From 2007 to 2026

Ilia Binder

Crash course

What we knew in 2007.

What we know now.

Riemann map

Caratheodory Extension

Harmonic measure

What is left.

Computability and Complexity in Complex Analysis: from 2007 to 2026.

Ilia Binder
University of Toronto

based on joint works with Mark Braverman, Adi Glucksam, Cristobal Rojas, and Michael Yampolsky

New Interactions between Probability and Geometry
IPAM
January 28, 2026

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1. Computability and Complexity: the necessary crash course.

2. What we knew in 2007.

3. What we know now.

   - Complexity of computing the Riemann map

   - Computability of Caratheodory extension

   - Computability of Harmonic Measure

4. What is left.

A function $f : \mathbb{N} \to \mathbb{N}$ is called *computable* if there exists an algorithm (a Turing machine) which, upon input n, outputs $f(n)$.

Computability of natural numbers and functions on natural numbers

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A complement of lower-computable set is called *upper-computable*. A set is computable iff it is simultaneously upper- and lower-computable.

Works the same way if $\mathbb{N}$ is replaced by any explicitly enumerable countable set, like $\mathbb{Q}^d$.

Computability of reals and functions

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$x \in \mathbb{R}^d$ is called

- *computable* if there is a computable function $f : \mathbb{N} \to \mathbb{Q}^d$ such that $|f(n) - x| < 2^{-n}$;
- *lower-computable, $d = 1$* if there is a computable function $f : \mathbb{N} \to \mathbb{Q}$ such that $f(n) \uparrow x$;
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A function $f : S \to \mathbb{R}^k$ ($S \subset \mathbb{R}^d$) is called *computable* if there exists an algorithm with an oracle for $x \in S$ and an input $n \in \mathbb{N}$ which outputs a rational point s_n such that $|s_n - f(x)| < 2^{-n}$. An algorithm may *query* an oracle by reading the values of the function $\phi(m)$ for an arbitrary $m \in \mathbb{N}$.

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We will partition $\mathbb{R}^d$ into dyadic cubes of the form

$$\prod_{l=1}^{d} \left[k_l 2^{-m}, (k_l + 1) 2^{-m} \right), \text{ where } m, k_1, \ldots k_d \in \mathbb{Z}.$$

A connected interior of a finite union of dyadic cubes of the same size 2^{-m} is called a *dyadic polygon of rank m*.

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Dyadic Polygons

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A sequence of dyadic polygons $\{P_n\}$ is *uniformly computable* if there exists an algorithm which, upon input $n \in \mathbb{N}$, outputs the size 2^{-m} and the finite collection of d-tuples $(k_1, \ldots k_d)$, which constitutes a finite description of P_n.

Computability of sets in $\mathbb{R}^d$.

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An open set $\Omega \subset \mathbb{R}^d$ is called *lower computable* if there exists an increasing uniformly computable sequence of dyadic polygons $\{P_n\}$ such that $\Omega = \bigcup_n P_n$.

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$$\text{Hdist}(P_n, E) < 2^{-n} \text{ for all } n.$$

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Another equivalent definition: the distance function $d : \mathbb{R}^d \to \mathbb{R}_+$ defined by $d(x) = \text{dist}(x, E)$ is computable.

A measure μ on a Borel set in $\mathbb{R}^d$ is called a *computable measure* if for every dyadic cube $C \subset \mathbb{R}^d$ and every sequence $(f_j : \mathbb{R}^d \to \mathbb{R})_{j \in \mathbb{N}}$ of uniformly computable functions in $\mathbb{R}^d$ there exists an algorithm which on input $(j, n) \in \mathbb{N}^2$ outputs a rational $I_{j,n}$ satisfying

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A family of measures $\{\mu_x\}_{x \in \Omega}$ is called *uniformly computable* if there exists an algorithm that computes measure μ_x using an oracle for $x \in \Omega$.

P – computable in time polynomial in the length of the input.

NP – solution can be checked in polynomial time.

$\#P$ – can be reduced to counting the number of satisfying assignments for a given propositional formula ($\#SAT$).

$PSPACE$ – solvable in *space* polynomial in the input size.

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$KNOWN:$ P $\neq$ EXP.

$CONJECTURED:$P $\subsetneq$ NP $\subsetneq$ #P $\subsetneq$ PSPACE $\subsetneq$ EXP.

