $E_n(k) = E_m(k)$ for some k. The ultimate breakthrough can be found in [5] where the analysis includes external and self-consistent potentials. Finally, it is worth mentioning that the Wigner formalism has became a powerful and general tool for studying oscillations in PDEs, according to [44].

2.3 From Schrödinger to Boltzmann: homogenization aspects

Let us now describe another amazing question involving the semi-classical regime. Classical particles subject to a random potential, under a suitable time-space rescaling, can be described by a Brownian motion. This classical question has been treated in full details in the reference paper [63]. Surprinsingly enough, the similar scaling when adapted to the quantum picture leads to a linear Boltzmann equation. The analysis is due to [40], at the price of a highly technical proof. Recently, Poupaud and Vasseur revisted the problem and proposed an alternative approach. Actually, they changed the problem since they introduce a somehow artificial time-dependence of the potential. Compared to [40], the result might be questionable since the physical model has been changed drastically by introducing randomness with respect to time. Indeed, in the standard problem, the coefficients oscillates randomly with respect to space and the decorrelation properties should be analyzed along the trajectories, which makes the analysis particularly tough, see e. g. [63]. Anyway, the analysis of [82] provides a statement with the same flavor, with a smart and quite simple proof. Besides, the method introduced by [82] is very flexible and it can be readily adapted to various physical situations where the time decorrelation appears naturally, see [17, 64, 54, 55, 21]. We shall now detail this approach.

We start with a toy model, inspired by [77]. Let us consider the simple ODE

$$\frac{\mathrm{d}}{\mathrm{d}t} u_\varepsilon(t) = \mathrm{i}\frac{1}{\varepsilon} a(t/\varepsilon^2) u_\varepsilon(t) \tag{2.25}$$

where $a(t) : \Omega \to \mathbb{R}$ is a random variable with zero mean

$$\mathbb{E}a = 0. \tag{2.26}$$

We also suppose it has finite variance, and fulfils the following stationarity property

$$\mathbb{E}a(t)a(s) = R(t-s) \tag{2.27}$$

As a preliminary remark, note that the equation preserves the modulus:

$$|u_\varepsilon(t)| = \left| \exp\left(\mathrm{i}\int_0^t a(s/\varepsilon^2)\,\mathrm{d}s\right)\right| \, |u(0)| = |u(0)|.$$

We assume that the initial data is deterministic (it does not depend on the alea variable $\omega \in \Omega$). We wish to determine the asymptotic behavior of the expection value $\mathbb{E}u_\varepsilon(t)$ as ε goes to 0. The crucial assumption consists in the following finite time decorrelation hypothesis:

$$\begin{cases} a(t) \text{ and } a(s) \text{ decorrelate when } |t-s| \geq 1: \\ \mathbb{E}\big(a(t)\ a(s)\big) = 0 \quad \text{for } |t-s| \geq 1. \end{cases} \tag{2.28}$$

The analysis is based on the Duhamel formula:

$$u_\varepsilon(t) = u_\varepsilon(s) + \frac{1}{\varepsilon}\int_s^t \mathrm{i}a(\sigma/\varepsilon^2) u_\varepsilon(\sigma)\,\mathrm{d}\sigma. \tag{2.29}$$

It has two immediate consequences:

a) first of all, with $s = 0$ and bearing in mind that $u(0)$ is deterministic, we realize that $u_\varepsilon(t)$ depends only on realization of $a(s/\varepsilon^2)$ for $0 \le s \le t$; hence due to (2.28), $u_\varepsilon(t)$ and $a(t'/\varepsilon^2)$ are independent when $t - t' \ge \varepsilon^2$;

b) second of all, it already provides the estimate

$$u_\varepsilon(t) - u_\varepsilon(s) = \mathcal{O}\Big(\frac{|t-s|}{\varepsilon}\Big). \tag{2.30}$$

Then, let us specialize (2.29) to $s = t - \varepsilon^2$, which involves the decorrelation time scale:

$$u_\varepsilon(t) = u_\varepsilon(t-\varepsilon^2) + \frac{1}{\varepsilon}\int_{t-\varepsilon^2}^t \mathrm{i}a(\sigma/\varepsilon^2)u_\varepsilon(\sigma)\,\mathrm{d}\sigma \tag{2.31}$$

It allows to rewrite

$$\frac{a(t/\varepsilon^2)}{\varepsilon}u_\varepsilon(t) = \frac{a(t/\varepsilon^2)}{\varepsilon}u_\varepsilon(t-\varepsilon^2) + \frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^t \mathrm{i}a(t/\varepsilon^2)a(\sigma/\varepsilon^2)u_\varepsilon(\sigma)\,\mathrm{d}\sigma.$$

In the right hand side, due to (2.26) and the decorrelation, the expectation of the first term vanishes

$$\begin{aligned}
\mathbb{E}\Big(\frac{a(t/\varepsilon^2)}{\varepsilon}u_\varepsilon(t-\varepsilon^2)\Big) &= \mathbb{E}\frac{a(t/\varepsilon^2)}{\varepsilon}\ \mathbb{E}u_\varepsilon(t-\varepsilon^2) && \text{by decorrelation}\\
&= 0 && \text{by (2.26)},
\end{aligned}$$

while the second term is of order

$$(\frac{1}{\varepsilon^2}\times \text{length of the integration interval}) = (\frac{1}{\varepsilon^2}\times\varepsilon^2) = \mathcal{O}(1).$$

In particular, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}u_\varepsilon(t) = \mathrm{i}\mathbb{E}\frac{a(t/\varepsilon^2)}{\varepsilon}u_\varepsilon(t) = -\mathbb{E}\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^t a(t/\varepsilon^2)a(\sigma/\varepsilon^2)u_\varepsilon(\sigma)\,\mathrm{d}\sigma = \mathcal{O}(1)$$

which already tells us, by applying the Arzela-Ascoli theorem. that $\mathbb{E}u_\varepsilon$ belongs to a compact set of $C^0([0,T])$. The next step consists in replacing in the last integral $u_\varepsilon(s)$ by $\mathbb{E}u_\varepsilon(t)$. The error can indeed be controlled and vanishes as ε goes to 0, as a consequence of (2.28) and (2.30). To this end, we write

$$\begin{aligned}
\mathbb{E}\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^t a(t/\varepsilon^2)a(\sigma/\varepsilon^2)u_\varepsilon(\sigma)\,\mathrm{d}\sigma &= \mathbb{E}\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^t a(t/\varepsilon^2)a(\sigma/\varepsilon^2)\,\mathrm{d}\sigma\ \mathbb{E}u_\varepsilon(t)\\
&+\mathbb{E}\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^t a(t/\varepsilon^2)a(\sigma/\varepsilon^2)\big(u_\varepsilon(\sigma) - u_\varepsilon(t-2\varepsilon^2)\big)\,\mathrm{d}\sigma\\
&+\mathbb{E}\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^t a(t/\varepsilon^2)a(\sigma/\varepsilon^2)\big(u_\varepsilon(t-2\varepsilon^2) - \mathbb{E}u_\varepsilon(t-2\varepsilon^2)\big)\,\mathrm{d}\sigma\\
&+\mathbb{E}\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^t a(t/\varepsilon^2)a(\sigma/\varepsilon^2)\big(\mathbb{E}u_\varepsilon(t-2\varepsilon^2) - \mathbb{E}u_\varepsilon(t)\big)\,\mathrm{d}\sigma.
\end{aligned}$$

Since, by (2.28), $u_\varepsilon(t-2\varepsilon^2)$ is independent of $\{a(\sigma/\varepsilon^2),\ t-\varepsilon^2 \le \sigma \le t\}$, the third term can be recast as

$$\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^{t} \mathbb{E}\Big(a(t/\varepsilon^2)a(\sigma/\varepsilon^2)\Big)\mathbb{E}\big(u_\varepsilon(t-2\varepsilon^2) - \mathbb{E}u_\varepsilon(t-2\varepsilon^2)\big)\,\mathrm{d}\sigma = 0$$

and it vanishes. The second term is estimated by using (2.30): for any $t-\varepsilon^2 \le \sigma \le t$, we have $|u_\varepsilon(\sigma) - u_\varepsilon(t-2\varepsilon^2)| \le |\sigma - (t-2\varepsilon^2)|/\varepsilon \le C\varepsilon$ hence we dominate the second term by

$$\frac{1}{\varepsilon^2} \times C\varepsilon \times \varepsilon^2 = C\varepsilon.$$

A similar estimate holds for the last term. We thus have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}u_\varepsilon(t) = \mathrm{i}\mathbb{E}\Big(\frac{a(t/\varepsilon^2)}{\varepsilon}u_\varepsilon(t)\Big) = -\mathbb{E}\left(\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^{t} a(t/\varepsilon^2)a(\sigma/\varepsilon^2)\,\mathrm{d}\sigma\right)\ \mathbb{E}u_\varepsilon(t) + r_\varepsilon, \qquad r_\varepsilon \xrightarrow[\varepsilon\to 0]{} 0. \tag{2.32}$$

We end up with the following statement.

Theorem 3 *The expectation $\mathbb{E}u_\varepsilon$ converges uniformly on $[0,T]$ to u, solution of the ODE*

$$\frac{\mathrm{d}}{\mathrm{d}t}u = -\lambda u$$

where the effective coefficient is

$$\lambda = \frac{1}{2}\int_{-\infty}^{+\infty} R(\tau)\,\mathrm{d}\tau \ge 0.$$

Proof. It only remains to identify the coefficient λ. First, let us check the positivity of λ which is not completely direct. The proof relies on the following observation: for any $F \in L^1(\mathbb{R})$, we have

$$\int_{\mathbb{R}} F(\tau)\,\mathrm{d}\tau = \lim_{R\to\infty}\frac{1}{2R}\int_{-R}^{+R}\int_{-R}^{+R} F(\sigma-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau. \tag{2.33}$$

Therefore, λ becomes

$$\begin{aligned}
\lambda &= \lim_{R\to\infty}\frac{1}{4R}\int_{-R}^{+R}\int_{-R}^{+R} R(\sigma-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau = \lim_{R\to\infty}\frac{1}{4R}\int_{-R}^{+R}\int_{-R}^{+R} \mathbb{E}\big(a(\sigma)a(\tau)\big)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\
&= \lim_{R\to\infty}\mathbb{E}\left(\frac{1}{4R}\int_{-R}^{+R} a(\sigma)\,\mathrm{d}\sigma\right)^2 \ge 0.
\end{aligned}$$

Consequently, note that the modulus of the limit u is not conserved anymore, but it decays as time grows. It indicates that the passage to the limit and the stochasticity effects have induced a loss of irreversibility. We prove (2.33) by writing

$$\int_{\mathbb{R}} F(s)\,\mathrm{d}s = \frac{1}{2R}\int_{-R}^{+R}\int_{\mathbb{R}} F(s)\,\mathrm{d}s\,\mathrm{d}t = \frac{1}{2R}\int_{-R}^{+R}\left(\int_{\mathbb{R}} F(\sigma-t)\,\mathrm{d}\sigma\right)\,\mathrm{d}t.$$

Therefore, it suffices to show that

$$\lim_{R\to\infty} \frac{1}{2R} \left(\int_{-R}^{+R} \int_{|\sigma|\geq R} |F(\sigma - t)| \, \mathrm{d}\sigma \right) \mathrm{d}t = 0.$$

Changing variables again, we reduce the problem to investigating the behavior of

$$\frac{1}{2R} \left(\int_{-R}^{+R} \int_{s+t\geq R} |F(s)| \, \mathrm{d}s \right) \mathrm{d}t,$$

for large R's, and similarly for the quantity obtained by replacing $s+t \geq R$ by $s+t \leq -R$. The Fubini theorem yields

$$\begin{aligned} &\frac{1}{2R} \int_0^\infty |F(s)| \left(\int_{\mathbb{R}} \mathbb{1}_{-R\leq t\leq R} \mathbb{1}_{R-s\leq t} \, \mathrm{d}t \right) \mathrm{d}s \\ &\quad = \frac{1}{2R} \int_0^{2R} |F(s)| \left(\int_{R-s}^{R} \mathrm{d}t \right) \mathrm{d}s + \frac{1}{2R} \int_0^{2R} |F(s)| \left(\int_{-R}^{R} \mathrm{d}t \right) \mathrm{d}s \\ &\quad = \int_0^{2R} |F(s)| \frac{s}{2R} \, \mathrm{d}s + \int_{2R}^{\infty} |F(s)| \, \mathrm{d}s, \end{aligned}$$

and we conclude by applying the Lebesgue theorem.

Then, we go back to the definition of the effective coefficient. In (2.32), we make the following quantity appear

$$\frac{1}{\varepsilon^2} \mathbb{E} \int_{t-\varepsilon^2}^{t} a(t/\varepsilon^2) a(s/\varepsilon^2) \, \mathrm{d}s = \int_{t-\varepsilon^2}^{t} R\Big(\frac{t-s}{\varepsilon^2}\Big) \, \mathrm{d}s = \int_0^1 R(\tau) \, \mathrm{d}\tau.$$

as a consequence of (2.27). Hence this quantity does not depend on ε anymore and it can be rewritten as

$$\mathbb{E} \int_0^1 a(\tau) a(0) \, \mathrm{d}\tau = \mathbb{E} \int_0^1 a(0) a(\tau) \, \mathrm{d}\tau = \mathbb{E} \int_0^1 R(-\tau) \, \mathrm{d}\tau = \frac{1}{2} \int_{-1}^{+1} R(\tau) \, \mathrm{d}\tau = \frac{1}{2} \int_{-\infty}^{+\infty} R(\tau) \, \mathrm{d}\tau$$

due to the support property of the function R in (2.28). ∎

We now apply this strategy for investigating the limit $\hbar \to 0$ in the following set of Schrödinger equations

$$\mathrm{i}\hbar \partial_t \psi_{m,\hbar} = -\frac{\hbar^2}{2} \Delta_x \psi_{m,\hbar} + \sqrt{\hbar} V(t/\hbar, x/\hbar) \psi_{m,\hbar}$$

We suppose that the set initial data $\{\psi_{m,\hbar}(t=0),\ m \in \mathbb{N}\}$ is a deterministic orthonormal system of $L^2(\mathbb{R}^N)$. Consequently for any $t \geq 0$, the system $\{\psi_{m,\hbar}(t),\ m \in \mathbb{N}\}$ is also orthonormal. We associate to the index n an occupation probability $\lambda_{m,\hbar}$. The particle is then described by the mixed state

$$n(t,x) = \sum_{m\in\mathbb{N}} \lambda_{m,\hbar} |\psi_{m,\hbar}(t,x)|^2.$$

We suppose that

$$\lambda_{m,\hbar} \geq 0, \qquad \sum_{m\in\mathbb{N}} \lambda_{n,\hbar} = 1, \qquad \sum_{m\in\mathbb{N}} |\lambda_{n,\hbar}|^2 \leq M\ h^N.$$

