Length Spectrum of Random Metric Maps

A Teichmüller theory approach

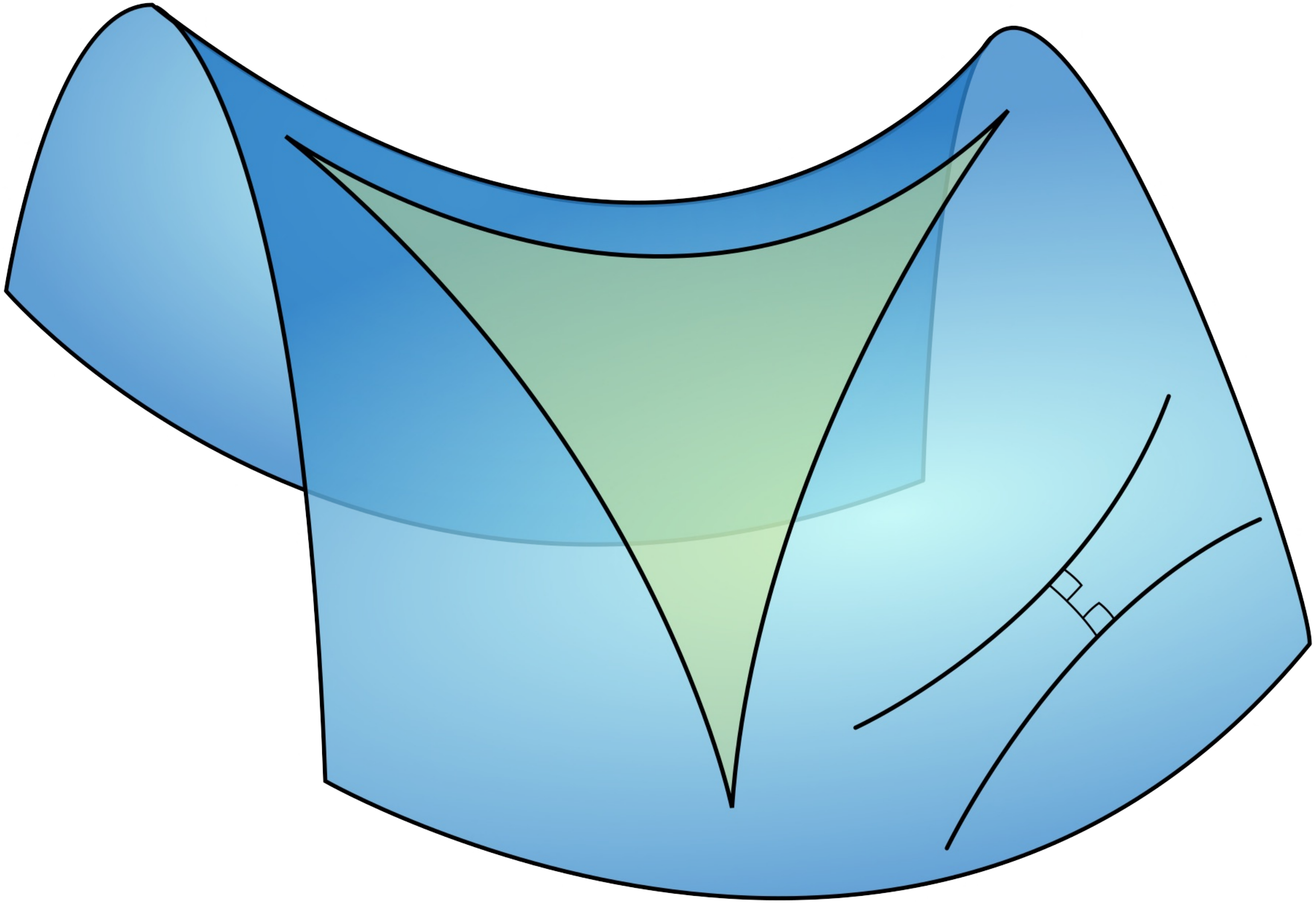
ipam 26,1,2026

joint work with

Simon Barazer

Alessandro Giacchetto

Definition a hyperbolic surface is a surface which locally looks like the hyperbolic plane



G. Meyer

PANTS DECOMPOSITIONS OF RANDOM SURFACES

LARRY GUTH, HUGO PARLIER†, AND ROBERT YOUNG

goal is to … ndom” surfaces
First we … of genus g for
equires cu … eover, we prove
etrics in … quipped with th
then con … ng euclidean tri
ther and … property.