What information about (Ω, w_0) does one need to compute the Riemann map $f : (\mathbb{D}, 0) \mapsto (\Omega, w_0)$, $f'(0) > 0$ at a given point?

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Computing the Riemann map.

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Theorem (**Constructive Riemann Mapping Theorem.** Hertling, 1997)

f is computable for every point $z \in \mathbb{D}$ and f^{-1} is computable for every point $w \in \Omega$ **if and only if** Ω
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Idea of the proof: Let (P_n) be the increasing sequence of dyadic polygons with $\cup P_n = \Omega$. The maps
$f_n : \mathbb{D} \mapsto P_n$ are explicitly computable and converge to f. To approximate the rate of convergence, just
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A lower bound on computational complexity

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Theorem (B-Braverman-Yampolsky, 2005)

Suppose there is an algorithm A that given a simply-connected domain Ω with a linear-time computable boundary, a point $w_0 \in \Omega$ with $\text{dist}(w_0, \partial\Omega) > \frac{1}{2}$ and a number n, computes $20n$ digits of the conformal radius $f'(0)$), then we can use one call to A to solve any instance of a $\#SAT(n)$ with a linear time overhead.

In other words, $\#P$ is poly-time reducible to computing the conformal radius of a set.

Any algorithm computing values of the uniformization map will also compute the conformal radius with the same precision, by Distortion Theorem.

For a propositional formula Φ with n variables, let $L \subset \{0, 1, \ldots, 2^n - 1\}$ be the set of numbers corresponding to its satisfying instances. Let k be the number of elements of L.

The proof of lower bound

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For a propositional formula Φ with n variables, let $L \subset \{0, 1, \ldots, 2^n - 1\}$ be the set of numbers corresponding to its satisfying instances. Let k be the number of elements of L.

Let Ω_L be defined as

$$\mathbb{D} \setminus \cup_{l \in L} \{|z - \exp(2\pi i l/2^{-n})| \leq 2^{-10n}\},$$

the unit disk with k very small and spaced out half balls removed.

The proof of lower bound

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The key estimate:

if $f : (\mathbb{D}, 0) \to (\Omega_L, 0)$ is conformal, $f'(0) > 0$ and n is large enough, then

$$\left| f'(0) - 1 + k 2^{-20n-1} \right| < \frac{1}{100} 2^{-20n}.$$

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The boundary of Ω_L is computable in linear time, given the access to Φ. The estimate implies that using the algorithm A we can evaluate $|L| = k$, and solve the $\#SAT$ problem on Φ.

Theorem (B-Braverman-Yampolsky, 2005)

There is an algorithm A that computes the uniformizing map in the following sense:
Let Ω be a bounded simply-connected domain, and $w_0 \in \Omega$. Assume that the boundary of a simply connected domain Ω, $\partial\Omega$, $w_0 \in \Omega$, and $w \in \Omega$ are provided to A by an oracle. Then A computes $g(w)$ with precision n with complexity PSPACE(n).

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The algorithm uses solution of Dirichlet problem with random walk and de-randomization. Why only *PSPACE* and not $\#P$?

An upper bound on computational complexity

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The algorithm uses solution of Dirichlet problem with random walk and de-randomization. Why only *PSPACE* and not $\#P$? Too many random bits ($\approx \varepsilon^{-2}$ to reach ε-neighborhood of $\partial\Omega$).

Walk on Spheres.

Walk On Spheres (WoS) started at $X_0 \in \Omega$ is defined by

$$X_{n+1} = X_n + d(X_n, \partial\Omega)/2\gamma_n,$$

where (γ_n) is a sequence of i.i.d.s uniformly distributed on the unit circle. X_n converges to X_∞ - random point on $\partial\Omega$ sampled from the harmonic measure.

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(a)   (b)

When does it reach ε-neighborhood of $\partial\Omega$?

Walk on Spheres: rate of convergence and optimal upper bound on complexity.

Definition.

A domain $\Omega \subset \mathbb{R}^d$ is said to be α-thick $0 \leq \alpha \leq d$ if there exists a constant $C > 0$ such that for every $x \in \partial \Omega$

$$H^{d-\alpha}(B(x,r) \setminus \Omega) \geq Cr^{d-\alpha}, \qquad r < 1.$$

($H^{d-\alpha}$ is the $d - \alpha$-dimensional Hausdorff content)

Walk on Spheres: rate of convergence and optimal upper bound on complexity.