(It is realized with a finite number $N_\hbar$ of states, with $N_\hbar = (1/\hbar^N)$ and $\lambda_{m,\hbar} = 1/N_\hbar$ if $m \leq N_\hbar$, 0 otherwise.) Hence, the Wigner transfrom

$$W_\hbar(t, x\xi) = \sum_{n\in\mathbb{N}} \lambda_{n,\hbar} \int_{\mathbb{R}^N} \psi_{m,\hbar}(t, x - \hbar y/2)\ \overline{\psi_{m,\hbar}(t, x + \hbar y/2)}\ \mathrm{e}^{\mathrm{i}y\cdot\xi} \frac{\mathrm{d}y}{(2\pi)^N}$$

is bounded in $L^\infty(0,\infty; L^2(\mathbb{R}^N \times \mathbb{R}^N))$. It satisfies

$$\partial_t W_\hbar + \xi \cdot \nabla_x W_\hbar = \Theta_\hbar(t)[W_\hbar] \tag{2.34}$$

with

$$\Theta_\hbar(t)[f](x, \xi) = \frac{\mathrm{i}}{\sqrt{\hbar}} \int_{\mathbb{R}^N} \big(V(t/\hbar, x/\hbar + y/2) - V(t/\hbar, x/\hbar - y/2)\big) \widehat{f}(x, y)\ \mathrm{e}^{\mathrm{i}y\cdot\xi} ds \frac{\mathrm{d}y}{(2\pi)^N}.$$

The requirements on the potential are the following

$$\left\{\begin{array}{l} \bullet\ V \text{ is a smooth random variable,} \\ \bullet \text{ For any } t, x,\ V \text{ is centered: } \mathbb{E}V(t, x) = 0, \\ \bullet\ \mathbb{E}V(t, x)V(s, y) = R(t - s, x - y) \text{ with furthermore } R(\tau, \cdot) = 0 \text{ when } |\tau| \geq 1. \end{array}\right.$$

Theorem 4 *Up to a subsequence, $W_\hbar$ converges, as $\hbar \to 0$ to $f \geq 0$ in $C^0([0,T]; L^2(\mathbb{R}^N \times \mathbb{R}^N) -$ weak), which is solution of the following linear Boltzmann equation*

$$\partial_t f + \xi \cdot \nabla_x f = \int_{\mathbb{R}^N} b(\xi, \xi_\star) f(\xi_\star)\, \mathrm{d}\xi_\star - \Lambda(\xi) f(\xi)$$

with

$$b(\xi, \xi_\star) = \frac{1}{(2\pi)^N} \widehat{R}\Big(\frac{\xi_\star^2 - \xi^2}{2}, \xi - \xi_\star\Big), \qquad \Lambda(\xi) = \int_{\mathbb{R}^N} b(\xi, \xi_\star)\, \mathrm{d}\xi_\star,$$

where

$$\widehat{R}(\sigma, \xi) = \int_{\mathbb{R}\times\mathbb{R}^N} R(t, y)\ \mathrm{e}^{-\mathrm{i}(\sigma t + y\cdot\xi)}\, \mathrm{d}y\, \mathrm{d}t.$$

Proof. We sketch the adaptations of the steps described for the toy model, skipping the tedious functional details. We see (2.34) as a perturbation of the free transport equation by the source term $\Theta_\hbar(W_\hbar)$. Therefore, integrating along the characteristics $x + t\xi$, we obtain the Duhamel formula

$$W_\hbar(t, x, \xi) = W_\hbar(s, x - (t - s)\xi, \xi) + \int_s^t \Theta_\hbar(\sigma)[W_\hbar(\sigma)](x - (t - \sigma)\xi, \xi)\, \mathrm{d}\sigma. \tag{2.35}$$

It provides the basic estimate and the decorrelation property, analog of a) and b) for the toy model. It is convenient to introduce the operator

$$S_t f(x, \xi) = f(x - t\xi, \xi)$$

the adjoint being

$$S_t^\star = S_{-t}.$$

Then, (2.35) recasts with the shorthand notation as

$$W_\hbar(t,x,\xi) = S_{t-s}W_\hbar(s)(x,\xi) + \int_s^t S_{t-\sigma}\Theta_\hbar(\sigma)[W_\hbar(\sigma)](x,\xi)\,\mathrm{d}\sigma. \tag{2.36}$$

(Roughly speaking passing form the toy model to (2.34), $\partial_t + \xi\cdot\nabla_x$ replaces the derivation $\frac{\mathrm{d}}{\mathrm{d}t}$, integration along the characteristics involving the operator S_t replaces the standard integration on time intervals while $\Theta_\hbar(t)$ replaces multiplication by $a(t)$.) Reasoning as above, we reduce to investigating the limit $\hbar\to 0$ in

$$\begin{aligned}
\partial_t\mathbb{E}W_\hbar + \xi\cdot\nabla_x\mathbb{E}W_\hbar &= \mathbb{E}\int_{t-\hbar}^t \Theta_\hbar(t)S_{t-\sigma}\Theta_\hbar(\sigma)[W_\hbar(\sigma)](x,\xi)\,\mathrm{d}\sigma\\
&= \mathbb{E}\int_{t-\hbar}^t \Theta_\hbar(t)S_{t-\sigma}\Theta_\hbar(\sigma)S_{\sigma-t}[\mathbb{E}W_\hbar(t)](x,\xi)\,\mathrm{d}\sigma + r_\hbar
\end{aligned}$$

where the remainder $r_\hbar$ can be shown to tend to 0 as $\hbar$ goes to 0, in some appropriate functional space. It is easier to derive the limit operator by reasoning by duality. Namely, multiplying the leading term in the right hand side by a trial function $\varphi\in C_c^\infty(\mathbb{R}^N\times\mathbb{R}^N)$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N\times\mathbb{R}^N}\mathbb{E}W_\hbar(t,x,\xi)\;\mathbb{E}\left(\int_{t-\hbar}^t S_{t-\sigma}\Theta_\hbar(\sigma)S_{\sigma-t}\Theta_\hbar(t)[\varphi](x,\xi)\,\mathrm{d}\sigma\right)\mathrm{d}\xi\,\mathrm{d}x&\\
:= \int_{\mathbb{R}^N\times\mathbb{R}^N}\mathbb{E}W_\hbar\;L_\hbar[\varphi](t,x,\xi)\,\mathrm{d}\xi\,\mathrm{d}x.&
\end{aligned}$$

More precisely, the operator $L_\hbar$ reads

$$\begin{aligned}
L_\hbar[\varphi](x,\xi) = -\mathbb{E}\frac{1}{\hbar}\int\int_{t-\hbar}^t \mathrm{e}^{\mathrm{i}y\cdot\xi}\Big[V\Big(\frac{\sigma}{\hbar},\frac{x-(t-\sigma)\xi+\hbar y/2}{\hbar}\Big) - V\Big(\frac{\sigma}{\hbar},\frac{x-(t-\sigma)\xi-\hbar y/2}{\hbar}\Big)\Big]\\
\times\mathrm{e}^{-\mathrm{i}y\cdot\xi_\star}\mathrm{e}^{\mathrm{i}z\cdot\xi_\star}\Big[V\Big(\frac{t}{\hbar},\frac{x-(t-\sigma)(\xi-\xi_\star)+\hbar z/2}{\hbar}\Big) - V\Big(\frac{t}{\hbar},\frac{x-(t-\sigma)(\xi-\xi_\star)-\hbar z/2}{\hbar}\Big)\\
\times\mathscr{F}_{\zeta\to z}\varphi(x-(t-\sigma)(\xi-\xi_\star),z)\frac{\mathrm{d}z}{(2\pi)^N}\,\mathrm{d}\xi_\star\frac{\mathrm{d}y}{(2\pi)^N}\,\mathrm{d}\sigma.
\end{aligned}$$

(Note that we multiply $1/\hbar$ by an integral over a time interval of size $\hbar$.) By using the self-correlation function, we obtain

$$\begin{aligned}
L_\hbar[\varphi](x,\xi) &= -\int\int_{t-\hbar}^t \mathrm{e}^{\mathrm{i}(y\cdot\xi-y\cdot\xi_\star+z\cdot\xi_\star)}\mathscr{F}_{\zeta\to z}\varphi(x-(t-\sigma)(\xi-\xi_\star),z)\\
&\times\Big[R\Big(\frac{\sigma-t}{\hbar},\frac{\sigma-t}{\hbar}\xi_\star+\frac{y-z}{2}\Big)+R\Big(\frac{\sigma-t}{\hbar},\frac{\sigma-t}{\hbar}\xi_\star-\frac{y-z}{2}\Big)\\
&-R\Big(\frac{\sigma-t}{\hbar},\frac{\sigma-t}{\hbar}\xi_\star-\frac{y+z}{2}\Big)-R\Big(\frac{\sigma-t}{\hbar},\frac{\sigma-t}{\hbar}\xi_\star+\frac{y+z}{2}\Big)\Big]\frac{\mathrm{d}z}{(2\pi)^N}\,\mathrm{d}\xi_\star\frac{\mathrm{d}y}{(2\pi)^N}\frac{\mathrm{d}\sigma}{\hbar}.
\end{aligned}$$

and with the change of variables $\tau = (\sigma - t)/\hbar$, $y_\star = (\sigma - t)\xi_\star/\hbar \pm (y \pm z)/2$, we arrive at

$$\begin{aligned} L_\hbar[\varphi](x,\xi) &= \int\int_0^1 \mathrm{e}^{\mathrm{i}z\cdot\xi_\star} \mathscr{F}_{\zeta\to z}\varphi(x+\hbar\tau(\xi-\xi_\star), z)\ R(\tau, y_\star) \\ &\times\Big(- \mathrm{e}^{-\mathrm{i}2\tau\xi_\star\cdot(\xi-\xi_\star)}\mathrm{e}^{\mathrm{i}(2y_\star+z)\cdot(\xi-\xi_\star)} - \mathrm{e}^{+\mathrm{i}2\tau\xi_\star\cdot(\xi-\xi_\star)}\mathrm{e}^{\mathrm{i}(-2y_\star+z)\cdot(\xi-\xi_\star)} \\ &+\mathrm{e}^{+\mathrm{i}2\tau\xi_\star\cdot(\xi-\xi_\star)}\mathrm{e}^{\mathrm{i}(-2y_\star-z)\cdot(\xi-\xi_\star)} + \mathrm{e}^{-\mathrm{i}2\tau\xi_\star\cdot(\xi-\xi_\star)}\mathrm{e}^{\mathrm{i}(2y_\star-z)\cdot(\xi-\xi_\star)}\Big)\frac{\mathrm{d}z}{(2\pi)^N}\,\mathrm{d}\xi_\star\frac{2^N\,\mathrm{d}y_\star}{(2\pi)^N}\,\mathrm{d}\tau. \end{aligned}$$

From now on, we approach $\varphi(x+\hbar\tau\xi_\star,\cdot)$ by $\varphi(x,\cdot)$ (which induces a new error term of order $\mathcal{O}(\hbar)$, owing to the regularity of the test function) and we get

$$\begin{aligned} L_\hbar[\varphi](x,\xi) &\simeq -\int \mathrm{e}^{\mathrm{i}z\cdot\xi}\mathscr{F}_{\zeta\to z}\varphi(x,z)\frac{\mathrm{d}z}{(2\pi)^N} \\ &\times\int_0^1\int R(\tau,y_\star)\Big(\mathrm{e}^{-\mathrm{i}2\tau\xi_\star\cdot(\xi-\xi_\star)}\mathrm{e}^{\mathrm{i}2y_\star\cdot(\xi-\xi_\star)} + \mathrm{e}^{+\mathrm{i}2\tau\xi_\star\cdot(\xi-\xi_\star)}\mathrm{e}^{-\mathrm{i}2y_\star\cdot(\xi-\xi_\star)}\Big)\,\mathrm{d}\xi_\star\frac{2^N\,\mathrm{d}y_\star}{(2\pi)^N}\,\mathrm{d}\tau \\ &+\int\int_0^1 \mathrm{e}^{\mathrm{i}z\cdot(2\xi_\star-\xi)}\mathscr{F}_{\zeta\to z}\varphi(x,z)\ R(\tau,y_\star) \\ &\times\Big(\mathrm{e}^{+\mathrm{i}2\tau\xi_\star\cdot(\xi-\xi_\star)}\mathrm{e}^{-\mathrm{i}2y_\star\cdot(\xi-\xi_\star)} + \mathrm{e}^{-\mathrm{i}2\tau\xi_\star\cdot(\xi-\xi_\star)}\mathrm{e}^{\mathrm{i}2y_\star\cdot(\xi-\xi_\star)}\Big)\frac{\mathrm{d}z}{(2\pi)^N}\,\mathrm{d}\xi_\star\frac{2^N\,\mathrm{d}y_\star}{(2\pi)^N}\,\mathrm{d}\tau \\ &\simeq -\varphi(x,\xi)\int\int_0^1 R(\tau,y_\star)\Big(\mathrm{e}^{-\mathrm{i}\tau(\xi^2-\zeta^2)/2}\mathrm{e}^{\mathrm{i}y_\star\cdot(\xi-\zeta)} + \mathrm{e}^{+\mathrm{i}\tau(\xi^2-\zeta^2)/2}\mathrm{e}^{-\mathrm{i}y_\star\cdot(\xi-\zeta)}\Big)\,\mathrm{d}\zeta\frac{\mathrm{d}y_\star}{(2\pi)^N}\,\mathrm{d}\tau \\ &+\int\int_0^1 \varphi(x,\zeta)\ R(\tau,y_\star)\Big(\mathrm{e}^{-\mathrm{i}\tau(\xi^2-\zeta^2)/2}\mathrm{e}^{\mathrm{i}y_\star\cdot(\xi-\zeta)} + \mathrm{e}^{+\mathrm{i}\tau(\xi^2-\zeta^2)/2}\mathrm{e}^{-\mathrm{i}y_\star\cdot(\xi-\zeta)}\Big)\,\mathrm{d}\zeta\frac{\mathrm{d}y_\star}{(2\pi)^N}\,\mathrm{d}\tau \end{aligned}$$

by using the change of variable $\zeta = 2\xi_\star - \xi$. Now we make use of symmetry properties satisfied by the function R, precisely, we have

$$R(t,y) = \mathbb{E}V(t,y)V(0,0) = \mathbb{E}V(0,0)V(t,y) = R(-t,-y)$$

which implies

$$\int_{\mathbb{R}^N} \mathrm{e}^{-\mathrm{i}y\cdot\xi}R(\tau,y)\,\mathrm{d}y := Q(\tau,\xi) = \int_{\mathbb{R}^N} \mathrm{e}^{-\mathrm{i}y\cdot\xi}R(-\tau,-y)\,\mathrm{d}y = \int_{\mathbb{R}^N} \mathrm{e}^{\mathrm{i}y\cdot\xi}R(-\tau,y)\,\mathrm{d}y = Q(-\tau,-\xi).$$

