Miracles of holomorphic motions

Malik Younsi, University of Hawaii

(Joint works with A. Furher and T. Ransford)

The basics: quasiconformal mappings

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Definition

An orientation-preserving homeomorphism $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is **quasiconformal** if it belongs to $W^{1,2}_{loc}(\widehat{\mathbb{C}})$ and satisfies the **Beltrami equation**

$$\frac{\partial f}{\partial \overline{z}} = \mu \frac{\partial f}{\partial z}$$

almost everywhere, for some measurable function μ on $\widehat{\mathbb{C}}$ with $\|\mu\|_\infty < 1$.

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- $\mu = \mu_f$ is the **Beltrami coefficient** or **complex dilatation** of f (measure of non-conformality).

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- $\mu = \mu_f$ is the **Beltrami coefficient** or **complex dilatation** of f (measure of non-conformality).
- k-quasiconformal: $\|\mu_f\|_\infty \leq k < 1$.

The measurable Riemann mapping theorem

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Theorem (Morrey (1938))

Let $\mu : \widehat{\mathbb{C}} \to \mathbb{D}$ *be a measurable function with* $\|\mu\|_\infty < 1$.

- *There exists a quasiconformal mapping* f *such that*

$$\frac{\partial f}{\partial \overline{z}} = \mu \frac{\partial f}{\partial z}$$

  almost everywhere on $\widehat{\mathbb{C}}$.
- *The map* f *is unique up to composition by a Möbius transformation.*

Quasiconformal distortion: some classical theorems

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Theorem (Astala (Acta Math.,1994))

Let $F:\mathbb{C}\to\mathbb{C}$ *be k-quasiconformal, and let A be a subset of* $\mathbb{C}$ *such that* $\dim_H(A)>0$*. Then*

$$\frac{1}{K}\Big(\frac{1}{\dim_H A}-\frac{1}{2}\Big)\leq\Big(\frac{1}{\dim_H F(A)}-\frac{1}{2}\Big)\leq K\Big(\frac{1}{\dim_H A}-\frac{1}{2}\Big),$$

where $K:=(1+k)/(1-k)$*.*

Quasiconformal distortion: some classical theorems

Theorem (Astala (Acta Math.,1994))

The estimates are sharp:
for each $0 < t < 2$ *and* $K \geq 1$, *there exist a set* $A \subset \mathbb{C}$ *with* $\dim_H(A) = t$ *and* $F : \mathbb{C} \to \mathbb{C}$ *k-quasiconformal such that*

$$\Big(\frac{1}{\dim_H F(A)} - \frac{1}{2}\Big) = K\Big(\frac{1}{t} - \frac{1}{2}\Big),$$

where $K := (1+k)/(1-k)$.

Quasiconformal distortion: some classical theorems

Theorem (Smirnov (Acta Math., 2010))

If $F : \mathbb{C} \to \mathbb{C}$ *is* k*-quasiconformal, then*

$$\dim_H(F(\mathbb{R})) \leq 1 + k^2.$$

Quasiconformal distortion: some classical theorems

Theorem (Astala (1994), Eremenko–Hamilton (1995))

Let $F : \mathbb{C} \to \mathbb{C}$ be a k-quasiconformal homeomorphism which is conformal on $\mathbb{C} \setminus \overline{\mathbb{D}}$, with $F(z) = z + o(1)$ near ∞. If $A \subset \overline{\mathbb{D}}$ is a Borel set then

$$|F(A)| \leq K\pi^{1-1/K}|A|^{1/K},$$

where again $K = (1+k)/(1-k)$.

A common framework?

Question

Is there a unified approach to quasiconformal distortion theorems?

A common framework?

Question

Is there a unified approach to quasiconformal distortion theorems?

- Yes! Using holomorphic motions and inf-harmonic functions.

Miracles of holomorphic motions

"What draws us to mathematics and maintains its spell upon us? [...] A beautiful example of such phenomena are no doubt the Holomorphic Motions, discovered by Mané, Sad and Sullivan and then developed by many others, that have hold their magic since their invention in the mid 80's. The notion can be explained in few lines to anyone interested in mathematics, and with an intuitive interpretation even beyond - a deformation of the space with a nonstandard concept of time; time is assumed to vary holomorphically. Nothing more is assumed and yet the conclusions are very strong and unexpected. [...] One cannot avoid an everlasting astonishment!"