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Theorem

Let Ω be a bounded α-thick domain in $\mathbb{R}^d$. Then the expected rate of convergence of the WoS from any $x \in \Omega$ until termination at distance $< \varepsilon$ to the boundary is given by the following table:

	Rate of convergence
$\alpha < 2$	$O(\log 1/\varepsilon)$
$\alpha = 2$	$O(\log^2 1/\varepsilon)$
$\alpha > 2$	$O((1/\varepsilon)^{2-4/\alpha})$

(1)

Moreover, the rates of convergence above are tight.

Walk on Spheres: optimal upper bound on complexity.

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Any simply-connected domain in $\mathbb{C}$ is 1-thick. So the convergence rate is $O(\log 1/\varepsilon)$ and we need this number of random bits.

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Let Ω be a bounded simply-connected domain, and $w_0 \in \Omega$. Assume that the boundary of a simply connected domain Ω, $\partial\Omega$, $w_0 \in \Omega$, and $w \in \Omega$ are provided to A by an oracle. Then A computes $g(w)$ with precision n with complexity $\#P(n)$.

Carathéodory extension of f to $\partial\Omega$ is given by

Caratheodory Theorem.

Let $\Omega \subset \mathbb{C}$ be a simply-connected domain. A conformal map $f : \mathbb{D} \mapsto \Omega$ extends to a continuous map $\overline{\mathbb{D}} \mapsto \overline{\Omega}$ iff $\partial\Omega$ is locally connected.

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A set $K \subset \mathbb{C}$ is called *locally connected* if there exists **modulus of local connectivity** $m(\delta)$: a non-decreasing function decaying to 0 as $\delta \to 0$ and such that for any $x, y \in K$ with $|x - y| < \delta$ one can find a connected $C \subset K$ containing x and y with diam $C < m(\delta)$.

Carathéodory extension

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f extends to a homeomorphism $\overline{\mathbb{D}} \mapsto \overline{\Omega}$ iff $\partial\Omega$ is a Jordan curve.

What information about Ω does one need to compute f up to the boundary?

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Carathéodory extension.

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$\eta(\delta) \leq m(\delta)$, but $\eta(\delta)$ exists iff $m(\delta)$ exists. Closer related to the Modulus of local connectivity $m'(\delta)$ of $\mathbb{C} \setminus \Omega$: $m'(\delta) \leq 2\eta(\delta) + \delta$.

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Theorem (B.-Rojas-Yampolsky, 2014).

The Carathéodory extension of $f : \mathbb{D} \to \Omega$ is computable iff f is computable and there exists a computable Carathéodory modulus of Ω.

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Theorem (B.-Rojas-Yampolsky, 2014).

The Carathéodory extension of $f : \mathbb{D} \to \Omega$ is computable iff f is computable and there exists a computable Carathéodory modulus of Ω. Furthermore, there exists a domain Ω with computable Carathéodory modulus but no computable modulus of local connectivity.

General simply-connected domains: Carathéodory metric.

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Carthéodory metric on (Ω, w):
$$\mathrm{dist}_C(z_1, z_2) = \inf \mathrm{diam}(\gamma),$$

where γ is a closed curve or crosscut in Ω separating $\{z_1, z_2\}$ from w_0. (Defined as continuous extension when one of the points is equal to w_0.)

General simply-connected domains: Carathéodory metric.

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The closure of Ω in Carathéodory metric is called the **Carathéodory compactification,** $\hat{\Omega}$. It is obtained from Ω by adding the prime ends.

General simply-connected domains: Carathéodory metric.

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f is extendable to a homeomorphism $\hat{f} : \overline{\mathbb{D}} \mapsto \hat{\Omega}$.

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Theorem (B.-Rojas-Yampolsky, 2014). (Computable Carathéodory Theorem)

In the presence of oracles for w_0 and for $\partial\Omega$, both $\hat{f}$ and $\hat{g} = \hat{f}^{-1}$ are computable.