us g, g … e-holed sphe
at a sur … n be divided
$\leq l$. W … nts length $\leq$
curves … The pants l
length measure the size an
divide the surface into simp
how big the pants length
understand how big the tot

h.GT]

Growth of Weil-Petersson volumes and random hyperbolic surfaces of large genus

Maryam Mirzakhani*

December 13, 2010

1 Introduction

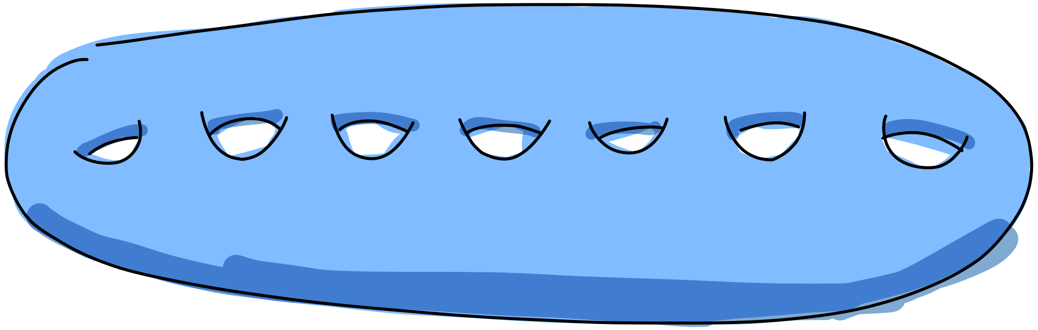
In this paper, we investigate the geometric properties of hyperbolic surfaces by studying the lengths of simple closed geodesics. The moduli space $\mathcal{M}_{g,n}$ of complete hyperbolic surfaces of genus $g \geq 2$ with n punctures, is equipped with a natural notion of measure, which is induced by the *Weil-Petersson* symplectic form $\omega_{g,n}$ (§2). By a theorem of Wolpert, this form is the symplectic form of a Kähler noncomplete metric on the moduli space $\mathcal{M}_{g,n}$. We describe the

What does
a random hyperbolic surface
of LARGE genus
look like?

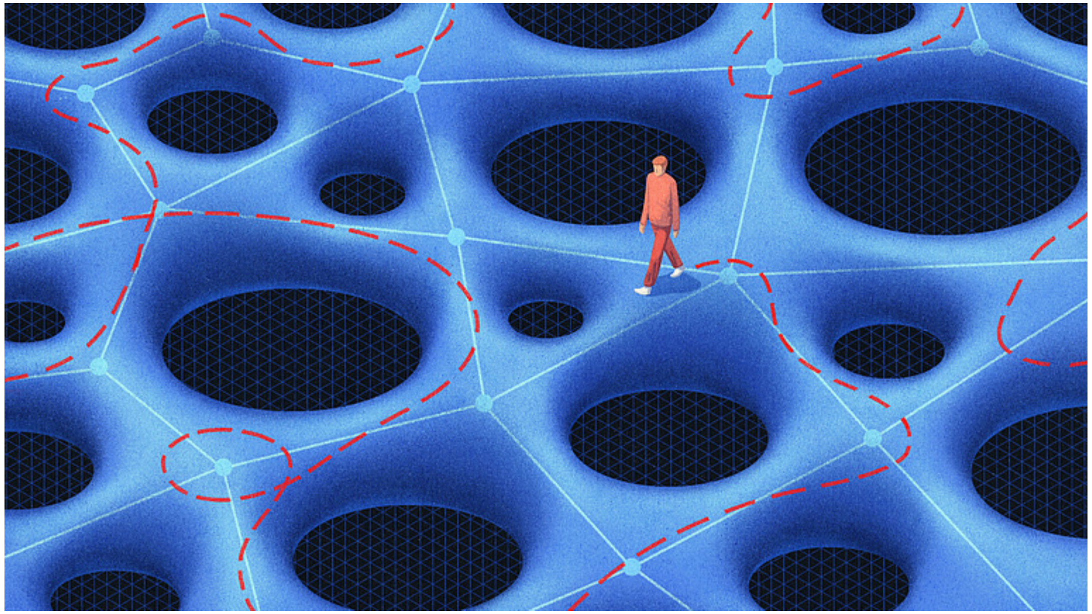
Theorem (Mirzakhani, 2010)