- **Kari Astala** in **Miracles of holomorphic motions**.

Holomorphic motions

Holomorphic motions

Definition

A **holomorphic motion** of a set $A \subset \widehat{\mathbb{C}}$ is a map $f : \mathbb{D} \times A \to \widehat{\mathbb{C}}$ such that

(i) for each fixed $z \in A$, the map $\lambda \mapsto f(\lambda, z)$ is holomorphic on $\mathbb{D}$,

(ii) for each fixed $\lambda \in \mathbb{D}$, the map $z \mapsto f(\lambda, z)$ is injective on A,

(iii) $f(0, z) = z$ for all $z \in A$.

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(iii) $f(0, z) = z$ for all $z \in A$.

- We write

$$f_\lambda(z) := f(\lambda, z) \qquad (\lambda \in \mathbb{D}, z \in A).$$

- We assume each f_λ fixes $0, 1, \infty$.

An Important Observation

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Observation (Mané–Sad–Sullivan (1983))

Julia sets often move holomorphically with holomorphic variations of the coefficients.

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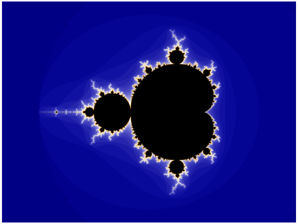

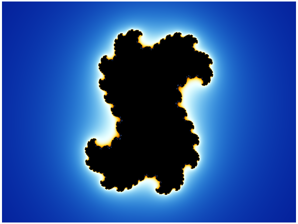

Another Astala quote!

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"In practical terms, we are discussing deformations with no assumptions at all on the space variable - continuity or even measurability - while the key property is the holomorphic dependence required from the time parameter. As innocent as this mere last assumption may appear, let us see how far it can take us - be ready for surprises!"

- **Kari Astala** in **Miracles of holomorphic motions**.

The λ-lemma

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Theorem (λ-lemma, Mané–Sad–Sullivan (1983))

Every holomorphic motion $f : \mathbb{D} \times A \to \widehat{\mathbb{C}}$ has an extension to a holomorphic motion $F : \mathbb{D} \times \overline{A} \to \widehat{\mathbb{C}}$.
Moreover, the map F is jointly continuous in (λ, z).

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Moreover, the map F is jointly continuous in (λ, z).

- *"The Mañé–Sad–Sullivan proof was so simple and the result so unexpected, that - to be honest - upon seeing the argument each of us complex analysts stood in bewilderment - how is this possible?"* - **Kari Astala**.

Słodkowski's theorem

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Every holomorphic motion $f : \mathbb{D} \times A \to \widehat{\mathbb{C}}$ has an extension to a holomorphic motion of $\widehat{\mathbb{C}}$.

- For each $\lambda \in \mathbb{D}$, the map $f_\lambda : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a homeomorphism. What is its precise degree of regularity?

Holomorphic motions and quasiconformal mappings

Holomorphic motions and quasiconformal mappings

Theorem (Extended λ-lemma)

Let $f : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. The following are equivalent:

- *The map f is a holomorphic motion.*
- *For each $\lambda \in \mathbb{D}$, the map $f_\lambda : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is $|\lambda|$-quasiconformal. Moreover, the $L^\infty(\mathbb{C})$-valued map $\lambda \mapsto \mu_{f_\lambda}$ is holomorphic on $\mathbb{D}$.*

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- Every quasiconformal mapping can be embedded into a holomorphic motion.
- Holomorphic motions $\longleftrightarrow$ families of quasiconformal mappings depending holomorphically on the parameter $\lambda \in \mathbb{D}$.

Holomorphic motions and Loewner theory

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  Solving $\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - W_t^\alpha}$ where $\alpha \in \mathbb{D}$, $W_t^\alpha := \alpha\lambda(t)$ and $\lambda : [0,1] \to \mathbb{C}$ is a suitable 1/2-Hölder function, gives a holomorphic motion of a line segment.

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Question

What if $W_t^\alpha := \alpha B_t$? SLE with complex κ?

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- *Loewner evolution driven by complex Brownian motion* by Gwynne and Pfeffer (2023).