Accordingly, we observe that

$$\begin{aligned} L_\hbar[\varphi](x,\xi) &\simeq -\varphi(x,\xi)\int\int_0^1 \Big(\mathrm{e}^{-\mathrm{i}\tau(\xi^2-\zeta^2)/2}Q(\tau,\zeta-\xi) + \mathrm{e}^{+\mathrm{i}\tau(\xi^2-\zeta^2)/2}Q(\tau,\xi-\zeta)\Big)\frac{\mathrm{d}\zeta}{(2\pi)^N}\,\mathrm{d}\tau \\ &+\int\int_0^1 \varphi(x,\zeta)\ \Big(\mathrm{e}^{-\mathrm{i}\tau(\xi^2-\zeta^2)/2}Q(\tau,\zeta-\xi)) + \mathrm{e}^{+\mathrm{i}\tau(\xi^2-\zeta^2)/2}Q(\tau,\xi-\zeta)\frac{\mathrm{d}\zeta}{(2\pi)^N}\,\mathrm{d}\tau \\ &\simeq -\varphi(x,\xi)\int\int_{-1}^0 \Big(\mathrm{e}^{\mathrm{i}\tau(\xi^2-\zeta^2)/2}Q(-\tau,\zeta-\xi) + \mathrm{e}^{-\mathrm{i}\tau(\xi^2-\zeta^2)/2}Q(-\tau,\xi-\zeta)\Big)\frac{\mathrm{d}\zeta}{(2\pi)^N}\,\mathrm{d}\tau \\ &+\int\int_{-1}^0 \varphi(x,\zeta)\ \Big(\mathrm{e}^{\mathrm{i}\tau(\xi^2-\zeta^2)/2}Q(-\tau,\zeta-\xi)) + \mathrm{e}^{-\mathrm{i}\tau(\xi^2-\zeta^2)/2}Q(-\tau,\xi-\zeta)\frac{\mathrm{d}\zeta}{(2\pi)^N}\,\mathrm{d}\tau \\ &\simeq -\varphi(x,\xi)\frac{1}{2}\int\int_{-\infty}^{+\infty} \Big(\mathrm{e}^{\mathrm{i}\tau(\xi^2-\zeta^2)/2}Q(\tau,\xi-\zeta) + \mathrm{e}^{-\mathrm{i}\tau(\xi^2-\zeta^2)/2}Q(\tau,\zeta-\xi)\Big)\frac{\mathrm{d}\zeta}{(2\pi)^N}\,\mathrm{d}\tau \\ &+\frac{1}{2}\int\int_{-\infty}^{+\infty} \varphi(x,\zeta)\ \Big(\mathrm{e}^{\mathrm{i}\tau(\xi^2-\zeta^2)/2}Q(\tau,\xi-\zeta) + \mathrm{e}^{-\mathrm{i}\tau(\xi^2-\zeta^2)/2}Q(\tau,\zeta-\xi)\frac{\mathrm{d}\zeta}{(2\pi)^N}\,\mathrm{d}\tau \end{aligned}$$

by using the fact that $\tau \to R(\tau, y)$ is supported in $(-1, +1)$. Using again that $(\tau, \xi) \mapsto Q(\tau, \xi)$ is even, we arrive at

$$\begin{aligned} L_\hbar[\varphi](x, \xi) \simeq & -\varphi(x, \xi) \int_{-\infty}^{\infty} \int_{\mathbb{R}^N} Q(\tau, \xi - \zeta)\ \mathrm{e}^{\mathrm{i}\tau(\xi^2 - \zeta^2)/2} \frac{\mathrm{d}\zeta}{(2\pi)^N}\, \mathrm{d}\tau \\ & + \int_{\mathbb{R}^N} \varphi(x, \zeta) \int_{-\infty}^{\infty} \int_{\mathbb{R}^N} Q(\tau, \xi - \zeta)\ \mathrm{e}^{\mathrm{i}\tau(\xi^2 - \zeta^2)/2} \frac{\mathrm{d}\zeta}{(2\pi)^N}\, \mathrm{d}\tau\, \mathrm{d}\zeta, \end{aligned}$$

which is exactly the linear Boltzmann operator with collision kernel given by

$$b(\xi, \zeta) = \frac{1}{(2\pi)^N}\ \mathscr{F}_{(\tau,y)\to(\omega,\xi)} R\Big(\frac{\zeta^2 - \xi^2}{2}, \xi - \zeta\Big).$$

The kernel is indeed symmetric since

$$\begin{aligned} b(\xi, \zeta) &= \frac{1}{(2\pi)^N} \int_{-\infty}^{+\infty} Q(\tau, \xi - \zeta) \mathrm{e}^{\mathrm{i}\tau(\xi^2 - \zeta^2)/2}\, \mathrm{d}\tau \\ &= \frac{1}{(2\pi)^N} \int_{-\infty}^{+\infty} Q(-\tau, \xi - \zeta) \mathrm{e}^{-\mathrm{i}\tau(\xi^2 - \zeta^2)/2}\, \mathrm{d}\tau && \text{by changing } \tau \mapsto -\tau \\ &= \frac{1}{(2\pi)^N} \int_{-\infty}^{+\infty} Q(\tau, \zeta - \xi) \mathrm{e}^{\mathrm{i}\tau(\zeta^2 - \xi^2)/2}\, \mathrm{d}\tau && \text{since } (\tau, \xi) \mapsto Q(\tau, \xi) \text{ is even} \\ &= b(\zeta, \xi). \end{aligned}$$

It only remains to check that $b \geq 0$.

The proof follows those of Proposition 3. Indeed, up to some irrelevant constant showing $b \geq 0$ reduces to show

$$\mathscr{F}^{-1}_{(\tau,y)\to(\omega,\xi)} R(\omega, -\xi) \geq 0.$$

To this end it suffices to justify that

$$\int_{\mathbb{R}\times\mathbb{R}^N} \mathscr{F}^{-1}_{(\tau,y)\to(\omega,\xi)} R(\omega, \xi)\ \big|\mathscr{F}_{(\tau,y)\to(\omega,\xi)} \phi(\omega, \xi)\big|^2\, \mathrm{d}\omega\, \mathrm{d}\xi \geq 0$$

holds for any $\phi \in \mathscr{S}(\mathbb{R} \times \mathbb{R}^N)$. However, the latter can be recast as

$$\begin{aligned} & \int_{\mathbb{R}\times\mathbb{R}^N} \int_{\mathbb{R}\times\mathbb{R}^N} R(\tau, y)\ \phi(\sigma, z) \overline{\phi(\sigma - \tau, z - y)}\, \mathrm{d}z\, \mathrm{d}\sigma\, \mathrm{d}y\, \mathrm{d}\omega \\ & = \int_{\mathbb{R}\times\mathbb{R}^N} \int_{\mathbb{R}\times\mathbb{R}^N} R(\sigma - \tau - \sigma, z - y - z)\ \phi(\sigma, z) \overline{\phi(\sigma - \tau, z - y)}\, \mathrm{d}z\, \mathrm{d}\sigma\, \mathrm{d}y\, \mathrm{d}\omega \\ & = \mathbb{E} \int_{\mathbb{R}\times\mathbb{R}^N} \int_{\mathbb{R}\times\mathbb{R}^N} V(\sigma - \tau, z)\ V(\sigma, z)\ \phi(\sigma, z) \overline{\phi(\sigma - \tau, z - y)}\, \mathrm{d}z\, \mathrm{d}\sigma\, \mathrm{d}y\, \mathrm{d}\omega \\ & = \mathbb{E} \left| \int_{\mathbb{R}\times\mathbb{R}^N} \int_{\mathbb{R}\times\mathbb{R}^N} V(\sigma, z)\ \phi(\sigma, z)\, \mathrm{d}z\, \mathrm{d}\sigma \right|^2 \geq 0. \end{aligned}$$

∎

3 Analysis of the Vlasov-Poisson system

This Section is devoted to the analysis of the PDE

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \Phi \cdot \nabla_v f = 0 \tag{3.37}$$

where the potential is defined by

$$-\Delta_x \Phi = n(t,x) = \int_{\mathbb{R}^N} f(t,x,v)\,\mathrm{d}v.$$

As said above, see (1.3), the latter equation should actually be understood as the convolution formula

$$\Phi(t,x) = \int_{\mathbb{R}^N} E_N(x-y) n(t,y)\,\mathrm{d}y = \int_{\mathbb{R}^N} E_N(x-y) f(t,y,v)\,\mathrm{d}y\,\mathrm{d}v \tag{3.38}$$

where E_N stands for the elementary solution of $-\Delta$ (and one has to care about formal integration by parts that would use directly the Poisson equation). Of course the difficulty comes from the quadratic non linearity in $\nabla_x \Phi \cdot \nabla_v f$. In fact the difficulty is two–fold: first, we should discuss the relevant functional framework and determine how this acceleration term makes sense (this is not so clear since the kernel E_N has some singularity at the origin); and second we need to establish some compactness property which will be necessary for proving the existence of solutions through a suitable approximation procedure.

3.1 A priori estimates and sketch of the existence proof

Of course, the keypoint relies on the derivation of a priori estimates satisfied by the solution. The first obvious estimate is nothing but the charge conservation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f\,\mathrm{d}v\,\mathrm{d}x = 0,$$

which offers the L^1 estimate. Next, neglecting any difficulty related to a possible lack of regularity of the field (which can be fixed by a regularization step), we can write

$$f(t,x,v) = f^{\mathrm{Init}}(X(0;t,x,v), V(0;t,x,v)),$$

where the characteristics $(X(s;t,x,v), V(s;t,x,v))$ are defined by the ODE system

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s} X(s) = V(s), & \dfrac{\mathrm{d}}{\mathrm{d}s} V(s) = -\nabla_x \Phi(s, X(s)), \\ X(t) = x & V(t) = v. \end{cases}$$

Hence, we also obtain

$$\|f(t)\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^N)} \le \|f^{\mathrm{Init}}\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^N)}.$$

The last "easy" estimate comes from energy conservation. Indeed, define the kinetic energy as

$$E_c(t) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{v^2}{2}\, f\,\mathrm{d}v\,\mathrm{d}x$$

and the potential energy as

$$E_p(t) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Phi f \,\mathrm{d}v\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^N} \Phi n \,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} E_N(x-y)n(t,y)n(t,x)\,\mathrm{d}y\,\mathrm{d}x.$$

It is worth pointing that E_p is non negative when $N \geq 3$. Then, by using integration by parts, we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}(E_c(t)+E_p(t)) = -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} v\cdot\nabla_x\Phi f\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} E_N(x-y)n(t,y)\ \partial_t n(t,x)\,\mathrm{d}y\,\mathrm{d}x$$

where in the last term we have used the symmetry of the kernel $E_N(z) = E_N(-z)$. Precisely, the last integral reads

$$\begin{aligned} -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} & E_N(x-y)n(t,y)\ v\cdot\nabla_x f(t,x,v)\,\mathrm{d}v\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} vf\cdot\nabla_x\Big(\int_{\mathbb{R}^N} E_N(x-y)n(t,y)\,\mathrm{d}y\Big)\,\mathrm{d}v\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} vf\cdot\nabla_x\Phi\,\mathrm{d}v\,\mathrm{d}x. \end{aligned}$$

We conclude that the total energy is conserved

$$\frac{\mathrm{d}}{\mathrm{d}t}(E_c(t)+E_p(t)) = 0.$$

The discussion of further a priori estimates will make use of the following claim, referred to as the Hardy-Litelwood-Sobolev inequality [86].

Lemma 3 *Let $1<\alpha<\infty$ and consider the operator*

$$T : f \mapsto \int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N/\alpha}}\,\mathrm{d}y.$$

Then, T is a bounded operator from $L^p(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$ for

$$1 < p < \alpha' = \frac{\alpha}{\alpha-1}, \qquad \frac{1}{q} = \frac{1}{p} + \frac{1}{\alpha} - 1.$$

Remark 4 *It would be tempting to perform some integration by parts and to write potential energy as the integral of $|\nabla_x\Phi|^2$. However, such a computation is misleading. This is the case in dimension 2. Indeed, let $n \in L^1(\mathbb{R}^2)$, with $n \geq 0$ and let Φ be a solution of $-\Delta\Phi = n$ in $\mathbb{R}^2$. Then $\nabla\Phi$ belongs to $L^2(\mathbb{R}^2)$ iff $n = 0$. This is clear by Fourier transform since we know that $\xi^2\widehat{\Phi}(\xi) = \widehat{n}(\xi) \in L^\infty(\mathbb{R}^2)$. Then, we get $|\widehat{\nabla_x\Phi}(\xi)| = |\xi|\ |\widehat{\Phi}(\xi)| = \frac{|\xi|}{\xi^2}\ |\widehat{n}(\xi)|$, which is non square integrable in the neighborhood of the origin where this quantity behaves like $|\widehat{n}(0)|/|\xi|$, with $\widehat{n}(0) = \int \rho(x)\,\mathrm{d}x > 0$ by assumption.*

Coming back to Lemma 3

$$\nabla_x\Phi(x) = \int_{\mathbb{R}^N} \frac{x-y}{|x-y|}\ \frac{n(y)}{|x-y|^{N-1}}\,\mathrm{d}y$$

corresponds to set $\alpha = N/(N-1) \in (1,\infty)$. Hence obtaining $\nabla_x\Phi \in L^2$ is not affordable in dimension 1 and 2 (we find $p = 2/3$ and $p = 1$ respectively which are not amissible); in dimension 3 it requires $p = 6/5$.

Let us make a short break by mentioning the twin situation where the sign in the Poisson equation is changes, namely, the potential is defined by

$$+\Delta_x\Phi = n$$

and thus instead of (3.38) we get

$$\Phi(t,x) = -\int_{\mathbb{R}^N} E_N(x-y)n(t,y)\,\mathrm{d}y = -\int_{\mathbb{R}^N} E_N(x-y)f(t,y,v)\,\mathrm{d}y\,\mathrm{d}v. \tag{3.39}$$

The model is physically relevant and it arises in astrophysics: the self-consistent force field now corresponds to the gravitational attractive force exerted by the particles instead of the repulsive electric forces between charges of similar sign. (But we are not aware of a similar relevant quantum model.) From the view point of mathematical analysis, the attractive case is more difficult than the repulsive one. For the time being, let us only use the attractive case to illustrate by a simple computation why the singularity of the kernel E_N is not completely harmless. We can indeed show that solutions can develop singularities in finite time.