Oscillation of f near boundary:

$$\omega(r) := \sup_{|z_0|=1, |z_1|<1,\ |z_2|<1, |z_1-z_0|<r, |z_2-z_0|<r} |f(z_1) - f(z_2)|.$$

Oscillation of f near boundary:

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Warshawski's Theorem (1950): $\omega(r) \leq \eta\left(\left(\frac{2\pi A}{\log 1/r}\right)^{1/2}\right)$, for all $r \in (0,1)$.

Here A is the area of Ω, and $\eta(\delta)$ is Carathéodory modulus.

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Warshawski's Theorem (1950): $\omega(r) \leq \eta\left(\left(\frac{2\pi A}{\log 1/r}\right)^{1/2}\right)$, for all $r \in (0,1)$.

Here A is the area of Ω, and $\eta(\delta)$ is Carathéodory modulus.

The estimate $|f(z) - f((1-r)z)| \leq \omega(r)$ for $|z| = 1$ allows one to compute $f(z)$ using $f(rz)$ for r close to 1.

A refinement of Lavrentieff estimate(1936) (Also proven by Ferrand(1942) and Beurling in the 50ties). Let $M = \text{dist}(\partial\Omega, w_0)$, γ be a crosscut with $\text{dist}(\partial\Omega, w_0) \geq M/2$, $\epsilon^2 < M/4$. Then

$$\text{diam}(\gamma) < \epsilon^2 \implies \text{diam}(f^{-1}(N_\gamma)) \leq \frac{30\epsilon}{\sqrt{M}}.$$

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Essentially, $\hat{f}^{-1}$ is $1/2$-Hölder as a map from $\hat{\Omega}$ to $\overline{\mathbb{D}}$.

The estimate implies that

$$\text{diam}(N_\gamma) \leq 2\omega(\text{diam}(f^{-1}(N_\gamma))) \leq 2\omega\left(\frac{30\epsilon}{\sqrt{M}}\right).$$

Thus, if f is computable up to the boundary, $2\omega\left(\frac{30\epsilon}{\sqrt{M}}\right)$ is a computable Carathéodory modulus.

Let $B \subset \mathbb{N}$ be a lower-computable, non-computable set. Set $x_i = 1 - 1/2i$.

An example of a domain with computable Caratheodory extension and non-computable modulus of local connectivity

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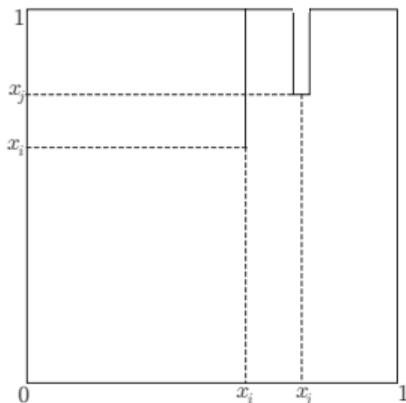What we knew in 2007.

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Let $B \subset \mathbb{N}$ be a lower-computable, non-computable set. Set $x_i = 1 - 1/2i$.

The domain Ω is constructed by modifying the square $(0,1) \times (0,1)$ as follows.

An example of a domain with computable Caratheodory extension and non-computable modulus of local connectivity

Let $B \subset \mathbb{N}$ be a lower-computable, non-computable set. Set $x_i = 1 - 1/2i$.

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If $i \notin B$, then we add a straight line (*i-line*) to I going from $(x_i, 1)$ to (x_i, x_i).

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If $i \notin B$, then we add a straight line (*i-line*) to I going from $(x_i, 1)$ to (x_i, x_i).
If $i \in B$ and it is enumerated in stage s, we remove *i-fjord*, i.e. the rectangle

$$[(x_i - s_i, (x_i + s_i] \times [x_i, 1]$$

where $s_i = \min\{2^{-s}, 1/(3i^2)\}$.