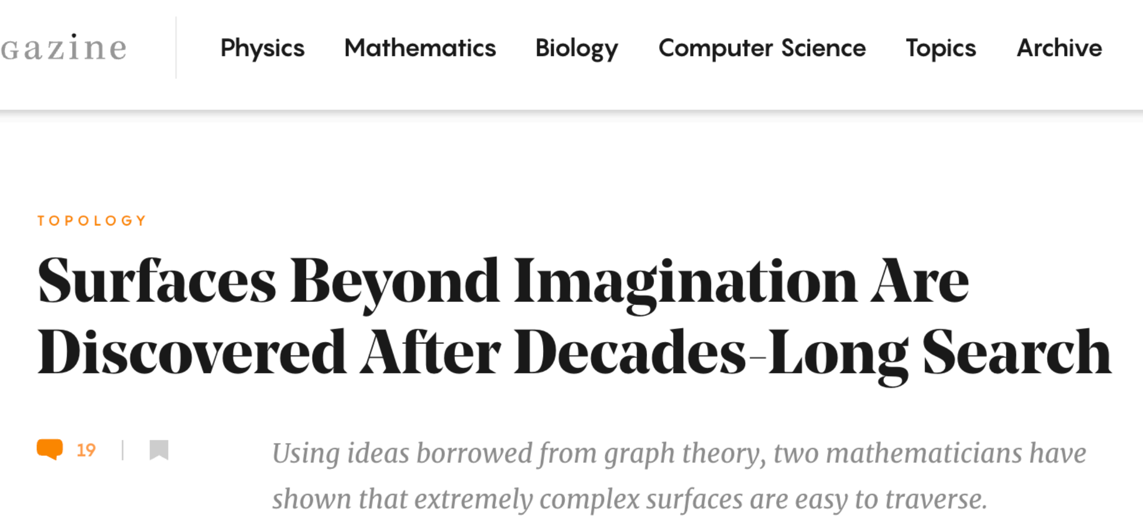
With proba → 1 when the genus $g \to \infty$ a random WP hyperbolic surface of genus g has

1) diameter $< 40 \log g$



diam $\propto g$

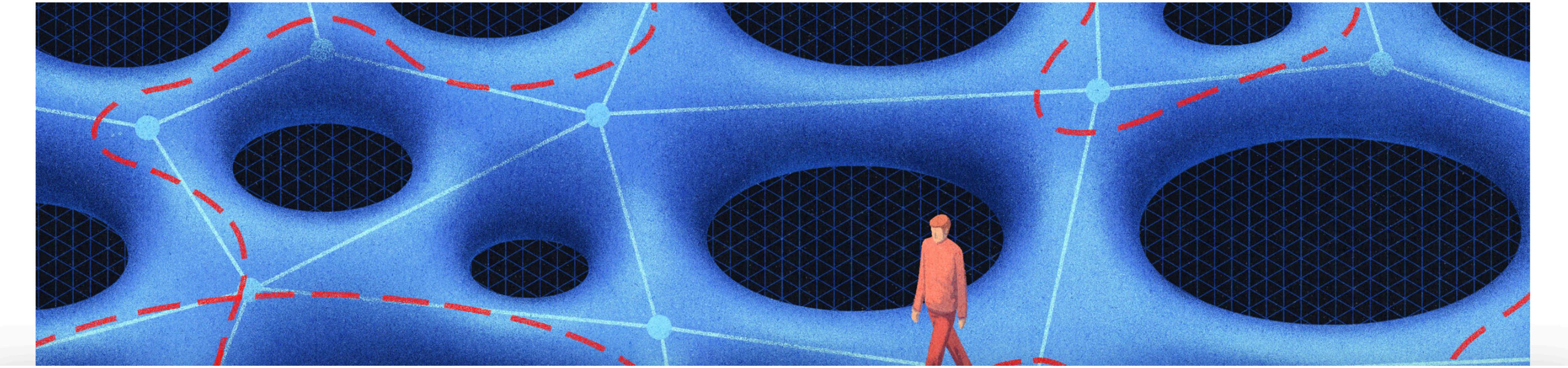




TOPOLOGY

Surfaces Beyond Imagination Are Discovered After Decades-Long Search

Using ideas borrowed from graph theory, two mathematicians have shown that extremely complex surfaces are easy to traverse.