Theorem 5 *Consider $N = 4$ and the attractive potential (3.39). Then there exists initial data $f^{\mathrm{Init}} \in C^2_c(\mathbb{R}^4 \times \mathbb{R}^4)$ such that the solution does not remain of class C^2 for any positive time.*

Proof. We remind that $E_4(x) = C/|x|^2$. Let us compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^4\times\mathbb{R}^4} \frac{x^2}{2} f\,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{R}^4\times\mathbb{R}^4} v\cdot x f\,\mathrm{d}v\,\mathrm{d}x$$

and next

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{\mathbb{R}^4\times\mathbb{R}^4} \frac{x^2}{2} f\,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{R}^4\times\mathbb{R}^4} v^2 f\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^4\times\mathbb{R}^4} \nabla_x\Phi\cdot x f\,\mathrm{d}v\,\mathrm{d}x.$$

However, the last integral recasts as

$$\begin{aligned}
&\int_{\mathbb{R}^4\times\mathbb{R}^4} \nabla_x\Phi\cdot x f\,\mathrm{d}v\,\mathrm{d}x = -2C\int_{\mathbb{R}^4\times\mathbb{R}^4} \frac{x-y}{|x-y|^4} n(t,y)\cdot x n(t,x)\,\mathrm{d}y\,\mathrm{d}x\\
&= -C\int_{\mathbb{R}^4\times\mathbb{R}^4} \frac{1}{|x-y|^3} n(t,y)n(t,x)\ \frac{x-y}{|x-y|}\cdot(x-y)\,\mathrm{d}y\,\mathrm{d}x\\
&= -C\int_{\mathbb{R}^4\times\mathbb{R}^4} \frac{1}{|x-y|^2} n(t,y)n(t,x)\ \frac{x-y}{|x-y|}\cdot(x-y)\,\mathrm{d}y\,\mathrm{d}x = -\int_{\mathbb{R}^4} n\Phi\,\mathrm{d}x.
\end{aligned}$$

Hence, we arrive at

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int_{\mathbb{R}^4\times\mathbb{R}^4} \frac{x^2}{2} f\,\mathrm{d}v\,\mathrm{d}x = 2(E_c + E_p)(t) = 2\times(\text{Initial Total Energy } \mathscr{E}_0).$$

In the attractive case, the potential energy contributes negatively and $\mathscr{E}_0$ can be negative. In such a situation we are led to a contradiction since the previous manipulations lead to

$$\int_{\mathbb{R}^4\times\mathbb{R}^4} \frac{x^2}{2} f \,\mathrm{d}v\,\mathrm{d}x = \mathscr{E}_0 t^2 + I_1 t + I_0$$

where the right hand side vanishes and becomes negative for a finite time $T_\star$, while, of course, the second moment of the density f should remain non negative. The contradiction comes from the fact that the computations above, and in particular the formal integrations by parts we made, are not licit when f becomes singular.

For obtaining a smooth data with negative energy, we proceed as follows: pick a non negative $f_0 \in C_c^\infty(\mathbb{R}^4\times\mathbb{R}^4)$ and compute

$$\mathscr{E}_c(f_0) = \int_{\mathbb{R}^4\times\mathbb{R}^4} \frac{v^2}{2} f_0 \,\mathrm{d}v\,\mathrm{d}x, \qquad \mathscr{E}_p(f_0) = \int_{\mathbb{R}^4\times\mathbb{R}^4} \frac{C}{|x-y|^2} f_0(x,v) f_0(y,w)\,\mathrm{d}w\,\mathrm{d}y\,\mathrm{d}v\,\mathrm{d}x$$

which are both positive. In the attractive case, the total energy reads

$$\mathscr{E}_0 = \mathscr{E}_c(f_0) - \mathscr{E}_p(f_0).$$

Set $f_0^\lambda = \lambda f_0$ for some $\lambda > 0$ and remark that the associated total energy is

$$\mathscr{E}_c(\lambda f_0) - \mathscr{E}_p(\lambda f_0) = \lambda \mathscr{E}_c(f_0) - \lambda^2 \mathscr{E}_p(f_0)$$

which becomes negative for large enough λ's. ∎

The one-dimension case is interesting since it allows to perform a proof that uses only elementary tools (and works for both the attractive and the repulsive potential). In this case we can indeed take advantage of the simple expression of the force

$$F(t,x) = \partial_x \Phi(t,x) = -\frac{1}{2}\Big(\int_{-\infty}^{x} n(t,y)\,\mathrm{d}y - \int_x^{+\infty} n(t,y)\,\mathrm{d}y\Big).$$

In particular, the L^1 bound on n immediatly imply that F is continuous and bounded. Furthermore when n belongs to L^∞, F is uniformly lipschitizian. Then the proof uses the a priori estimates, sharp estimates on the characteristics and the preservation of support properties (which provides the L^∞ estimate on n on finite time interval...).

Theorem 6 *Let $f^{\mathrm{Init}} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $f^{\mathrm{Init}} \geq 0$ be an integrable and bounded function, compactly supported. Then there exists a unique weak solution $f \in C^0([0,\infty); L^p(\mathbb{R}^2))$, for any $1 \leq p < \infty$ solution o the Vlasov-Poisson ystem with initial data f^{Init}. For any $0 \leq t \leq T < \infty$, the support of $f(t)$ is compact in $\mathbb{R}^2$, f belongs to $L^\infty((0,\infty)\times\mathbb{R}^2)$ and the total energy is conserved. Furthermore, if f^{Init} belongs to C^k, the solution is C^k too.*

Let usrather focus on the 3-dimensional case for the repulsive potential. The first step consists in replacing the singular kernel $E_3(x) = \frac{1}{4\pi\ |x|}$ by a smooth approximation in the definition of the potential. Introduce a sequence of mollifiers

$$\zeta_\varepsilon \in C_c^\infty(\mathbb{R}^3), \qquad 0 \le \zeta_\varepsilon(x) \le 1, \qquad \int_{\mathbb{R}^3} \zeta_\varepsilon(x)\,\mathrm{d}x = 1.$$

We set

$$U_\varepsilon(x) = \zeta_\varepsilon \star E_3(x).$$

We note that, for any $0 < R < \infty$,

$$E_3 \in L^q(B(0,R)),\ \text{for}\ 1 \le q < 3, \qquad \nabla_x E_3(x) = -\frac{1}{4\pi}\frac{x}{|x|^3} \in L^q(B(0,R)),\ \text{for}\ 1 \le q < 3/2$$

and we check readily that U_ε and $\nabla_x U_\varepsilon$ converge to E_3 and $\nabla_x E_3$ in these spaces, respectively.

Owing to a fixed point argument we construct a sequence of solutions to the following non linear problem

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon - \nabla_x \Phi_\varepsilon \cdot \nabla_v f_\varepsilon = 0, \\ \Phi_\varepsilon = U_\varepsilon \star_x n_\varepsilon, \qquad n_\varepsilon(t,x) = \displaystyle\int_{\mathbb{R}^3} f_\varepsilon(t,x,v)\,\mathrm{d}v, \\ f_{\varepsilon,|t=0} = f^{\mathrm{Init}} \ge 0 \end{cases} \tag{3.40}$$

The solutions verify

- $f_\varepsilon \ge 0$,
- $\|f_\varepsilon(t)\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)} = \|f^{\mathrm{Init}}\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)}$, for any $1 \le p \le \infty$,
- and the energy conservation

$$\int_{rit^3}\int_{\mathbb{R}^3} \frac{v^2}{2} f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int_{rit^3}\int_{\mathbb{R}^3} \Phi_\varepsilon f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x = E_c(0) + E_p(0) < \infty.$$

Note again that the both the kinetic and the potential energy are non negative, a specific feature of the repulsive case for dimension $N \ge 3$. Thanks to these estimates, we can suppose, possibly at the price of extracting a subsequence, that

$$f_\varepsilon \rightharpoonup f \quad \text{weakly in } L^p((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)$$

for finite $1 \le p < \infty$ and weakly -$\star$in $L^\infty((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)$. and we wish to pass to the limit $\varepsilon \to 0$ in the weak formulation

$$-\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f^{\mathrm{Init}}\varphi(0,x,v)\,\mathrm{d}v\,\mathrm{d}x - \int_0^\infty\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon\ (\partial_t + v\cdot\nabla_x)\varphi(t,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^\infty\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon\ \nabla_x\Phi_\varepsilon\cdot\nabla_v\varphi(t,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t = 0$$

which holds for any $\varphi \in C_c^\infty([0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3)$. Clearly, the difficulty concentrates on the last term.

For constructing solutions of (3.40), we remark that for any $n \in L^1(\mathbb{R}^N)$, the potential $U_\varepsilon \star n \in C^2(\mathbb{R}^N)$ with $\nabla_x(U_\varepsilon \star n)$ uniformly lipschitzian on $\mathbb{R}^N$ (but with a Lipschitz constant that blows up when ε goes to 0). Therefore, we define the mapping

$$\mathscr{T} : g \mapsto f$$

so that

$$\partial_t f + v\cdot\nabla_x f = \nabla_x\Phi_g\cdot\nabla_v f, \qquad \Phi_g(t,x) = U_\varepsilon \star \int_{\mathbb{R}^N} g(t,x,v)\,\mathrm{d}v, \qquad f_{|t=0} = f^{\mathrm{Init}}.$$

The regularity of Φ_g, for g given in $C^0([0,\infty);L^1(\mathbb{R}^\times\mathbb{R}^N))$ allows to define the associated characteristics, and thus we obtain the solution f by integrating the kinetic equation along characteristics. It remains to show that $\mathscr{T}$ is a contraction mapping on $C^0([0,T];L^1(\mathbb{R}^N\times\mathbb{R}^N))$ for any initial data $f^{\mathrm{Init}} \in C_c^1(\mathbb{R}^N\times\mathbb{R}^N)$ (but more general data can then be considered by a regularizing argument).

It is necessary now to make precise the estimates on the force field. To this end, we need an interpolation lemma which will allow to take advantage of the energy conservation.

Lemma 4 *Let $f : \mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$ such that, for some $m > 0$,*

$$f \in L^\infty(\mathbb{R}^N\times\mathbb{R}^N) \qquad \text{and} \qquad \int\!\!\int |v|^m\ |f(x,v)|\,\mathrm{d}v\,\mathrm{d}x < \infty.$$

Then $n(x) = \int_{\mathbb{R}^N} f(x,v)\,\mathrm{d}v$ belongs to $L^p(\mathbb{R}^N)$ for $p = (N+m)/N$.

Proof. We split as follows

$$\begin{aligned} |n(x)| &\le \int_{\mathbb{R}^N} |f(x,v)|\,\mathrm{d}v \le \int_{|v|\le R} |f(x,v)|\,\mathrm{d}v + \int_{|v|\ge R} |f(x,v)|\,\mathrm{d}v \\ &\le \|f\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)}\mathrm{meas}(B_{\mathbb{R}^N}(0,R)) + \frac{1}{R^m}\int_{\mathbb{R}^N} |v|^m\ |f(x,v)|\,\mathrm{d}v \\ &\le C_N\|f\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)}R^N + \frac{1}{R^m}\int_{\mathbb{R}^N} |v|^m\ |f(x,v)|\,\mathrm{d}v. \end{aligned}$$

Optimizing with respect to R yields

$$|n(x)| \le C_N\ \|f\|^{m/(m+N)}_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)}\ \left(\int_{\mathbb{R}^N} |v|^m\ |f(x,v)|\,\mathrm{d}v\right)^{N/(N+m)}.$$

Therefore, we end up with

$$\int_{\mathbb{R}^N} |n(x)|^{(N+m)/N}\,\mathrm{d}x \le C_N\ \|f\|^{m/N}_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)}\ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v|^m\ |f(x,v)|\,\mathrm{d}v\,\mathrm{d}x < \infty.$$

∎

With $m = 2$ and $N = 3$ one gets that the macroscopic density n lies in $L^{5/3}(\mathbb{R}^3)$. The energy conservation tells us that $\int_{\mathbb{R}^3\times\mathbb{R}^3} v^2 f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x$ is bounded and we deduce the following a priori estimates.

Corollary 2 *The sequence* $\big(n_\varepsilon\big)_{\varepsilon>0}$ *is bounded in* $L^\infty(0,\infty; L^{5/3}(\mathbb{R}^3))$.

Corollary 3 *The sequence* $\big(\nabla_x \Phi_\varepsilon\big)_{\varepsilon>0}$ *is bounded in* $L^\infty(0,\infty; L^q(\mathbb{R}^3))$, *for* $3/2 < q < 15/4$.

Proof. Clearly we have

$$\|\nabla_x \Phi_\varepsilon\|_{L^q} \leq \|\zeta_\varepsilon\|_{L^1}\ \|\nabla_x E_3 \star n_\varepsilon\|_{L^q} \leq \frac{1}{4\pi}\Big\| \int_{\mathbb{R}^3} \frac{n_\varepsilon(t,y)}{|x-y|^2}\,\mathrm{d}y\Big\|_{L^q}.$$

Hence, we apply the Hardy-Littlewood-Sobolev inequality with $\alpha = 3/2$ and we obtain $1/q = 3/5 + 2/3 - 1 = 4/15$. Since all moments $0 \leq m \leq 2$ are bounded, we obtain estimates on a family of L^q spaces.

In particular, we remark that $m = 3/5$ gives $n_\varepsilon(t,x)$ bounded in $L^\infty(0,\infty; L^{6/5}(\mathbb{R}^3))$ and thus $\nabla_x \Phi_\varepsilon$ bounded in $L^2(\mathbb{R}^3)$. ∎

Consequently the product $f_\varepsilon \nabla_x \Phi_\varepsilon$ is bounded in $L^\infty(0,\infty; L^q(\mathbb{R}^3))$, for $3/2 < q < 15/4$, and we can suppose that it admits a limit. It remains to establish that

$$f_\varepsilon \nabla_x \Phi_\varepsilon \rightharpoonup f \nabla_x \Phi, \qquad \Phi = E_3 \star \int_{\mathbb{R}^3} f\,\mathrm{d}v.$$

It requires improved compactness properties, that is to show some strong convergence property. To this end, a first attempt would be to exploit the smoothing effects of the Poisson equation, and precisely use that $n \mapsto \int_{\mathbb{R}^N} \frac{n(y)}{|x-y|^\alpha}\,\mathrm{d}y$ is a compact operator from $L^p(\mathbb{R}^N)$ into $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ for convenient pairs (p,q). This argument needs additionally some care concerning the treatment of the time variable. Hence, we shall use instead "Average Lemma" techniques which have became a very standard tool in the theory of kinetic equations. In particular, the argument is necessary for proving the existence of solutions for the more complete Vlasov-Maxwell model, see [33, 83]. The flavor of the argument can be summarized by saying that if f and $(\partial_t + v \cdot \nabla_x) f$ belong to L^p, $p > 1$, then average with respect to the v variable are smoother. It translates into stronger compactness properties when dealing with bounded sequences. We shall detail below the Average Lemma approach, which allows to finish the proof of the following claim.