The example: $\partial\Omega$ and Caratheodory modulus are computable.
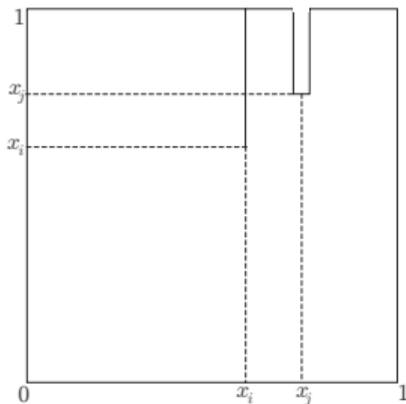
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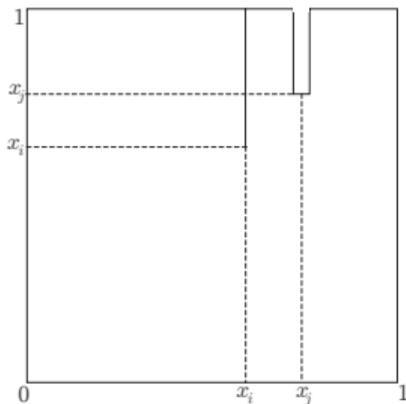
Computing a 2^{-s} Hausdorff approximation of
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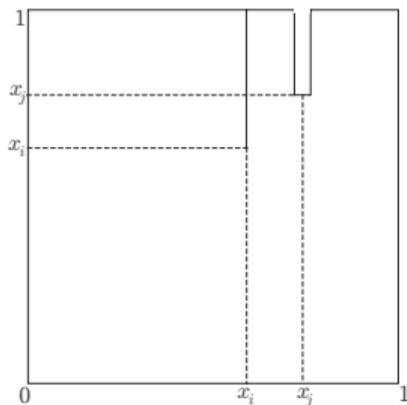
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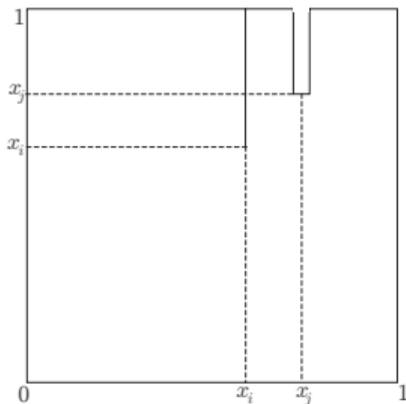
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Carathéodory modulus: $2\sqrt{r}$.

The example: Modulus of local connectivity $m(r)$ is not computable

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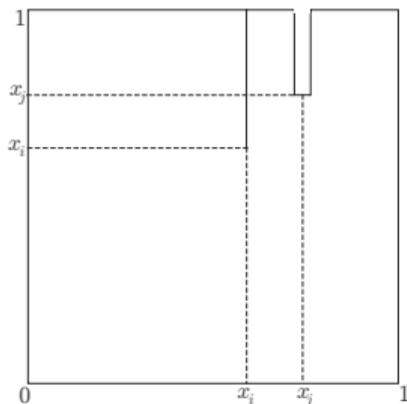
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Compute B using $m(r)$.

Compute B using $m(r)$. First, for $i \in \mathbb{N}$, compute $r_i \in \mathbb{Q}$ such that

$$m(2 \cdot 2^{-r_i}) < \frac{x_i}{2}.$$

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If $i \in B$ then i is enumerated in fewer than r_i steps. Our algorithm to compute B will emulate the algorithm for enumerating B for r_i steps.

Harmonic measure: planar simply connected domains.

Theorem (B.-Braverman-Rojas-Yampolsky, 2011).

Let Ω be a planar simply connected domain. The following are equivalent:

1. *Both Ω and $\partial\Omega$ are lower computable.*
2. *For any computable $w_0 \in \Omega$ the conformal maps $g : (\Omega, w_0) \mapsto (\mathbb{D}, 0)$ and $f : (\mathbb{D}, 0) \mapsto (\Omega, w_0)$ are both computable conformal bijections.*
3. *Ω has a computable harmonic measure at x_0 for some computable $x_0 \in \Omega$.*
4. *Ω has uniformly computable harmonic measures.*

Harmonic measure: planar simply connected domains.

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Let Ω be a planar simply connected domain. The following are equivalent:

1. *Both Ω and $\partial\Omega$ are lower computable.*
2. *For any computable $w_0 \in \Omega$ the conformal maps $g : (\Omega, w_0) \mapsto (\mathbb{D}, 0)$ and $f : (\mathbb{D}, 0) \mapsto (\Omega, w_0)$ are both computable conformal bijections.*
3. *Ω has a computable harmonic measure at x_0 for some computable $x_0 \in \Omega$.*
4. *Ω has uniformly computable harmonic measures.*

On the other hand, outside of simply connected domains, things are more complicated:

Theorem (B.-Braverman-Rojas-Yampolsky, 2011).