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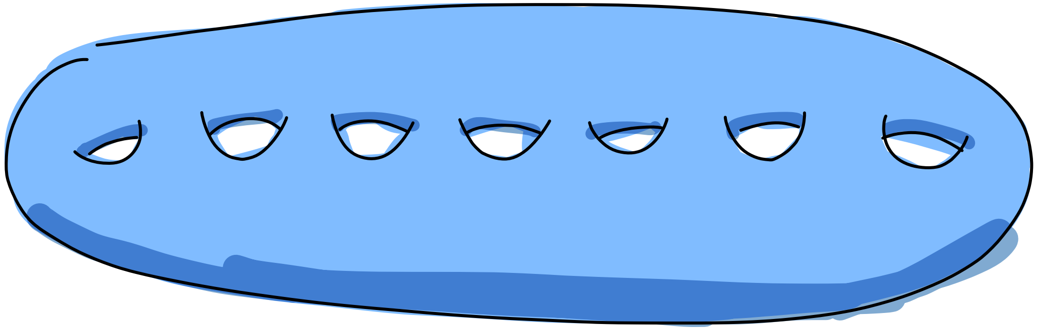
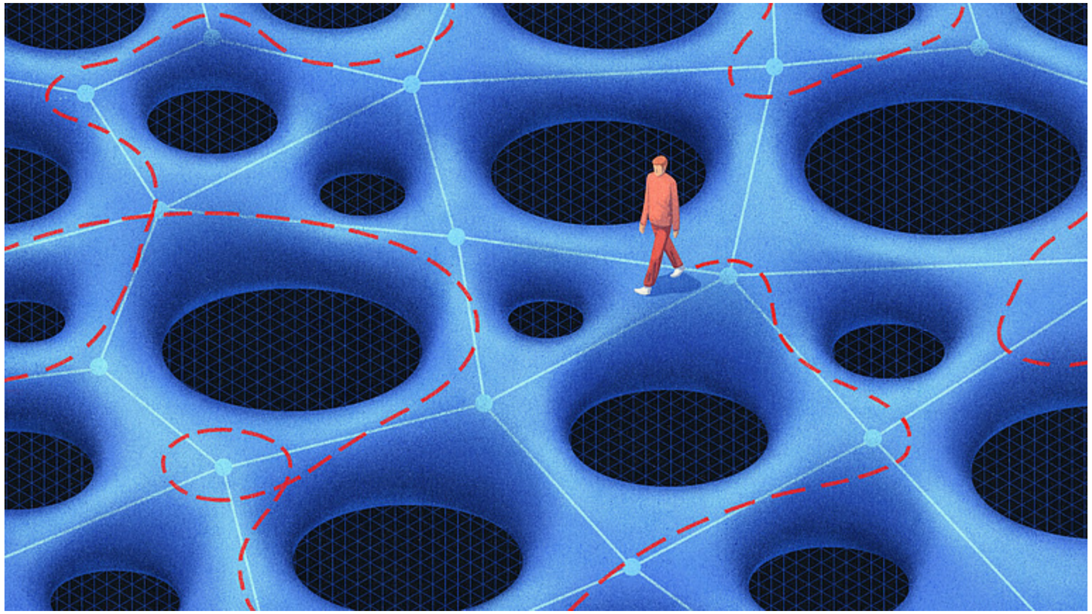


diam $\propto g$



diam $\propto \sqrt{g}$

Theorem (Mirzakhani, 2010)

With proba → 1 when the genus $g \to \infty$

a random WP hyperbolic surface of genus g has

1) diameter $< 40 \log g$

2) spectral gap λ_1

smallest > 0 eigenvalue of Δ

- counting of closed geodesics
- Cheeger constant
- Brownian motion

Theorem (Mirzakhani, 2010)

With proba $\to 1$ when the genus $g \to \infty$ a random WP hyperbolic surface of genus g has

1) diameter $< 40 \log g$

2) spectral gap $\lambda_1 > 0.002$

smallest > 0 eigenvalue of Δ

Theorem (Mirzakhani, 2010)