Theorem 7 *Let* $N = 3$ *and let* $f^{\mathrm{Init}} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, $f^{\mathrm{Init}} \geq 0$, *such that*

$$v^2 f^{\mathrm{Init}} \in L^1(\mathbb{R}^3 \times \mathbb{R}^3) \qquad \frac{1}{|x|^2} \star \int_{\mathbb{R}^3} f^{\mathrm{Init}} \in L^2(\mathbb{R}^3).$$

Then, there exists $f \in C^0([0,\infty); L^\infty(\mathbb{R}^3 \times \mathbb{R}^3) - \mathrm{weak} - \star)$, *solution of the Vlasov-Poisson system with initial data* f^{Init}. *Furthermore, we have*

$$\|f(t)\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)} \leq \|f^{\mathrm{Init}}\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)}, \qquad (E_c + E_p)(t) \leq (E_c + E_p)(0).$$

3.2 Average lemma

The average lemma is a useful and powerful tool for obtaining compactness on solutions of kinetic equations. Roughly speaking the statement tells us that macroscopic quantities are smoother than the microscopic ones, which in turn leads to improved compactness properties for quantities involving averages on the velocity variable. The basic statement reads as follows.

Theorem 8 *Let* $f(x,v) \in L^2(\mathbb{R}^N \times \mathbb{R}^N)$ *verifying*

$$v \cdot \nabla_x f = g \in L^2(\mathbb{R}^N \times \mathbb{R}^N). \tag{3.41}$$

Let $\psi \in L^\infty(\mathbb{R}^N_v)$ *with compact support in* $\mathbb{R}^N_v$*. Then the macroscopic quantity*

$$\rho_\psi(x) = \int_V \psi(v)\ f(x,v)\,\mathrm{d}v$$

belongs to the Sobolev space $H^{1/2}(\mathbb{R}^N)$*. More precisely, we have*

$$\int_{\mathbb{R}\times\mathbb{R}^N} |\xi|\ \ |\widehat{\rho_\psi}(\xi)|^2\,\mathrm{d}\xi \le C(\psi)\|f\|_{L^2(\mathbb{R}^N\times V)}\ \|g\|_{L^2(\mathbb{R}^N\times V)}, \tag{3.42}$$

where $\widehat{\phi}$ *stands for the Fourier transform*

$$\widehat{\phi}(\xi) = \int_{\mathbb{R}^N} \phi(t,x)\mathrm{e}^{-\mathrm{i}x\cdot\xi}\,\mathrm{d}x$$

for $\xi \in \mathbb{R}^N$*.*

Remark 5 *The support assumption on* ψ *can be relaxed when we have estimates on high order moments (wrt* v*) of* f*. More generally we can deal with a set* $V \subset \mathbb{R}^N$ *endowed with a measure* $\mathrm{d}\mu(v)$*, but it is crucial to require that*

$$\begin{cases} \text{for any } R>0 \text{ there exists } C_R>0 \text{ and } \varepsilon_0>0 \text{ such that for any } \varepsilon_0 \ge \varepsilon > 0 \text{ we have} \\ \sup\limits_{\xi\in\mathbb{S}^{N-1}} \mu\big(\{v \in B(0,R),\ |v\cdot\xi| \le \varepsilon\}\big) \le C_R\varepsilon. \end{cases} \tag{3.43}$$

Note that (3.43) excludes discrete models (for example, $V = \{v_1, ..., v_M\}$*,* $\mathrm{d}\mu(v) = \sum_{i=1}^M \delta(v = v_i)$ *and pick* ξ *orthogonal to one of the* v^i*'s in* V*). Roughly speaking* (3.43) *means that we need "enough directions"* v *to make the information on* $v \cdot \nabla_x f$ *useful.*

Remark 6 *Of course, analogous results are available for the non stationary case. There are refined (and useful!) versions of this statement, see in particular [34], [12], [42], [43], [75]...: on the one hand we can relax the regularity required on the right hand side* g *and in particular we can deal with derivatives and on the other hand we can replace* v *by more general functions* $a(v)$*... This tool has been successfully applied to justify the Rosseland approximation in radiative transfer [3]. It is also a key ingredient in the proof of existence (in a suitable weak sense...) for the Boltzmann equation of gas dynamics [32], [66]. Further developments and applications can be found in the lecture notes [13]... Note however that some non linear hydrodynamic limit problems can be treated without average lemma techniques, but using instead a compensated compactness argument which allow to deal with discrete velocity models [67, 50, 31].*

Proof. We suppose that $\mathrm{supp}(\psi) \subset B(0,R)$. Clearly ρ_ψ belongs to $L^2(\mathbb{R}^N)$. The remarkable fact lies in the improved regularity, while the microscopic quantity f belongs only to L^2. Applying the Fourier transform to (3.41) yields

$$v \cdot \xi \ \widehat{f}(\xi, v) = -\mathrm{i}\widehat{g}(\xi, v).$$

When $|\xi \cdot v| \geq \delta |\xi| > 0$, this relation allows to estimate $|\xi| \widehat{f}$ nicely, while the "bad" set on which $|\xi \cdot v| \xi|$ is small has only a "small" contribution thanks to (3.43). Let us make this idea precise. We split the integral

$$\int_{\mathbb{R}^N} \psi(v) \widehat{f}(\xi, v) \,\mathrm{d}v = \int_{|\xi \cdot v| \leq \delta |\xi|} \psi(v) \widehat{f}(\xi, v) \,\mathrm{d}v + \int_{|\xi \cdot v| \geq \delta |\xi|} \psi(v) \widehat{f}(\xi, v) \,\mathrm{d}v.$$

We estimate the second integral by using (3.41):

$$\begin{aligned} \left| \int_{|\xi \cdot v| \geq \delta |\xi|} \psi(v) \widehat{f}(\xi, v) \,\mathrm{d}v \right| &= \left| \int_{|\xi \cdot v| \geq \delta |\xi|} \psi(v) \widehat{f}(\xi, v) \frac{\xi \cdot v}{\xi \cdot v} \,\mathrm{d}v \right| \\ &\leq \|\psi\|_{L^\infty} \left(\int_{\mathbb{R}^N} |\widehat{g}(\xi, v)|^2 \,\mathrm{d}v \right)^{1/2} \left(\int_{|\xi \cdot v| \geq \delta |\xi|} \frac{1}{|\xi \cdot v|^2} \mathbb{1}_{B(0,R)}(v) \,\mathrm{d}v \right)^{1/2} \end{aligned}$$

where

$$\int_{|\xi \cdot v| \geq \delta |\xi|} \frac{1}{|\xi \cdot v|^2} \mathbb{1}_{B(0,R)}(v) \,\mathrm{d}v = \frac{C(R)}{|\xi|^2} \int_\delta^\infty \frac{\mathrm{d}r}{r^2} = \frac{C(R)}{\delta \ |\xi|^2}.$$

Next, we estimate as follows

$$\left| \int_{|\xi \cdot v| \leq \delta |\xi|} \psi(v) \widehat{f}(\xi, v) \,\mathrm{d}v \right| \leq \|\psi\|_{L^\infty} \left(\int_{\mathbb{R}^N} (\widehat{f}(\xi, v))^2 \,\mathrm{d}v \right)^{1/2} \left(\int_{|\xi \cdot v| \leq \delta |\xi|} \mathbb{1}_{B(0,R)} \,\mathrm{d}v \right)^{1/2}$$

where (see (3.43))

$$\int_{|\xi \cdot v| \geq \delta |\xi|} \mathbb{1}_{B(0,R)} \,\mathrm{d}v = C(R) \ \int_0^\delta \mathrm{d}r = C(R) \ \delta.$$

Combining all the pieces, we obtain

$$\left| \int_{\mathbb{R}^N} \psi(v) \widehat{f}(\xi, v) \,\mathrm{d}v \right| \leq C(R) \|\psi\|_{L^\infty} \Big(F(\xi) \sqrt{\delta} + \frac{G(\xi)}{|\xi| \sqrt{\delta}} \Big),$$

where $G(\xi) = \|\widehat{g}(\xi, \cdot)\|_{L^2(\mathbb{R}^N_v)}$ and $F(\xi) = \|\widehat{f}(\xi, \cdot)\|_{L^2(\mathbb{R}^N_v)}$ belong to $L^2(\mathbb{R}^N_\xi)$. Since this relation holds for any δ, in particular we can choose $\delta = \frac{G(\xi)}{|\xi| \ F(\xi)}$ which finally leads to

$$\left| \int_{\mathbb{R}^N} \psi(v) \widehat{f}(\xi, v) \,\mathrm{d}v \right| \leq C(R) \|\psi\|_{L^\infty} \frac{1}{\sqrt{|\xi|}} \ \sqrt{FG(\xi)}.$$

■

We can now finish the proof of the existence theorem for the Vlasov-Poisson system. The delicate term can be recast as

$$I_\varepsilon = \int_0^\infty \int_{\mathbb{R}^N} \nabla_x \Phi_\varepsilon \cdot \left(\int_{\mathbb{R}^N} f_\varepsilon \nabla_v \varphi \, \mathrm{d}v \right) \mathrm{d}x \, \mathrm{d}t$$

which make naturally a velocity average appear. We can use the following refined version of the average lemma [75, 13]

Theorem 9 *Let f_n and g_n^α satisfy*

$$(\partial_t + v \cdot \nabla_x) f_n = \sum_{|\alpha| \le m} \partial_v^\alpha g_n^\alpha$$

for some $m \in \mathbb{N}$. Let Q be a open set in $\mathbb{R} \times \mathbb{R}^N$. We suppose that $\big(f_n\big)_{n\in\mathbb{N}}$ is bounded in $L^p(Q \times \mathbb{R}^N)$ for some $p > 1$ and the $\big(g_n^\alpha\big)_{n\in\mathbb{N}}$'s are bounded in $L^1(Q \times \mathbb{R}^N)$. Then, for any $\phi \in C_c^\infty(\mathbb{R}^N)$, the sequence defined by $\rho_n(t,x) = \int_{\mathbb{R}^N} f_n \phi \, \mathrm{d}v$ is relatively compact in $L^p(Q)$.

Accordingly, we obtain

$$\int_{\mathbb{R}^N} f_\varepsilon \nabla_v \varphi \, \mathrm{d}v \xrightarrow[\varepsilon \to 0]{} \int_{\mathbb{R}^N} f \nabla_v \varphi \, \mathrm{d}v \quad \text{strongly in } L^2((0,T) \times \mathbb{R}^N \times \mathbb{R}^N).$$

Besides, owing to Corollary 3, we can suppose

$$\nabla_x \Phi_\varepsilon \rightharpoonup \Psi \quad \text{weakly in } L^2 L^2((0,T) \times \mathbb{R}^N).$$

Therefore, we conclude that

$$I_\varepsilon \xrightarrow[\varepsilon \to 0]{} \int_0^\infty \int_{\mathbb{R}^N} \Psi(t,x) \int_{\mathbb{R}^N} f \nabla_v \varphi \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t.$$

Coming back to the Hardy-Littewood-Sobolev inequality, Φ_ε is bounded in some $L^p((0,T) \times \mathbb{R}^N)$ space so that, still extracting subsequences if necessary, we obtain $\Phi_\varepsilon \rightharpoonup \Phi$ and $\nabla_x \Phi_\varepsilon \rightharpoonup \Psi = \nabla_x \Phi$ while passing to the limit in the Poisson equation yields $-\Delta_x \Phi = n = \int_{\mathbb{R}^N} f \, \mathrm{d}v$. Remark that, using the average lemma again, we can actually prove the strong convergence of Φ_ε in L^p spaces.

Eventually, we justify compactness with respect to the time variable in order that the initial data for the limit equation makes sense. Since

$$\partial_t f_\varepsilon = -\mathrm{div}_x(v f_\varepsilon) + \mathrm{div}_v(\nabla_x \Phi_\varepsilon f_\varepsilon)$$

appears as derivatives with respect to x, v of functions bounded in $L^\infty(0,T; L^1(B(0,R))$, for any $0 < T, R < \infty$ we can indeed appeal to the following claim.

Lemma 5 *Let* $n_\varepsilon : (0,T) \times \mathbb{R}^D \to \mathbb{R}$ *such that*

$$\sup_{\varepsilon>0} \|n_\varepsilon(t)\|_{L^\infty((0,T)\times\mathbb{R}^D)} \leq M < \infty.$$

Furthermore, suppose that

$$\partial_t n_\varepsilon = \sum_{|\alpha|\leq k} \partial_x^\alpha g_\varepsilon^{(\alpha)},$$

where, for any compact set $K \subset \mathbb{R}^D$,

$$\sup_{\varepsilon>0} \left\{ \int_E \int_K |g_\varepsilon^{(\alpha)}| \, \mathrm{d}x \, \mathrm{d}t \right\} \longrightarrow 0 \qquad \text{as } |E| \to 0.$$

Then, the sequence n_ε *is compact in* $\mathrm{C}^0([0,T]; L^\infty(\mathbb{R}^D) - \mathrm{weak} - \star)$.

Proof. Pick $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^D)$. At first, we remark that

$$\sup_{\varepsilon>0,\ 0\leq t\leq T} \Big| \int_{\mathbb{R}^D} n_\varepsilon(t,x)\varphi(x) \, \mathrm{d}x \Big| \leq M\|\varphi\|_{L^\infty(\mathbb{R}^D)} < \infty. \tag{3.44}$$

Secondly, the estimate

$$\begin{aligned} &\Big| \int_{\mathbb{R}^D} n_\varepsilon(t+h,x)\varphi(x) \, \mathrm{d}x - \int_{\mathbb{R}^D} n_\varepsilon(t,x)\varphi(x) \, \mathrm{d}x \Big| \\ &\qquad \leq \sum_{|\alpha|\leq k} \|\partial^\alpha \varphi\|_{L^\infty(\mathbb{R}^D)} \Big| \int_t^{t+h} \int_{\mathrm{supp}(\varphi)} |g_\varepsilon^{(\alpha)}| \, \mathrm{d}x \, \mathrm{d}s \Big|, \end{aligned}$$

proves that the family

$$\left\{ \int_{\mathbb{R}^D} n_\varepsilon(t,x)\varphi(x) \, \mathrm{d}x,\ \varepsilon > 0 \right\}$$

is equicontinuous on $[0,T]$. Therefore, for a given test function φ, the family is compact in $\mathrm{C}^0([0,T])$, as a direct consequence of the Arzela-Ascoli theorem.

By density and using (3.44), the compactness property extends to any test function φ in $L^1(\mathbb{R}^D)$. Since $L^1(\mathbb{R}^D)$ is separable, by using a diagonal argument, we can extract a subsequence such that

$$\int_{\mathbb{R}^D} \varphi(x) n_{\varepsilon_\ell}(t,x) \, \mathrm{d}x \longrightarrow \int_{\mathbb{R}^D} \varphi(x) n(t,x) \, \mathrm{d}x \qquad \text{as } \ell \to \infty \tag{3.45}$$

in $\mathrm{C}^0([0,T])$, for any test function φ in D, a demombrable dense subset of $L^1(\mathbb{R}^D)$. Coming back to (3.44) and by density, we realize that the convergence (3.45) applies to any test function $\varphi \in L^1(\mathbb{R}^D)$. ∎

A natural and deep question consists in considering C^1 solutions. so that the equations (1.5) holds in a strong and pointwise sense. The question is clearly related to the possibility of defining the characteristics, and therefore to control the second derivatives of the field. The key estimate in this direction reads as follows.

