The polynomial method for spectral gaps of random hyperbolic surfaces

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Durham University

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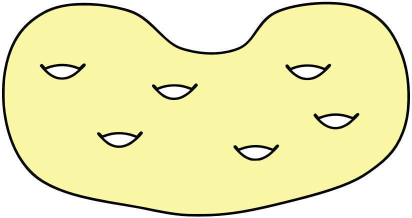
$$\mathbb{H} = \{x + iy : y > 0\},$$

with Riemannian metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}.$$



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- We are interested in the Laplacian Δ acting on $L^2(X)$. On $\mathbb{H}$ it is given by

$$\Delta_{\mathbb{H}} = -y^2 \left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} \right).$$

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- 0 is a simple eigenvalue arising from constant functions.

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- λ_1 is the spectral gap of the Laplacian. It is related to the geometry of the surface and the dynamics of its geodesic flow.

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So for surfaces of large genus (= large volume), $\frac{1}{4}$ is an asymptotically optimal spectral gap.

- Hide and Magee (2021) proved that there exists such a sequence $(X_i)_i$ with $\lambda_1(X_i) \to \frac{1}{4}$, confirming a conjecture of Buser.

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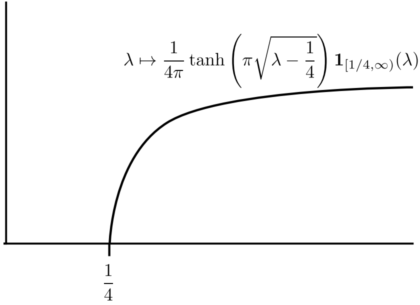
- For the hyperbolic plane this is the Plancherel measure:



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- The square-root behaviour of the Plancherel measure leads to the conjecture that for reasonable random surface models, there will be a constant $C > 0$ such that

$$Cg^{\frac{2}{3}}\left(\lambda_1 - \frac{1}{4}\right),$$

should be TW_1 distributed as $g \to \infty$.

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- Let $\mathcal{M}_g$ denote the collection of hyperbolic surfaces with genus g identified up to isometry.
- The moduli space arises as the quotient of the corresponding Teichmüller space $\mathcal{T}_g$ by the mapping class group. Teichmüller space carries a natural symplectic form ω_{WP} called the Weil-Petersson form which is invariant under the mapping class group action and so descends to $\mathcal{M}_g$.

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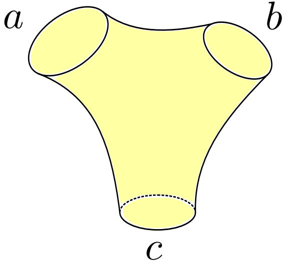
- Given $a, b, c > 0$, there is a unique (up to isometry) pair of pants with (geodesic) boundary lengths a, b, c.



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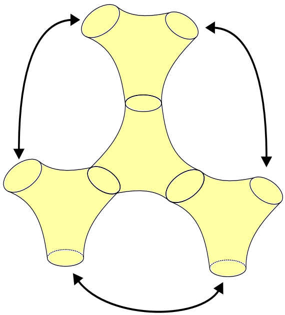

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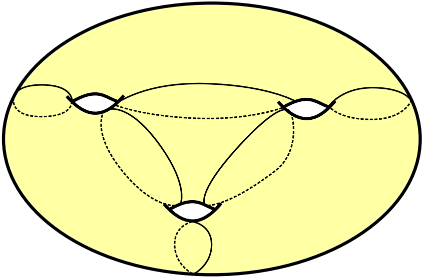

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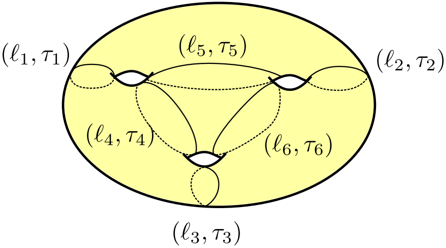
- We can vary the lengths of the pants boundaries ℓ_i and the choice of isometries τ_i (parametrised by a real number) to get coordinates on $\mathcal{T}_g \cong \mathbb{R}_{>0}^{3g-3} \times \mathbb{R}^{3g-3}$.