With proba → 1 when the genus $g \to \infty$

a random WP hyperbolic surface of genus g has

1) diameter $< 40 \log g$

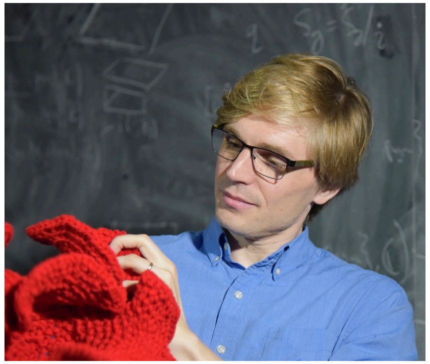
$\frac{3}{16} - \varepsilon$

2) spectral gap $\lambda_1 >$ ~~0.002~~

smallest > 0 eigenvalue of Δ

2021: Wu-Xue

Lipnowski-Wright



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2) spectral gap $\lambda_1 > \cancel{0.002}$ $\cancel{\frac{3}{16} - \varepsilon}$ $\frac{2}{9} - \varepsilon$

smallest > 0 eigenvalue of Δ

2021: Wu-Xue

Lipnowski-Wright '23 Anantharaman-Monk

Theorem (Mirzakhani, 2010)

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smallest > 0 eigenvalue of Δ

2021: Wu-Xue '25

Lipnowski-Wright '23

Anantharaman-Monk

Theorem (Mirzakhani, 2010)

With proba → 1 when the genus $g \to \infty$

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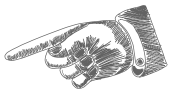
1) diameter $< 40 \log g$ $\quad \cancel{\frac{3}{16} - \varepsilon} \quad \cancel{\frac{2}{9}} \quad \frac{1}{4} - \varepsilon$



optimal

2) spectral gap $\lambda_1 > \cancel{0.002}$

smallest > 0 eigenvalue of Δ

a few months ago...

2021: Wu – Xue '25

Lipnowski – Wright '23 Anantharaman – Monk

Theorem (Mirzakhani, 2010)

With proba → 1 when the genus $g \to \infty$
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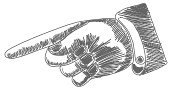

optimal

smallest > 0 eigenvalue of Δ

Hide – Mecera – Thomas

a few months ago

2021: Wu – Xue

Lipnowski – Wright

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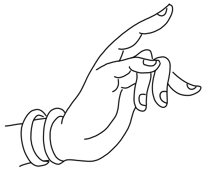
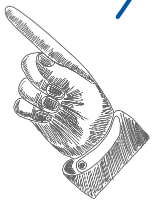
'23 Anantharaman – Monk

Weil-Petersson random hyperbolic surfaces

Model $(\Omega, \mathbb{P})$

sample space  Ω  proba measure on Ω

$$\| $$

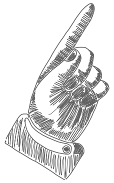
$$\{ \text{hyperbolic surface of genus } g \} / \text{isometry}$$

$$!!$$

$\mathcal{M}_g$ moduli space

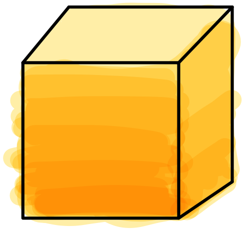
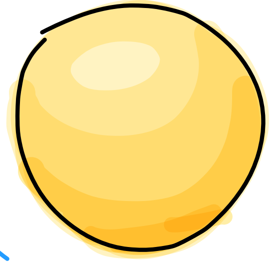
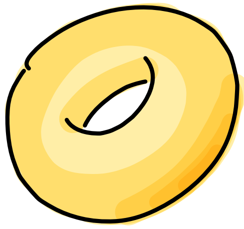
Moduli space M_g

orbifold



☺ M_g is "almost" a manifold (of $\dim_{\mathbb{R}} = 6g-6$

☹ It's a bit complicated … χ Euler char

χ () $=$ χ () $= 2$, χ () $= 0$

dim 2 : χ = #vertices − #edges + #faces $\overset{\text{Euler}}{=} 2 - 2g$

$$\chi(M_2) = -\frac{1}{240} \approx -4.17 \times 10^{-3}$$

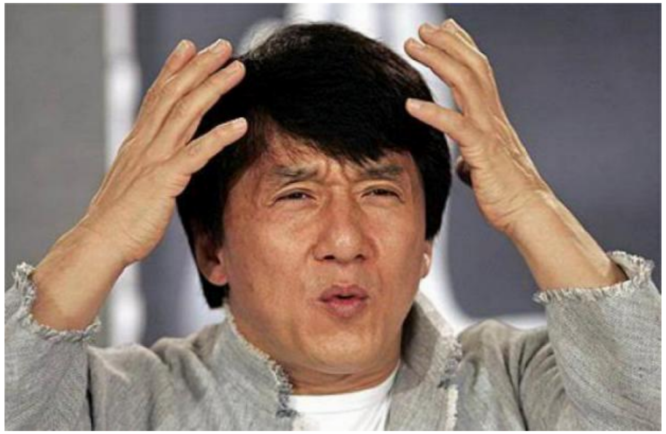
$$\chi(M_4) \approx -6.94 \times 10^{-4}$$