Lemma 6 *Let $n \in L^1 \cap L^\infty(\mathbb{R}^3)$ such that $\nabla_x n \in L^\infty(\mathbb{R}^3)$. Set $\Phi = \frac{1}{|x|}$. Then $D^2\Phi$ belongs to $L^\infty(\mathbb{R}^3)$ with*

$$\|D^2\Phi\|_{L^\infty} \leq C\Big(1 + \|n\|_{L^1} + \|n\|_{L^\infty}\big(1 + \ln(1 + \|\nabla_x n\|_{L^\infty})\big)\Big).$$

By combining this claim to properties of transport equations we get

Corollary 4 *Let f be a smooth solution of the Vlasov-Poisson system. We assume that for any $0 \leq t \leq T < \infty$, there exists a finite $R_T > 0$ such that*

$$f(t,x,v) = 0 \qquad \text{for } |v| \geq R_T.$$

Then, there exists a constant C depending on $\|f^{\mathrm{Init}}\|_{L^1}$, $\|f^{\mathrm{Init}}\|_{L^\infty}$, $\|\nabla_{x,v} f^{\mathrm{Init}}\|_{L^\infty}$ and T such that

$$\|D^2\Phi\|_{L^\infty} \leq C, \qquad \|\nabla_{x,v} f(t)\|_{L^\infty} \leq C$$

holds.

Hence, the question reduces to a control of the expansion of the support of the solutions of the Vlasov-Poisson system: the support should remain bounded on finite time intervals. Denoting $P(t) = \sup\big\{|v| \text{ such that there exists } x \in \mathbb{R}^N \text{ verifying } f(t,x,v) \neq 0\big\}$ we can derive quite easily an estimate on short time. Going further requires a very fine analysis; we refer to [85] and [69] for ultimate results in that direction.

3.3 Attractive case ($N = 3$)

The proof can be adapted to treat the attractive potential (3.39) as well, despite the fact that the energy has no definite sign a priori. Nevertheless, the methods detailed above allow to control the kinetic energy, which is the crucial point in the proof. We start with a simple interpolation inequality

Lemma 7 *Let $n : \mathbb{R}^3 \to \mathbb{R}$. Then, we have*

$$\int_{\mathbb{R}^3} |n|^{6/5}\,\mathrm{d}x \leq \left(\int_{\mathbb{R}^3} |n|\,\mathrm{d}x\right)^{7/10} \left(\int_{\mathbb{R}^3} |n|^{5/3}\,\mathrm{d}x\right)^{3/10}.$$

Proof. Write $6/5 = \theta + (1-\theta)5/3$, that is $\theta = 7/10$ and use the Hölder inequality on $|n|^\theta \in L^{1/\theta}$ and $|n|^{(1-\theta)5/3} \in L^{1/(1-\theta)}$. ∎

Then, for $\nabla_x\Phi = -\frac{1}{4\pi}\frac{x}{|x|^3} \star n$, where $\int_{\mathbb{R}^3} n\,\mathrm{d}x = 1$, the Hardy-Littlewood-Sobolev yields

$$\|\nabla_x\Phi\|_{L^2(\mathbb{R}^3)} \leq C\|n\|_{L^{6/5}(\mathbb{R}^3)} \leq C\|n\|^{5/12}_{L^{5/3}(\mathbb{R}^3)}.$$

We go back to Lemma 4 which imply for $n(x) = \int_{\mathbb{R}^3} f(x,v)\,\mathrm{d}v$,

$$\int_{\mathbb{R}^3} |n|^{5/3}\,\mathrm{d}x \leq C\|f\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} \int_{\mathbb{R}^3\times\mathbb{R}^3} v^2 f(x,v)\,\mathrm{d}v\,\mathrm{d}x$$

and we end up with

$$\|\nabla_x \Phi\|_{L^2(\mathbb{R}^3)} \leq C \left(\int_{\mathbb{R}^3\times\mathbb{R}^3} v^2 f(x,v)\,\mathrm{d}v\,\mathrm{d}x \right)^{1/4}$$

where C depends on the L^1 and L^∞ norms of f. Consequently, the associated total energy verifies

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} v^2 f(x,v)\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^3} |\nabla_x\Phi|^2\,\mathrm{d}x \geq \int_{\mathbb{R}^3\times\mathbb{R}^3} v^2 f(x,v)\,\mathrm{d}v\,\mathrm{d}x - C\left(\int_{\mathbb{R}^3\times\mathbb{R}^3} v^2 f(x,v)\,\mathrm{d}v\,\mathrm{d}x \right)^{1/2}.$$

Dealing with the sequence of solutions of approximated problems defined as above for the attractive case, we show that $\big(f_\varepsilon\big)_{\varepsilon>0}$ is bounded in $L^\infty((0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3)$ and in $L^\infty(0,\infty; L^1(\mathbb{R}^3\times\mathbb{R}^3))$, with furthermore

$$\sup_{t\geq 0} \int_{\mathbb{R}^3\times\mathbb{R}^3} v^2 f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x \quad \text{bounded uniformly wrt } \varepsilon$$

(since this quantity satisfies an inequality looking like $P(k) = k - C\sqrt{k} - E \leq 0$). Then, we can follow the arguments developped for the repulsive case and extend Theorem 7 to the attractive potential as well.

3.4 Dispersion inequalities

Let us finish by establishing some dispersion inequalities which provide interesting information on the large time behavior of the particles. The result is in agreement with the physical intuition: due to the repulsive effect of the self-consistent force, particles spread in space producing a decay of the macroscopic density.

Theorem 10 *Let (f,Φ) be a solution of the (repulsive) Vlasov-Poisson system. Then, there exists $C>0$ such that*

$$\begin{cases} \|\nabla_x\Phi(t,\cdot)\|_{L^2(\mathbb{R}^3)} \leq \dfrac{C}{\sqrt{1+t}}, \\ \displaystyle\int_{\mathbb{R}^3\times\mathbb{R}^3} |x-tv|^2 f\,\mathrm{d}v\,\mathrm{d}x \leq C(1+t), \\ \|n(t)\|_{L^{5/3}(\mathbb{R}^3)} \leq \dfrac{C}{(1+t)^{3/5}}. \end{cases}$$

Proof. We follow the tricky proof proposed in [60]. The first step of the proof consists in remarking that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|x-tv|^2}{2} f\,\mathrm{d}v\,\mathrm{d}x = -\frac{1}{2}\Big(t^2 \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_x\Phi(t,\cdot)\|^2_{L^2(\mathbb{R}^3)} + t\|\nabla_x\Phi(t,\cdot)\|^2_{L^2(\mathbb{R}^3)}\Big). \tag{3.46}$$

Indeed, integration by parts shows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|x-tv|^2}{2} f\,\mathrm{d}v\,\mathrm{d}x &= \int_{\mathbb{R}^3\times\mathbb{R}^3} \Big(x - tv\cdot v) - (x-tv)\cdot(-t\nabla_x\Phi - (x-tv)\cdot v\Big) f\,\mathrm{d}v\,\mathrm{d}x \\ &= t\int_{\mathbb{R}^3\times\mathbb{R}^3} x\cdot\nabla_x\Phi\ f\,\mathrm{d}v\,\mathrm{d}x - t^2\int_{\mathbb{R}^3\times\mathbb{R}^3} v\cdot\nabla_x\Phi\ f\,\mathrm{d}v\,\mathrm{d}x. \end{aligned}$$

Besides, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\frac{|\nabla_x\Phi|^2}{2}\,\mathrm{d}x=\int_{\mathbb{R}^3\times\mathbb{R}^3} v\cdot\nabla_x\Phi\ f\,\mathrm{d}v\,\mathrm{d}x.$$

Finally, we use the expression of the elementary solution E_3 to obtain

$$\begin{aligned}\int_{\mathbb{R}^3\times\mathbb{R}^3} x\cdot\nabla_x\Phi\ f\,\mathrm{d}v\,\mathrm{d}x &= -\frac{1}{4\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3} x\cdot\frac{x-y}{|x-y|^3}n(t,x)n(t,y)\,\mathrm{d}y\,\mathrm{d}x\\ &= -\frac{1}{8\pi}\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{n(t,x)n(t,y)}{|x-y|}\,\mathrm{d}y\,\mathrm{d}x=-\frac{1}{2}\int_{\mathbb{R}^3}|\nabla_x\Phi|^2\,\mathrm{d}x.\end{aligned}$$

These relations finally lead to (3.46).

We set

$$\Gamma(t)=\frac{t^2}{2}\int_{\mathbb{R}^3}|\nabla_x\Phi|^2\,\mathrm{d}x$$

which verifies

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(t)=t\int_{\mathbb{R}^3}|\nabla_x\Phi|^2\,\mathrm{d}x+\frac{t^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\nabla_x\Phi|^2\,\mathrm{d}x.$$

Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|x-tv|^2}{2}f\,\mathrm{d}v\,\mathrm{d}x+\Gamma(t)\Big)=\frac{t}{2}\int_{\mathbb{R}^3}|\nabla_x\Phi|^2\,\mathrm{d}x=\frac{\Gamma(t)}{t}.$$

Integration between, say 1 and t, yields

$$\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|x-tv|^2}{2}f\,\mathrm{d}v\,\mathrm{d}x+\Gamma(t)=C_1+\int_1^t\frac{\Gamma(s)}{s}\,\mathrm{d}s\geq\Gamma(t)\geq 0$$

and the Gronwmall lemma allows us to deduce that

$$\Gamma(t)\leq C_1\exp\Big(\int_1^t\frac{\Gamma(s)}{s}\,\mathrm{d}s\Big)=C_1 t$$

that is $\|\nabla_x\Phi(t)\|^2_{L^2(\mathbb{R}^3)}\leq C_1/t$. Then it follows that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|x-tv|^2}{2}f\,\mathrm{d}v\,\mathrm{d}x\leq C_1+\int_1^t C_1\,\mathrm{d}s\leq C_1(1+t).$$

Eventually, we write

$$\begin{aligned}0\leq n(t,x) &= \int_{\mathbb{R}^3}f(t,x,v)\,\mathrm{d}v=\int_{|x-tv|\leq R}f(t,x,v)\,\mathrm{d}v+\int_{|x-tv|\leq R}f(t,x,v)\,\mathrm{d}v\\ &\leq \|f\|_{L^\infty}\mathrm{meas}\big(\{v\in\mathbb{R}^3,\ |x/t-v|\leq R/t\}\big)+\frac{1}{R^2}\int_{\mathbb{R}^3\times\mathbb{R}^3}|x-tv|^2 f\,\mathrm{d}v\,\mathrm{d}x\\ &\leq \|f^{\mathrm{Init}}\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}\frac{4}{3}\pi(R/t)^3+\frac{1}{R^2}\int_{\mathbb{R}^3\times\mathbb{R}^3}|x-tv|^2 f\,\mathrm{d}v\,\mathrm{d}x.\end{aligned}$$

Optimizing with respect to R we find

$$0\leq n(t,x)\leq C\ t^{-6/5}\Big(\int_{\mathbb{R}^3\times\mathbb{R}^3}|x-tv|^2 f\,\mathrm{d}v\,\mathrm{d}x\Big)^{3/5},$$

which leads to the estimate on the $L^{5/3}$ norm of n. ∎

4 Hydrodynamic limits: from kinetic equation to drift-diffusion models

In this Section, we describe an example of hydrodynamic limit, obtaining drift-diffusion equations from a kinetic model as the mean free path becomes small (compared to a length of reference). The original point in the model devised below relies on the fact that the drift velocity depends on the macroscopic concentration through the Poisson equation. To this end, we start from a kinetic model which takes into account some collisional effects. Indeed, fluid approximation in collisionless situation can be questionable, as indicated in the deep numerical investigation [9]. Therefore, the Vlasov-Poisson system describes the evolution of electrons interacting through their self-consistent electric field and we consider additionaly friction forces and Brownian effects, embodied into the following linear Fokker-Planck operator

$$Lf = \mathrm{div}_v\Big(vf + \frac{k_B T_{th}}{m_e}\nabla_v f\Big).$$

Written in physical variables, the evolution of the particles density obeys

$$\partial_t f + v\cdot\nabla_x f - \frac{q}{m_e}\nabla_x\Phi\cdot\nabla_v f = \frac{1}{\tau}\nabla_v\cdot\Big(vf + \frac{k_B T_{th}}{m_e}\nabla_v f\Big).$$

The electrons are thus subject to the force $-\frac{q}{m_e}\nabla_x\Phi$ with the coupling

$$-\Delta\Phi = \frac{q}{\varepsilon_0}\int f\,\mathrm{d}v.$$

4.1 Dimension analysis

The problem is driven by the following set of physical (positive) quantities:
- ε_0, the vacuum permittivity,
- q, the elementary charge of the electrons,
- m_e the mass of the electrons,
- τ_e, the relaxation time characteristic of the interactions of the particles with the thermal bath,
- k_B, the Boltzmann constant,
- T_{th}, the temperature of the thermal bath.