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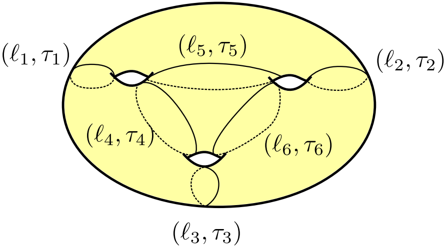

- In these coordinates, (Wolpert 1982)

$$\omega_{\mathrm{WP}} = \frac{1}{2} \sum_{i=1}^{3g-3} \mathrm{d}\ell_i \wedge \mathrm{d}\tau_i.$$

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- We can also define $\mathcal{M}_{g,n}(\ell_1, \ldots, \ell_n)$ and $V_{g,n}(\ell_1, \ldots, \ell_n)$ for surfaces with boundaries.

λ_1 for Weil-Petersson random surfaces: some history

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Theorem (Hide, Macera, T. 2025)

There exists $c > 0$ such that

$$\lim_{g \to \infty} \mathbb{P}_g \left(X : \lambda_1(X) > \frac{1}{4} - \frac{1}{g^c} \right) = 1.$$

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- Our proof develops the polynomial method for use with the Selberg trace formula.
- In doing so, we are required to develop new asymptotic expansions of Weil-Petersson volumes of moduli space (showing that they lie in a Gevrey class).

The trace formula

The Selberg trace formula for a closed hyperbolic surface for an even $f \in C_c^\infty(\mathbb{R})$ is given by

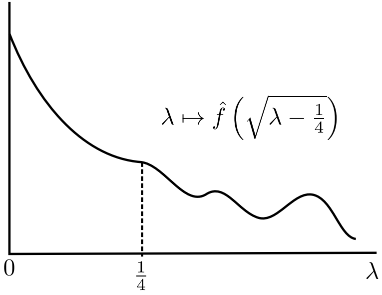
$$\sum_{j=0}^{\infty} \hat{f}\left(\sqrt{\lambda_j - \frac{1}{4}}\right) = (g-1)\int_{\frac{1}{4}}^{\infty} \hat{f}\left(\sqrt{r-\frac{1}{4}}\right) \tanh\left(\pi\sqrt{r-\frac{1}{4}}\right) \mathrm{d}r$$
$$+ \sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^{\infty} \frac{\ell_\gamma(X)}{2\sinh\left(\frac{k\ell_\gamma(X)}{2}\right)} f(k\ell_\gamma(X)),$$

where

- $\mathcal{P}(X)$ is the collection of primitive oriented closed geodesics on X,
- $\ell_\gamma(X)$ is the length of γ.

Our choice of f

Basic properties of the Fourier transform lets us choose $f \geq 0$ to be even, have support equal to $(-1, 1)$ and $\hat{f}$ to look like



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- This results in having to understand geodesics of lengths up to scales $c \log(g)$ on the right-hand side.

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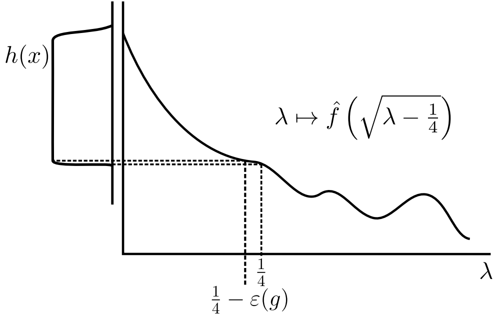
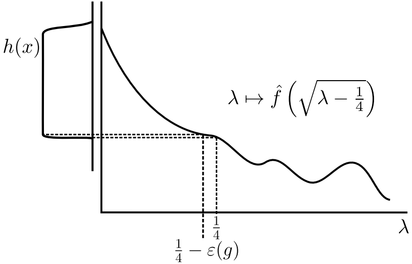


Choose h to be a smooth bump function that is 1 on $[\hat{f}(i\sqrt{\varepsilon(g)}), \hat{f}(\frac{i}{2})]$ and zero out of $[\hat{f}(0), \hat{f}(i)]$.