A general question

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- $\mathcal{F}$: real-valued function $\mathcal{F}$ defined on subsets of $\mathbb{C}$
- $f : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ holomorphic motion, $A \subset \mathbb{C}$.

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How does the function

$$\lambda \mapsto \mathcal{F}(f_\lambda(A)) \qquad (\lambda \in \mathbb{D})$$

behave?

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Question

How does the function

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behave?

- Studied by Ruelle, Ransford, Astala, Astala–Zinsmeister, Eremenko–Hamilton, Earle–Mitra, Pouliasis, Zakeri, Ransford–Y.–Ai, Y., Furher–Ransford–Y., etc.

Some results

Some results

$\mathcal{F}$	**continuous?**	**Regularity known?**
$\mathcal{H}^s \quad (0 < s < 2)$	no	
Area	yes	no
Hausdorff dimension	yes	almost!
Packing dimension	yes	yes
(upper) Minkowski dimension	yes	almost!
Logarithmic capacity	yes	no
Condenser capacity	yes	no
Analytic capacity	no	
Continuous analytic capacity	no	

A first example: logarithmic capacity

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Definition

For $E \subset \mathbb{C}$ compact and $n \geq 2$, we define the **n-th diameter** of E by

$$\delta_n(E) := \sup \left\{ \prod_{j<k} |w_j - w_k|^{2/n(n-1)} : w_1, \ldots, w_n \in E \right\}.$$

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Theorem (Fekete–Szegő)

The sequence $(\delta_n(E))_{n \geq 2}$ is decreasing, and

$$\lim_{n \to \infty} \delta_n(E) = c(E),$$

where $c(E)$ is the **logarithmic capacity** *or* **transfinite diameter** *of E.*

Continuity of logarithmic capacity under holomorphic motions

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Theorem (Ransford–Y.–Ai (2020))

Let $A \subset \mathbb{C}$ be compact and let $f : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a holomorphic motion. Then

$$\lambda \mapsto c(f_\lambda(A)) \qquad (\lambda \in \mathbb{D})$$

is continuous.

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- For $z_1, \ldots, z_n \in A$ distinct, $\lambda \mapsto \log \left| \prod_{j<k} (f_\lambda(z_j) - f_\lambda(z_k)) \right|$ is harmonic on $\mathbb{D}$

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- $\lambda \mapsto -\log c(f_\lambda(A))$ is continuous on $\mathbb{D}$ (Harnack's inequality!).

Variation of dimension

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Question

If $f : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a holomorphic motion and if A is a subset of $\mathbb{C}$, what kind of function is

$$\lambda \mapsto \dim(f_\lambda(A)) \qquad (\lambda \in \mathbb{D})?$$

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Question

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$$\lambda \mapsto \dim(f_\lambda(A)) \qquad (\lambda \in \mathbb{D})?$$

- Here dim denotes any reasonable notion of dimension, such as Hausdorff, packing or (upper) Minkowski.

Hausdorff dimension

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Definition

Let $A \subset \mathbb{C}$. For $s \geq 0$, we define the s-**dimensional Hausdorff measure** of A by

$$\mathcal{H}^s(A) := \lim_{\delta \to 0} \left(\inf \left\{ \sum_{j=1}^{\infty} \mathrm{diam}(A_j)^s : A \subset \bigcup_j A_j, \mathrm{diam}(A_j) \leq \delta \right\} \right).$$

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Definition

The **Hausdorff dimension** of A is the unique real number $\dim_H(A) \in [0, 2]$ such that

$$\mathcal{H}^s(A) = \begin{cases} \infty, & s < \dim_H(A), \\ 0, & s > \dim_H(A). \end{cases}$$

Packing dimension

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Definition

Let $A \subset \mathbb{C}$. The *s*-**dimensional pre-packing measure** of A is

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Minkowski dimension

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Definition

Let $A \subset \mathbb{C}$ be bounded. The **(upper) Minkowski dimension** of A is

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where $N_\delta(A)$ is the smallest number of sets of diameter at most δ needed to cover A.