We introduce time and length units T, L, respectively. They define a velocity unit L/T which has to be compared to the thermal velocity

$$\mathrm{V} = \sqrt{k_B T_{th}/m_e}$$

like the time unit will be compared to the relaxation time τ_e. The quantity

$$\ell_e = \tau_e\sqrt{\frac{k_B T_{th}}{m_e}}$$

is the mean free path of the particles: it is the typical distance travelled by the particles during the time τ_e. The plasma is also characterized by $\mathcal{N}$, the typical number of electrons. The quantity

$$\Lambda = \sqrt{\frac{\varepsilon_0 k_B T_{th} \mathrm{L}}{q^2 \mathcal{N}}}\ \mathrm{L}$$

defines the so–called Debye length, that is the typical length of perturbations of a quasi-neutral plasma. We define dimensionless variables and unknowns by the following relations

$$\begin{cases} t = \mathrm{T}\ t', \qquad x = \mathrm{L}\ x', \qquad v = \mathrm{V}\ v', \\ f(t,x,v) = \dfrac{\mathcal{N}}{L^3 \mathrm{V}^3}\ f'(t/\mathrm{T}, x/\mathrm{L}, v/\mathrm{V}). \end{cases}$$

Eventually, for the potential we set

$$\Phi(t,x,v) = \chi\ \frac{k_B T_{th}}{q}\ \Phi'(t/\mathrm{T}, x/\mathrm{L})$$

where $\chi > 0$ is a dimensionless parameter. Accordingly, the dimensionless form of the equations reads as follows (where the primes have been dropped),

$$\begin{cases} \partial_t f + \dfrac{\mathrm{VT}}{\mathrm{L}}\ v \cdot \nabla_x f - \dfrac{q\mathrm{T}}{m\mathrm{LV}} \chi\ \dfrac{k_B T_{th}}{q}\ \nabla_x \Phi \cdot \nabla_v f = \dfrac{\mathrm{T}}{\tau}\ \nabla_v \cdot \Big(v f + \nabla_v f \Big), \\ \qquad = \partial_t f + \dfrac{\mathrm{VT}}{\mathrm{L}}\ v \cdot \nabla_x f - \chi\ \dfrac{\mathrm{VT}}{\mathrm{L}} \nabla_x \Phi \cdot \nabla_v \\ \\ -\varepsilon_0\ \chi\ \dfrac{k_B T_{th}}{q}\ \dfrac{\mathrm{L}}{q\mathcal{N}} \dfrac{1}{L^2}\ \Delta_x \Phi = -\chi\ \dfrac{\varepsilon_0 k_B T_{th} \mathrm{L}}{q^2 \mathcal{N}} \Delta_x \Phi = -\chi\ \left(\dfrac{\Lambda}{L}\right)^2 \Delta_x \Phi = \displaystyle\int f\ \mathrm{d}v. \end{cases}$$

Hence, we have at hand three free parameters $\chi, \mathrm{T}, \mathrm{L}$ corresponding to the physical quantities $\mathcal{N}, T_{th}$ and τ which characterize the plasma. The scaling we consider assumes that

$$\chi \Big(\frac{\Lambda}{L}\Big)^2 = 1$$

which serves as a definition of χ. Next, since we consider hydrodynamic regimes where the microscopic distribution relaxes to an equilibrium we have

$$\tau \ll \mathrm{T}.$$

Given $0 < \varepsilon \ll 1$, we distinguish two relevant regimes:

- either, we set

$$\frac{\tau}{\mathrm{T}} = \varepsilon^2, \qquad \frac{\mathrm{L}}{\mathrm{VT}} = \varepsilon, \qquad \chi \text{ fixed.} \tag{4.47}$$

Since $\ell = \tau \mathrm{V} = \frac{\tau}{\mathrm{T}} \frac{\mathrm{VT}}{\mathrm{L}} \mathrm{L}$ and $\Lambda/\mathrm{L} = 1/\sqrt{\chi}$ it means

$$\ell = \varepsilon \mathrm{L} \ll \mathrm{L} \simeq \Lambda;$$

- or we set

$$\frac{\tau}{\mathrm{T}} = \varepsilon, \qquad \frac{\mathrm{L}}{\mathrm{VT}} = 1, \qquad \chi = \frac{1}{\varepsilon}, \tag{4.48}$$

which means

$$\ell = \varepsilon \mathrm{L} \ll \Lambda = \sqrt{\varepsilon}\mathrm{L} \ll \mathrm{L}.$$

These regimes has been identified in [81], where the former is analyzed, with complements in [49]. The latter regime has been investigated in [74] for the one-dimensional case, and then in [51] for higher dimensions results. Similar investigations dealing with the non linear BGK or the Fermi-Dirac collision operator are due to [8] and [6, 7]. For attempts with the Maxwell equations instead of the electrostatic coupling, we refer to [15, 16].

4.2 Formal asymptotics: diffusive scaling

In what follows, we focus on the scaling (4.47). We are thus interested in the behavior as $\varepsilon \to 0$ of the solution $(f_\varepsilon, \Phi_\varepsilon)$ of the following Vlasov-Poisson-Fokker-Planck system (VPFP for short)

$$\begin{cases} \partial_t f_\varepsilon + \dfrac{1}{\varepsilon}\, v \cdot \nabla_x f_\varepsilon - \dfrac{\chi}{\varepsilon}\, \nabla_x \Phi_\varepsilon \cdot \nabla_v f_\varepsilon = \dfrac{1}{\varepsilon^2}\, L f_\varepsilon \quad \text{for } t \geq 0,\, x \in \mathbb{R}^N,\, v \in \mathbb{R}^N, \\ Lf = \nabla_v \cdot (vf + \nabla_v f), \\ -\Delta \Phi_\varepsilon = \rho_\varepsilon, \qquad \rho_\varepsilon(t,x) = \displaystyle\int_{\mathbb{R}^N} f_\varepsilon(t,x,v)\,\mathrm{d}v. \end{cases} \tag{4.49}$$

The problem is supplemented with an initial data

$$f_{\varepsilon,|t=0} = f_\varepsilon^{\mathrm{Init}} \geq 0. \tag{4.50}$$

As said above the Poisson equation should be understood as

$$\Phi_\varepsilon(t,x) = \big(E_N \star_x \rho_\varepsilon(t,\cdot)\big)(x). \tag{4.51}$$

The relaxation effect due to the collisions can be understood by writing

$$Lf = \nabla_v \cdot \big(\mathrm{e}^{-v^2/2} \nabla_v (f \mathrm{e}^{+v^2/2})\big),$$

the kernel of which is spanned by the Maxwellian distributions. Since the asymptotic $\varepsilon \to 0$ makes the collision operator vanish ($Lf_\varepsilon \to 0$), it would lead to

$$f_\varepsilon(t,x,v) \simeq \rho(t,x)\, \frac{\mathrm{e}^{-v^2/2}}{(2\pi)^{N/2}}. \tag{4.52}$$

The limit equation satisfied by the macroscopic density $\rho(t,x)$ can be obtained by looking at the evolution equations satisfied by the moments ρ_ε and $J_\varepsilon = \int_{\mathbb{R}^N} \frac{v}{\varepsilon} f_\varepsilon \,\mathrm{d}v$. We have

$$\begin{cases} \partial_t \rho_\varepsilon + \nabla_x \cdot J_\varepsilon = 0, \\ \varepsilon^2 \partial_t J_\varepsilon + \mathrm{Div}_x \mathbb{P}_\varepsilon = -\chi \rho_\varepsilon \nabla_x \Phi_\varepsilon - J_\varepsilon \end{cases} \tag{4.53}$$

where furthermore

$$\mathbb{P}_\varepsilon(t,x) = \int_{\mathbb{R}^N} v\otimes v\ f_\varepsilon \,\mathrm{d}v.$$

Using (4.52) yields $\mathbb{P}_\varepsilon(t,x) \simeq \rho(t,x)\ \mathbb{I}$, and thus, passing to the limit in nonlinearities, we are led to

$$\left\{\begin{array}{l}\partial_t \rho + \nabla_x \cdot J = 0,\\ J = -\chi\rho\nabla_x\Phi - \nabla_x\rho,\end{array}\right. \tag{4.54}$$

coupled to the Poisson relation

$$-\Delta\Phi = \rho. \tag{4.55}$$

4.3 Analysis of the diffusive scaling when $N = 2$

The difficulty is two–fold:

- first we should establish a priori estimates, uniform with respect to ε; to this end, entropy dissipation induced by collisions is crucial,

- then, we shall see that exploiting these estimates so that the non linear acceleration term makes sense and passes to the limit can be a bit subtle.

The convergence of solutions of (4.49) to those of (4.54)-(4.55) is established in [81] on small enough interval of time $(0, T_\star)$, under a fast decay assumption on the initial data. Precisely, if, for some $p > N$, $\mathrm{e}^{(p-1)v^2/2} f_\varepsilon^{\mathrm{Init}}$ is bounded in $L^p(\mathbb{R}^N\times\mathbb{R}^N)$, then, there exists $T_\star > 0$ such that a subsequence satisfies

$$\left\{\begin{array}{l}\rho_\varepsilon \to \rho \quad \text{in } L^q(0,T_\star; L^r(\mathbb{R}^N)), \quad 1\le r<p,\ 1\le q<\infty,\\ f_\varepsilon(t,x,v) \to \rho(t,x)\ (2\pi)^{-N/2}\mathrm{e}^{-v^2/2} \quad \text{in } L^q(0,T_\star; L^r(\mathbb{R}^N\times\mathbb{R}^N))\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2\le r<p,\ 2\le q<\infty.\end{array}\right.$$

We shall describe below an alternative approach which applies to the specific case of dimension two and gives the result without any restriction on the time interval nor on the data, considering only bound on entropy and energy.

Theorem 11 *Set $N = 2$. Let $f_\varepsilon^{\mathrm{Init}} \ge 0$ satisfy*

$$\left\{\begin{array}{l}\displaystyle\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\varepsilon^{\mathrm{Init}} \,\mathrm{d}v\,\mathrm{d}x = 1,\\ \displaystyle\sup_{\varepsilon>0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\varepsilon^{\mathrm{Init}}(1+v^2+|x|+|\ln(f_\varepsilon^{\mathrm{Init}})|)\,\mathrm{d}v\,\mathrm{d}x = M_0 < \infty.\end{array}\right. \tag{4.56}$$

Let $0 < T < \infty$. Then, up to a subsequence, ρ_ε converges in $\mathrm{C}^0([0,T]; L^1(\mathbb{R}^2)-weak)$ to ρ, solution of the limit system (4.54-4.55).

Remind that for $N = 2$, we have

$$E_2(x) = -\frac{1}{2\pi}\ln(|x|).$$

A typical feature of the 2D case is that E_2 has no definite sign. The starting point of the analysis consists in establishing a priori estimates, the clue being the energy conservation already remarked when dealing with the collisionless Vlasov-Poisson equation. However, due to the collision, we observe additionally for (4.49) a dissipation mechanism. The key computation reads

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\varepsilon \ln(f_\varepsilon)\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{v^2}{2} f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x + \frac{\chi}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\varepsilon \Phi_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\right\}\\
&\quad = -\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (v f_\varepsilon + \nabla_v f_\varepsilon)^2\,\frac{1}{f_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x = -\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (v\sqrt{f_\varepsilon} + 2\nabla_v\sqrt{f_\varepsilon})^2\,\mathrm{d}v\,\mathrm{d}x\\
&\quad = -\frac{4}{\varepsilon^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|\nabla_v\sqrt{f_\varepsilon e^{v^2/2}}\right|^2 e^{-v^2/2}\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}
\tag{4.57}
$$

It clearly reveals the relaxation effects, in agreement to (4.52).

We will also need some control on the behavior of f_ε at infinity (with respect to the space variable). To this end, we remark that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |x| f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x = \frac{1}{\varepsilon}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} v\cdot\frac{x}{|x|}\,f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\\
&\quad = \frac{1}{\varepsilon}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (v\sqrt{f_\varepsilon} + 2\nabla_v\sqrt{f_\varepsilon})\cdot\frac{x}{|x|}\,\sqrt{f_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x\\
&\quad \leq \left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\frac{v\sqrt{f_\varepsilon} + 2\nabla_v\sqrt{f_\varepsilon}}{\varepsilon}\right|^2\mathrm{d}v\,\mathrm{d}x\right)^{1/2}.
\end{aligned}
\tag{4.58}
$$

Next, the macroscopic entropy can be estimated by means of the microscopic entropy.

Lemma 8 *Let $f : \mathbb{R}^N \times \mathbb{R}^N \longmapsto \mathbb{R}$, with $f \geq 0$. Set $\rho(x) = \int_{\mathbb{R}^N} f(x,v)\,\mathrm{d}v$. Then, we have*

$$
\begin{aligned}
\int_{\mathbb{R}^N} \rho\ln(\rho)\,\mathrm{d}x \ \leq\ & \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f\ln(f)\,\mathrm{d}v\,\mathrm{d}x\\
&+\frac{N}{2}\ln(2\pi)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2}{2} f\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}
$$

Proof. The statement is a direct consequence of the Jensen inequality applied to the convex function $\Psi(s) = s\ln(s)$ and the probability measure on $\mathbb{R}^N$ $(2\pi)^{-N/2}e^{-v^2/2}\,\mathrm{d}v = M(v)\,\mathrm{d}v$. We get

$$
\begin{aligned}
\rho\ln(\rho) \ &=\ \Psi(\rho) = \Psi\Big(\int_{\mathbb{R}^N}\frac{f}{M}\,M\,\mathrm{d}v\Big)\\
&\leq\ \int_{\mathbb{R}^N}\Psi\Big(\frac{f}{M}\Big)\,M\,\mathrm{d}v = \int_{\mathbb{R}^N} f\Big(\frac{v^2}{2} + \ln(f)\Big)\,\mathrm{d}v + \frac{N}{2}\ln(2\pi)\int_{\mathbb{R}^N} f\,\mathrm{d}v.
\end{aligned}
$$

We conclude by integrating with respect to x. ∎

Accordingly, integrating (4.57) with respect to time, we are led to

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\rho_\varepsilon\ln(\rho_\varepsilon)\,\mathrm{d}x + \frac{\chi}{2}\int_{\mathbb{R}^2}\rho_\varepsilon\Phi_\varepsilon\,\mathrm{d}x\\
&\qquad + \frac{4}{\varepsilon^2}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\nabla_v\sqrt{f_\varepsilon e^{v^2/2}}\right|^2 e^{-v^2/2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \leq M_0 + M_0' + \ln(2\pi).
\end{aligned}
\tag{4.59}
$$

However, the potential energy has no sign. Nevertheless, it can be bounded from below, by using the following statement.

Lemma 9 *Let $\rho : \mathbb{R}^2 \to \mathbb{R}$ such that $\rho \geq 0$. Then, for any $k > \mathrm{e}$, we have*

$$\int_{\mathbb{R}^2} \rho\Phi \, \mathrm{d}x \geq -\frac{\ln(k)}{\pi}\left[\frac{1}{2}\left(\int_{\mathbb{R}^2} \rho \, \mathrm{d}x\right)^2 + \frac{1}{k}\int_{\mathbb{R}^2} \rho \, \mathrm{d}x \int_{\mathbb{R}^2} |x|\rho \, \mathrm{d}x\right].$$

Proof. We follow a trick in [35] by introducing the parameter $k > \mathrm{e}$. Since $k \longmapsto \frac{\ln(k)}{k}$ is non increasing on $(\mathrm{e}, +\infty)$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} \rho\Phi \, \mathrm{d}x &= -\frac{1}{2\pi}\iint_{|x-y|\leq k} \ln(|x-y|)\rho(y)\rho(x) \, \mathrm{d}y \, \mathrm{d}x - \frac{1}{2\pi}\iint_{|x-y|\geq k} \ln(|x-y|)\rho(y)\rho(x) \, \mathrm{d}y \, \mathrm{d}x \\
&\geq -\frac{\ln(k)}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho(x)\rho(y) \, \mathrm{d}y \, \mathrm{d}x - \frac{\ln(k)}{2\pi k}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |x-y|\rho(x)\rho(y) \, \mathrm{d}y \, \mathrm{d}x \\
&\geq -\frac{\ln(k)}{2\pi}\left(\int_{\mathbb{R}^2} \rho(x) \, \mathrm{d}x\right)^2 - \frac{\ln(k)}{\pi k}\int_{\mathbb{R}^2} \rho(x) \, \mathrm{d}x \int_{\mathbb{R}^2} |x|\rho(x) \, \mathrm{d}x.
\end{aligned}$$

∎

Coming back to (4.59) leads to

$$\begin{aligned}
&\int_{\mathbb{R}^2} \rho_\varepsilon \ln(\rho_\varepsilon) \, \mathrm{d}x + \frac{4}{\varepsilon^2}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|\nabla_v \sqrt{f_\varepsilon e^{v^2/2}}\right|^2 e^{-v^2/2} \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}s \\
&\leq M_0 + M_0' + \ln(2\pi) + \chi\frac{\ln(k)}{4\pi} + \chi\frac{\ln(k)}{2\pi k}\int_{\mathbb{R}^2} |x|\rho_\varepsilon(x) \, \mathrm{d}v \, \mathrm{d}x.
\end{aligned}$$

Using (4.58), we obtain, for some $\nu > 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^2} \rho_\varepsilon \ln(\rho_\varepsilon) \, \mathrm{d}x + \left(\nu - \chi\frac{\ln(k)}{2k\pi}\right)\int_{\mathbb{R}^2} |x|\rho \, \mathrm{d}x \\
&\qquad + (1-\nu/2)\frac{4}{\varepsilon^2}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|\nabla_v \sqrt{f_\varepsilon e^{v^2/2}}\right|^2 e^{-v^2/2} \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}s \\
&\leq (1+\nu)M_0 + M_0' + \chi\frac{\ln(k)}{4\pi} + \frac{\nu t}{2}.
\end{aligned}$$

We pick $0 < \nu < 2$, and then k large enough so that $\nu > \chi\ln(k)/(2k\pi)$.