Polynomial method



Then by Markov's inequality,

$$\begin{aligned}
&\mathbb{P}_g \left(X : \lambda_1(X) < \frac{1}{4} - \varepsilon(g) \right) \\
&\leq \mathbb{P}_g \left(X : \mathrm{Tr}\ h \circ \hat{f}\left(\sqrt{\Delta - \frac{1}{4}}\right) - h \circ \hat{f}(\tfrac{i}{2}) \geq 1 \right) \\
&\leq \mathbb{E}_g \left[\mathrm{Tr}\ h \circ \hat{f}\left(\sqrt{\Delta - \frac{1}{4}}\right) - h \circ \hat{f}(\tfrac{i}{2}) \right].
\end{aligned}$$

Polynomial method

The desire is to obtain an expansion for the expected trace in the following form: there exist compactly supported distributions ν_0 and ν_1 on $C^\infty(\mathbb{R})$ such that for any $\tilde{h} \in C^\infty(\mathbb{R})$, if $h(x) = x\tilde{h}(x)$, then

$$\left| \mathbb{E}_g \frac{1}{g} \mathrm{Tr}\ h \circ \hat{f}\left(\sqrt{\Delta - \tfrac{1}{4}}\right) - \frac{1}{g} h \circ \hat{f}(\tfrac{i}{2}) - \nu_0(\tilde{h}) - \frac{\nu_1(\tilde{h})}{g} \right| \leq \frac{\mathrm{Const}(\tilde{h})}{g^2}.$$

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- Since $\hat{f}(0) > 0$, our bump function h vanishes at zero so we can choose the smooth function $\tilde{h}(x) = \frac{h(x)}{x}$.
- The ν_0 and ν_1 will be explicit enough to show that they are supported in $[-\hat{f}(0), \hat{f}(0)]$ and hence vanish for our bump function.

Polynomial method

- The control over the error term $\mathrm{Const}(\tilde{h})$ is the **crucial** part.
- The polynomial method allows us to obtain an error term that depends only on the sup norm of some number of derivatives of $\tilde{h}$:

$$\mathrm{Const}(\tilde{h}) = C \left\| \tilde{h}\left(\hat{f}(\tfrac{i}{2}) \cos(\theta) \right) \right\|_{C^m([0,2\pi])}.$$

Since we wish for h to smooth out to zero on a scale $\varepsilon(g)$, this will be bounded by $C\varepsilon(g)^{-\frac{m}{2}}$ allowing for us to take $\varepsilon(g) = g^{-c}$.

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- The extensions rely on classical inequalities for polynomials. The coefficients in the expansion become compactly supported distributions.

- We use the random model and geometry to compute the support of the distributions, showing that the first two orders are supported only on functions with support on the eigenvalues above $\frac{1}{4}$.

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- The actual support can then be determined through moments: The support is contained in $[-\rho, \rho]$ if

$$\limsup_{p \to \infty} |\nu(x^p)|^{\frac{1}{p}} \leq \rho.$$

Polynomial method

The expansion is obtained from the Selberg trace formula and using Mirzakhani's integration formula to understand

$$\mathbb{E}_g \sum_{\gamma \in \mathcal{P}(X)} \sum_{k=1}^{\infty} \frac{\ell_\gamma(X)}{2\sinh\left(\frac{k\ell_\gamma(X)}{2}\right)} f^{*t}(k\ell_\gamma(X)).$$

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- The integration formula passes the expectation to an integral involving Weil-Petersson volumes of moduli spaces of surfaces with boundary.
- A key technical part of our work is obtaining new expansions for these volumes with explicit dependencies in error terms.