There exists a bounded regular domain $\Omega \subset \mathbb{C}$ such that $\partial\Omega$ is computable, but the harmonic measure is not computable at any point $x_0 \in \Omega$.

A point $x \in \partial\Omega$ is called *regular* if for any $f(z)$ continuous on $\partial\Omega$ with the Direchlet solution $u(z)$, we have

$$\lim_{z \to x, \ z \in \Omega} u(z) = f(x).$$

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One of the equivalent definitions of regular point: a point $x \in \partial\Omega$ is regular if

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Ω is called *computably regular* if there exists a computable positive function $\varepsilon(n) : \mathbb{N} \mapsto \mathbb{Q}$ such that

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Any computably regular domain is regular.

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Any computably regular domain is regular. Any simply connected planar domain is computably regular (by Beurling Projection Theorem).

A compact set $K \subset \mathbb{C}$ which contains at least two points is *uniformly perfect* if there exists some $C > 0$ such that for any $x \in K$ and $r > 0$, we have $(B(x, Cr) \setminus B(x, r)) \cap K = \emptyset \implies K \subset B(x, r)$.

In particular, every connected set is uniformly perfect.

Uniformly perfect sets

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**Harmonic
measure**

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In particular, every connected set is uniformly perfect. Every polynomial Julia set is uniformly perfect. Any planar uniformly perfect domain is computably regular (by a Theorem of Pommerenke).

Computable regularity and computability of harmonic measure

Theorem (B.-Glucksam-Rojas-Yampolsky, 2020).

Let Ω be a computably regular domain. Then the following are equivalent:

1. Ω *and* $\partial\Omega$ *are both lower computable.*

2. Ω *has a computable harmonic measure at* x_0 *for some computable* $x_0 \in \Omega$.

3. Ω *has uniformly computable harmonic measures.*

Computable regularity and computability of harmonic measure

For regular domains with **computable** boundary, computable regularity is necessary:

Theorem (B.-Glucksam-Rojas-Yampolsky, 2020).

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Regularity is important: $\mathbb{D} \setminus \{0\}$ has computable boundary and harmonic measure, but not computably regular.

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Regularity is important: $\mathbb{D} \setminus \{0\}$ has computable boundary and harmonic measure, but not computably regular.

There exists a lower computable regular domain Ω with a lower computable boundary and uniformly computable harmonic measures which is not computably regular.

Theorem (B.-Glucksam-Rojas-Yampolsky, 2020).

Let Ω be a domain and let $x_0 \in \Omega$. If Ω has a computable harmonic measure at x_0, then Ω has a computable harmonic measure at x, for every $x \in \Omega$.

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Theorem (B.-Glucksam-Rojas-Yampolsky, 2020).

There exists a regular domain Ω such that:

i) *Ω has a computable harmonic measure at x, for every $x \in \Omega$;*

ii) *The harmonic measure of Ω is not uniformly computable.*

Pointwise computability

Theorem (B.-Glucksam-Rojas-Yampolsky, 2020).

Let Ω be a domain and let $x_0 \in \Omega$. If Ω has a computable harmonic measure at x_0, then Ω has a computable harmonic measure at x, for every $x \in \Omega$.

Theorem (B.-Glucksam-Rojas-Yampolsky, 2020).

There exists a regular domain Ω such that:

i) *Ω has a computable harmonic measure at x, for every $x \in \Omega$;*

ii) *The harmonic measure of Ω is not uniformly computable.*

Theorem (B.-Glucksam-Rojas-Yampolsky, 2020).

There exists a regular domain Ω and a computable simple function $f : \partial\Omega \to \mathbb{R}$ such that the unique solution $u(x) : \Omega \to \mathbb{R}$ to the Dirichlet problem with the boundary data given by f satisfies:

i) *For each $x \in \Omega$ the value $u(x)$ is computable relative to x;*

ii) *u is not a computable function.*

Theorem (B.-Glucksam-Rojas-Yampolsky, 2025).

Let Ω be a domain which has a computable harmonic measure at x_0 for some computable $x_0 \in \Omega$. Then Ω has uniformly computable harmonic measures if and only if Ω is lower computable.

- Conformal welding. (Current project with M. Yampolsky and M. Younsi)

Open questions and current projects

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- Green maps.
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- SLE curves: the easiest approach is through a convergent model.
- And much more.

Thank You!

Questions?