We conclude by using a classical trick which will provide an estimate on the non negative quantity $\rho_\varepsilon|\ln(\rho_\varepsilon)|$. Let $\kappa > 0$. We have

$$\begin{aligned}
\rho|\ln(\rho)| &= \rho\ln(\rho) - 2\rho\ln(\rho)\chi_{e^{-\kappa|x|}\leq\rho\leq 1} - 2\rho\ln(\rho)\chi_{0\leq\rho\leq e^{-\kappa|x|}} \\
&\leq \rho\ln(\rho) + 2\kappa|x|\rho + K\sqrt{\rho}\chi_{0\leq\rho\leq e^{-\kappa|x|}} \\
&\leq \rho\ln(\rho) + 2\kappa|x|\rho + Ke^{-\kappa|x|/2},
\end{aligned}$$

for some $K > 0$. Hence, we are led to

$$\begin{aligned}
&\int_{\mathbb{R}^2} \rho_\varepsilon|\ln(\rho_\varepsilon)| \, \mathrm{d}x + \left(\nu - 2\kappa - \chi\frac{\ln(k)}{2k\pi}\right)\int_{\mathbb{R}^2} |x|\rho_\varepsilon \, \mathrm{d}x \\
&\qquad + \left(1 - \frac{\nu}{2}\right)\frac{4}{\varepsilon^2}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|\nabla_v \sqrt{f_\varepsilon e^{v^2/2}}\right|^2 e^{-v^2/2} \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}s \\
&\leq (1+\nu)M_0 + M_0' + \chi\frac{\ln(k)}{4\pi} + \frac{\nu t}{2} + K\int_{\mathbb{R}^2} e^{-\kappa|x|/2} \, \mathrm{d}x.
\end{aligned}$$

Summarizing, we proved

Proposition 6 *Suppose (4.56). Let* $0 < T < \infty$. *Then,*
i) $\rho_\varepsilon(1 + |x| + |\ln(\rho_\varepsilon)|)$ *is bounded in* $L^\infty(0, T; L^1(\mathbb{R}^2))$*;*
ii) $\left|\nabla_v \sqrt{f_\varepsilon e^{v^2/2}}\right|^2 e^{-v^2/2}$ *is bounded in* $L^1((0, T) \times \mathbb{R}^2 \times \mathbb{R}^2)$.

Corollary 5 *Let the assumptions of Proposition 6 be fulfilled. Then* $v^2 f_\epsilon$ *is bounded in* $\mathrm{L}^1((0,T) \times \mathbb{R}^2 \times \mathbb{R}^2)$.

Proof. We note that

$$\begin{aligned} 0 &\le \int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} v^2 f_\epsilon \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\ &= \int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Big(|v\sqrt{f_\epsilon} + 2\nabla_v\sqrt{f_\epsilon}|^2 - 4|\nabla_v\sqrt{f_\epsilon}|^2 - 4v\sqrt{f_\epsilon}\cdot\nabla_v\sqrt{f_\epsilon}\Big)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\ &\le \int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |v\sqrt{f_\epsilon} + 2\nabla_v\sqrt{f_\epsilon}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s + 0 + 4\int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\epsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s. \end{aligned}$$

By Proposition 6, it is dominated by $C_T\epsilon^2 + 4T$, where C_T depende only on T and the initial data.

∎

We go back to the moments system (4.53), where as a consequence of Corollary 5, J_ϵ and $\mathbb{P}_\epsilon$ are bounded in $\mathrm{L}^1((0,T) \times \mathbb{R}^2)$. We go a step further by remarking

$$\begin{aligned} \mathbb{P}_\epsilon &= \int_{\mathbb{R}^2} v\sqrt{f_\epsilon} \otimes (v\sqrt{f_\epsilon} + 2\nabla_v\sqrt{f_\epsilon})\,\mathrm{d}v - 2\int_{\mathbb{R}^2} v\sqrt{f_\epsilon}\otimes\nabla_v\sqrt{f_\epsilon}\,\mathrm{d}v \\ &= \int_{\mathbb{R}^2} v\sqrt{f_\epsilon} \otimes (v\sqrt{f_\epsilon} + 2\nabla_v\sqrt{f_\epsilon})\,\mathrm{d}v + \rho_\epsilon\,\mathbb{I} = \mathcal{O}_{L^1}(\varepsilon) + \rho_\varepsilon\mathbb{I}. \end{aligned}$$

We can suppose that

$$\rho_\varepsilon \rightharpoonup \rho, \qquad J_\varepsilon \rightharpoonup J,$$

weakly in $L^1((0,T) \times \mathbb{R}^2)$ (by virtue of the Dunford-Pettis theorem, see [39]) and in the vague topology for bounded measures on $(0,T) \times \mathbb{R}^2$, respectively. Furthermore, we can use a convenient adaptation of Lemma 5 and justify that ρ_ε converges to ρ in $C^0([0,T]; L^1(\mathbb{R}^N) - \text{weak})$. Letting $\varepsilon \to 0$ in (4.53) yields

$$\begin{cases} \partial_t\rho + \nabla_x \cdot J = 0, \\ J + \chi\nabla_x\rho = -\lim\limits_{\varepsilon\to 0} \rho_\varepsilon\nabla_x\Phi_\varepsilon \end{cases}$$

in the $\mathcal{D}'((0,T) \times \mathbb{R}^2)$ sense. It remains to identify the limit of the nonlinear term $\rho_\varepsilon\nabla_x\Phi_\varepsilon$, but we point out that there is no direct argument providing a clear functional framework for the potential by exploiting the available estimates.

The idea uses the symmetry properties of the Poisson kernel: for $\varphi \in \big(\mathrm{C}_c^\infty(\mathbb{R}^2)\big)^2$, we write

$$\begin{aligned} \big\langle \rho_\varepsilon\nabla_x\Phi_\varepsilon, \varphi\big\rangle &= \frac{-\gamma}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho_\varepsilon(t,x)\rho_\varepsilon(t,y)\,\frac{x-y}{|x-y|^2}\cdot\varphi(x)\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{-\gamma}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho_\varepsilon(t,x)\rho_\varepsilon(t,y)\,\frac{x-y}{|x-y|^2}\cdot\big(\varphi(x) - \varphi(y)\big)\,\mathrm{d}y\,\mathrm{d}x. \end{aligned}$$

This idea, which appears in [81], is reminiscient to the study of weak solutions of the two-dimensional Euler equations by Schochet [88], and for further developpements we refer to [79]. The advantage is that

$$\begin{array}{ccc} \mathbb{R}^2 \times \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x,y) & \longmapsto & \dfrac{x-y}{|x-y|^2} \cdot \big(\varphi(x) - \varphi(y)\big) \end{array}$$

belongs to $L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$: (it is bounded by $\|\nabla\varphi\|_{L^\infty}$ and not well-defined on the diagonal $\{(x,x),\ x \in \mathbb{R}^2\}$ which is a negligeable set of $\mathbb{R}^2 \times \mathbb{R}^2$. Therefore, we obtain

$$\lim_{\varepsilon \to 0} \langle \rho_\varepsilon \nabla_x \Phi_\varepsilon, \varphi \rangle = \frac{-\gamma}{4\pi} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho(t,x)\rho(t,y)\ \frac{x-y}{|x-y|^2} \cdot \big(\varphi(x) - \varphi(y)\big)\ \mathrm{d}y\ \mathrm{d}x,$$

for any time $t \in [0,T]$. We conclude that ρ is a solution of (4.54-4.55) in the sense that

$$\left\{ \begin{array}{l} \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_{\mathbb{R}^2} \rho(t,x)\varphi(x)\ \mathrm{d}x = \chi \int_{\mathbb{R}^2} \rho(t,x)\Delta\varphi(x)\ \mathrm{d}x \\ \qquad\qquad + \dfrac{\gamma}{4\pi} \displaystyle\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho(t,x)\rho(t,y)\ \frac{x-y}{|x-y|^2} \cdot \big(\nabla\varphi(x) - \nabla\varphi(y)\big)\ \mathrm{d}y\ \mathrm{d}x, \\ \displaystyle\int_{\mathbb{R}^2} \rho(t,x)\varphi(x)\ \mathrm{d}x\Big|_{t=0} = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^0_\varepsilon(x)\varphi(x)\ \mathrm{d}v\ \mathrm{d}x. \end{array} \right. \tag{4.60}$$

holds in $\mathcal{D}'([0,+\infty))$ for any test function $\varphi \in \mathrm{C}^\infty_c(\mathbb{R}^2)$.

4.4 Attractive potential

It is interesting to look at the same problem in the in the attractive case that is by reversing the sign in the Poisson relation: (4.49) is now coupled to

$$\Delta_x \Phi_\varepsilon = \rho_\varepsilon$$

that is

$$\Phi_\varepsilon(t,x) = -\int_{\mathbb{R}^N} E_N(x-y)\rho_\varepsilon(t,y)\,\mathrm{d}y.$$

The asymptotic problem is detailed on physical grounds in [24], motivated by stellar physics modelling. As $\varepsilon \to 0$ we are led to

$$\left\{ \begin{array}{l} \partial_t \rho - \mathrm{div}_x(\chi\rho\nabla_x\Phi + \nabla_x\rho) = 0 \\ \Delta_x \Phi = \rho. \end{array} \right. \tag{4.61}$$

which is referred to as the Smoluchowski system. Far later, the system (4.61) has also been proposed as a model describing the evolution of certain biological systems subject to aggregation dynamics by Keller-Segel [62]. The analysis of the system is amazing since the solution can exhibit concentration as a Dirac mass in finite time. The singular behavior depends on a threshold involving the parameter χ.

Proposition 7 *We suppose that the initial data for* (4.61) *is a smooth, say* C^1*, normalized function*

$$\int_{\mathbb{R}^2} \rho(0,x)\,\mathrm{d}x = 1$$

with a finite second moment. Then, if $\chi > 8\pi$*, the solution does not remain smooth for any time.*

Proof. Clearly, mass conservation holds

$$\int_{\mathbb{R}^2} \rho(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^2} \rho(0,x)\,\mathrm{d}x = 1.$$

We compute the time derivative of the second order moment

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} \frac{x^2}{2}\rho\,\mathrm{d}x &= -\int_{\mathbb{R}^2} x\cdot(\chi\rho\nabla_x\Phi + \nabla_x\rho)\,\mathrm{d}x \\
&= 2\int_{\mathbb{R}^2}\rho\,\mathrm{d}x - \frac{\chi}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} x\cdot\frac{x-y}{|x-y|^2}\rho(t,x)\rho(t,y)\,\mathrm{d}y\,\mathrm{d}x \\
&= 2\int_{\mathbb{R}^2}\rho\,\mathrm{d}x - \frac{\chi}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (x-y)\cdot\frac{x-y}{|x-y|^2}\rho(t,x)\rho(t,y)\,\mathrm{d}y\,\mathrm{d}x \\
&= 2\int_{\mathbb{R}^2}\rho\,\mathrm{d}x - \frac{\chi}{4\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho(t,x)\rho(t,y)\,\mathrm{d}y\,\mathrm{d}x = 2\Big(1-\frac{\chi}{8\pi}\Big).
\end{aligned}$$

When $\chi > 8\pi$, it contradicts the fact that $\int_{\mathbb{R}^2} x^2\rho\,\mathrm{d}x$ is non negative for any time. ∎

In dimension 2 $\chi = 8\pi$ splits really the behavior of the solutions, which are globally defined when $\chi < 8\pi$. We refer to [61, 41, 58] and to the crystal clean recent proof in [36]. The critical case is detailed in [4] and for a suitable measure-valued framework we mention [79] and [37]. The situation is more involved in higher dimensions or in bounded domains [59] As far as we are concerned with the hydrodynamic aymptotics question, we can prove the following result which guarantees global convergence provided there is no blow up in the limit equation.

Theorem 12 *We suppose* $\chi < 8\pi$*. Then Theorem 11 applies to the attractive case as well.*

The difficulty relies on the estimation of the contribution of the potential energy. Hence, the proof differs from the repulsive case by the replacement of Lemma 9 by the following claim, which is a direct application of Theorem 2 of [10] (see also [19]).

Lemma 10 *Let* $\rho : \mathbb{R}^2 \to \mathbb{R}$ *such that* $\rho \geq 0$ *and* $\int_{\mathbb{R}^2}\rho\ \mathrm{d}x = 1$*. Then, there exists a constant* $C_* > 0$ *such that*

$$-4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho(x)\rho(y)\ln(|x-y|)\mathrm{d}y\ \mathrm{d}x \leq C_* + 2\int_{\mathbb{R}^2}\rho\ln(\rho)\ \mathrm{d}x.$$

Accordingly, we get in the attractive case

$$\int_{\mathbb{R}^2} \rho\Phi \, \mathrm{d}x \geq -\frac{C_*}{8\pi} - \frac{1}{4\pi}\int_{\mathbb{R}^2} \rho \ln(\rho) \, \mathrm{d}x.$$

Then, using this inequality in the attractive version of (4.59), we get

$$\left(1-\frac{\chi}{8\pi}\right)\int_{\mathbb{R}^2} \rho_\varepsilon \ln(\rho_\varepsilon)\, \mathrm{d}x + \frac{4}{\varepsilon^2}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|\nabla_v \sqrt{f_\varepsilon e^{v^2/2}}\right|^2 e^{-v^2/2}\, \mathrm{d}v\, \mathrm{d}x\, \mathrm{d}s$$
$$\leq M_0 + M_0' + \ln(2\pi) + \chi\frac{C_*}{16\pi}$$

where the threshold clearly appears. The other arguments can be repeated *mutadis mutandis*. ∎

The scaling (4.48) leads to the same coupled system, up to the diffusion term. We refer to [74, 51] for the asymptotic analysis and to [73] for the derivation of the transport equation from the convection-diffusion system.

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