Proof sketch: expansion for powers

We show that there are constants $c > 0$ and $\{a_i^{(t)}\}_{i \in \mathbb{Z}_{\geq 0}}$ such that for all $k \geq 1$,

$$\left| \mathbb{E}_g \left[\frac{1}{g} \mathrm{Tr}\ \hat{f} \left(\sqrt{\Delta - \tfrac{1}{4}} \right)^t \right] - \sum_{i=0}^{k-1} \frac{a_i^{(t)}}{g^i} \right| \leq \frac{(ckt)^{c(k+t)}}{g^k},$$

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for all $g > (ck)^c$. Explicitly,

$$a_0^{(t)} = \int_{\frac{1}{4}}^{\infty} \hat{f}\left(\sqrt{r - \tfrac{1}{4}} \right)^t \tanh\left(\pi \sqrt{r - \tfrac{1}{4}} \right) \mathrm{d}r,$$

$$a_1^{(t)} = \int_0^{\infty} \sum_{k=1}^{\infty} 2 \frac{\sinh\left(\frac{\ell}{2}\right)^2}{\sinh\left(\frac{k\ell}{2}\right)} f^{*t}(k\ell) \mathrm{d}\ell - a_0^{(t)}.$$

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- There are several nuances to this that really do require an expansion of the trace up to orders greater than two... e.g. exponential dependence on the polynomial degree in the error that needs to be killed with large powers of g.

- Overall, we establish an expansion for general polynomials $\tilde{h}(x) = \sum_{t=0}^{q-1} s_t x^t$ of the form

$$\left| \mathbb{E}_g \frac{1}{g} \mathrm{Tr}\ h \circ \hat{f}\left(\sqrt{\Delta - \frac{1}{4}}\right) - \nu_0(\tilde{h}) - \frac{\nu_1(\tilde{h})}{g} \right| \leq \frac{Cq^C \|\tilde{h}\|_{[-\hat{f}(\frac{i}{2}),\hat{f}(\frac{i}{2})]}}{g^2},$$

for a universal $C > 0$.

Proof sketch: smooth functions

- The ν_0 and ν_1 are given by

$$\nu_0(\tilde{h}) := \int_{\frac{1}{4}}^{\infty} h \circ \hat{f}\left(\sqrt{r - \tfrac{1}{4}}\right) \tanh\left(\pi \sqrt{r - \tfrac{1}{4}}\right) \mathrm{d}r$$

$$\nu_1(\tilde{h}) := \sum_{t=0}^{q-1} s_t \int_0^{\infty} \sum_{k=1}^{\infty} 2 \frac{\sinh\left(\frac{\ell}{2}\right)^2}{\sinh\left(\frac{k\ell}{2}\right)} f^{*(t+1)}(k\ell)\mathrm{d}\ell - \nu_0(\tilde{h}).$$

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- The extension to smooth functions follows from converting the $q^C \|\tilde{h}\|_{[-\hat{f}(\frac{i}{2}),\hat{f}(\frac{i}{2})]}$ error into a sup norm of the derivatives of $\tilde{h}$ using polynomial inequalities and then taking limits along sequences of polynomials.

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- This results in the new error term

$$\frac{C\left\|\tilde{h}\left(\hat{f}(\frac{i}{2})\cos(\theta)\right)\right\|_{C^m([0,2\pi])}}{g^2}.$$

Proof sketch: support of the CSDs

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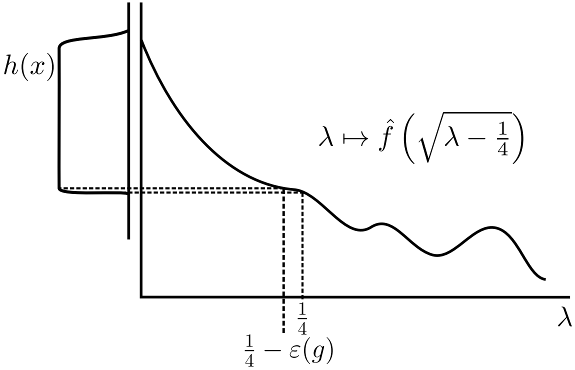
- We want them to be supported on $[-\hat{f}(0), \hat{f}(0)]$ due to our choice of h



Thank you for listening!