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- $\dim_H(A) \leq \dim_P(A) \leq \overline{\dim}_M(A)$

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Recall Astala's inequality:

$$\frac{1-k}{1+k}\Big(\frac{1}{\dim_H A}-\frac{1}{2}\Big) \leq \Big(\frac{1}{\dim_H F(A)}-\frac{1}{2}\Big) \leq \frac{1+k}{1-k}\Big(\frac{1}{\dim_H A}-\frac{1}{2}\Big).$$

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Embed F into a holomorphic motion $f : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$:

$$\frac{1-|\lambda|}{1+|\lambda|}\Big(\frac{1}{\dim_H A}-\frac{1}{2}\Big) \leq \Big(\frac{1}{\dim_H(f_\lambda(A))}-\frac{1}{2}\Big) \leq \frac{1+|\lambda|}{1-|\lambda|}\Big(\frac{1}{\dim_H A}-\frac{1}{2}\Big).$$

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- Harnack inequality applied to $\lambda \mapsto 1/\dim_H(f_\lambda(A)) - 1/2$!

Improved Harnack inequality

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In the special case $A = \mathbb{R}$, we are able to use symmetry in order to obtain an improved Harnack inequality:

$$\frac{1-|\lambda|^2}{1+|\lambda|^2}\Big(\frac{1}{\dim_H \mathbb{R}}-\frac{1}{2}\Big) \leq \Big(\frac{1}{\dim_H(f_\lambda(\mathbb{R}))}-\frac{1}{2}\Big) \leq \frac{1+|\lambda|^2}{1-|\lambda|^2}\Big(\frac{1}{\dim_H \mathbb{R}}-\frac{1}{2}\Big).$$

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Setting $\lambda = k$, the lower bound gives

$$\frac{1}{\dim_H(F(\mathbb{R}))} - \frac{1}{2} \ge \frac{1-k^2}{1+k^2}\Big(1-\frac{1}{2}\Big)$$

and

$$\frac{1}{\dim_H(F(\mathbb{R}))} \ge \frac{1}{2}\frac{1-k^2}{1+k^2} + \frac{1}{2} = \frac{1}{1+k^2}.$$

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$$\frac{1}{\dim_H(F(\mathbb{R}))} \geq \frac{1}{2}\frac{1-k^2}{1+k^2} + \frac{1}{2} = \frac{1}{1+k^2}.$$

This is Smirnov's theorem!

Inf-harmonic functions

Inf-harmonic functions

Definition

Let D be a domain in $\mathbb{C}$. A positive function $u : D \to [0, \infty)$ is called **inf-harmonic** if there exists a family $\mathcal{H}$ of harmonic functions on D such that

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- Inf-harmonic functions are continuous, superharmonic, and satisfy the Harnack inequality.

Variation of dimension under holomorphic motions

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Theorem (Fuhrer–Ransford–Y. (2023))

Let $f : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *be a holomorphic motion and let* $A \subset \mathbb{C}$ *with* $\dim(A) > 0$. *Then*

$$\lambda \mapsto \frac{1}{\dim(f_\lambda(A))} - \frac{1}{2} \qquad (\lambda \in \mathbb{D})$$

is inf-harmonic.

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Let $d : \mathbb{D} \to (0, 2]$ be a function such that $1/d$ is inf-harmonic on $\mathbb{D}$. Then there exists a holomorphic motion $f : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and a compact set $A \subset \mathbb{C}$ such that

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- Gives a complete characterization of the variation of the packing dimension of a set moving under a holomorphic motion.

An interesting consequence: lack of regularity

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Corollary (Fuhrer–Ransford–Y. (2023))

Given a concave function $\psi : \mathbb{D} \to [0, \infty)$*, there exists a holomorphic motion* $f : \mathbb{D} \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *and a compact set* $A \subset \mathbb{C}$ *such that*

$$\dim_H(f_\lambda(A)) = \dim_P(f_\lambda(A)) = \frac{2}{1 + \psi(\lambda)} \quad (\lambda \in \mathbb{D}).$$

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No Sobolev regularity needed.

An open problem: quasiconformal deformation of the Sierpinski carpet

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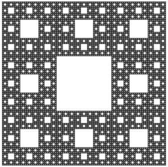

Figure: The Sierpinski carpet S

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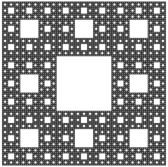

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Question

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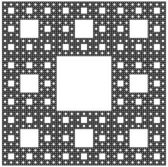

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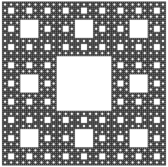

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- Inf-harmonic functions with global maximum at 0?

THANK YOU!