$$\chi(M_8) \approx -3.17 \times 10^{-2}$$

$$\chi(M_{16}) \approx -1.57 \times 10^{7}$$

$$\chi(M_{32}) \approx -5.28 \times 10^{34}$$

$$\chi(M_{64}) \approx -3.26 \times 10^{109}$$



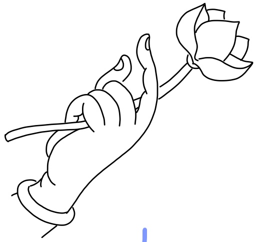
universal cover $\widetilde{M}_g = T_g$ Teichmüller space

$$\begin{tikzcd} T_g \arrow[d, dash, "\sim"', "\text{homeo}"] \\ (\mathbb{R}_{>0} \times \mathbb{R})^{3g-3} \end{tikzcd}$$

Fenchel–Nielsen coordinates

$$M_g = T_g / \pi_1(M_g)$$

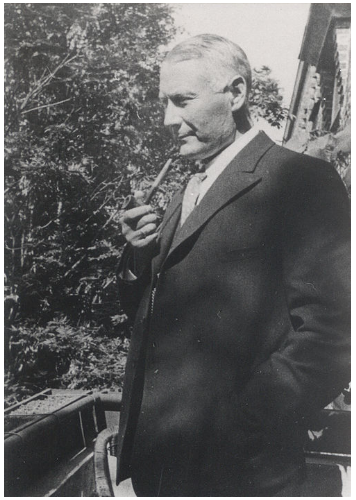
mapping class group



$$M_g \approx \mathbb{T}^2 \simeq \mathbb{R}^2 / \mathbb{Z}^2$$

$$T_g \approx \widetilde{\mathbb{T}}^2 \simeq \mathbb{R}^2$$

$$\pi_1(M_g) \approx \pi_1(\mathbb{T}^2) \simeq \mathbb{Z}^2$$

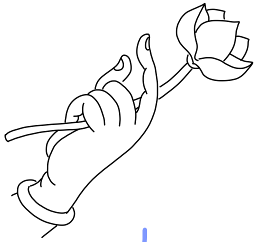


universal cover $\widetilde{M}_g = T_g$ Teichmüller space

M_g

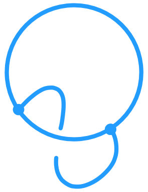
$\|$

$T_g / \pi_1(M_g)$



mapping class group

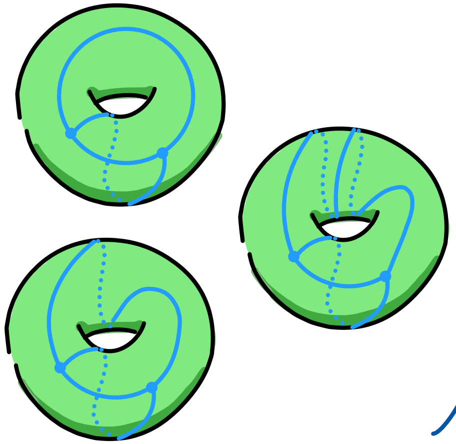
$M_g \approx \{$



$\}$

map

$T_g \approx \{$



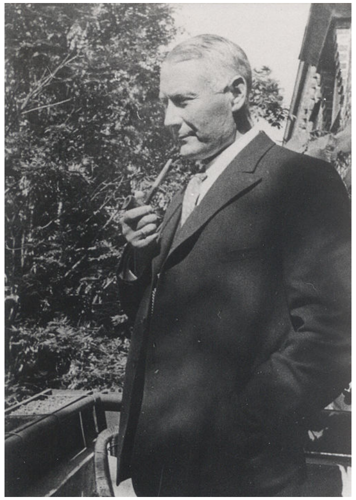
$\}$

map + embedding

$\sim \Big|$ homeo

$(\mathbb{R}_{>0} \times \mathbb{R})^{3g-3}$

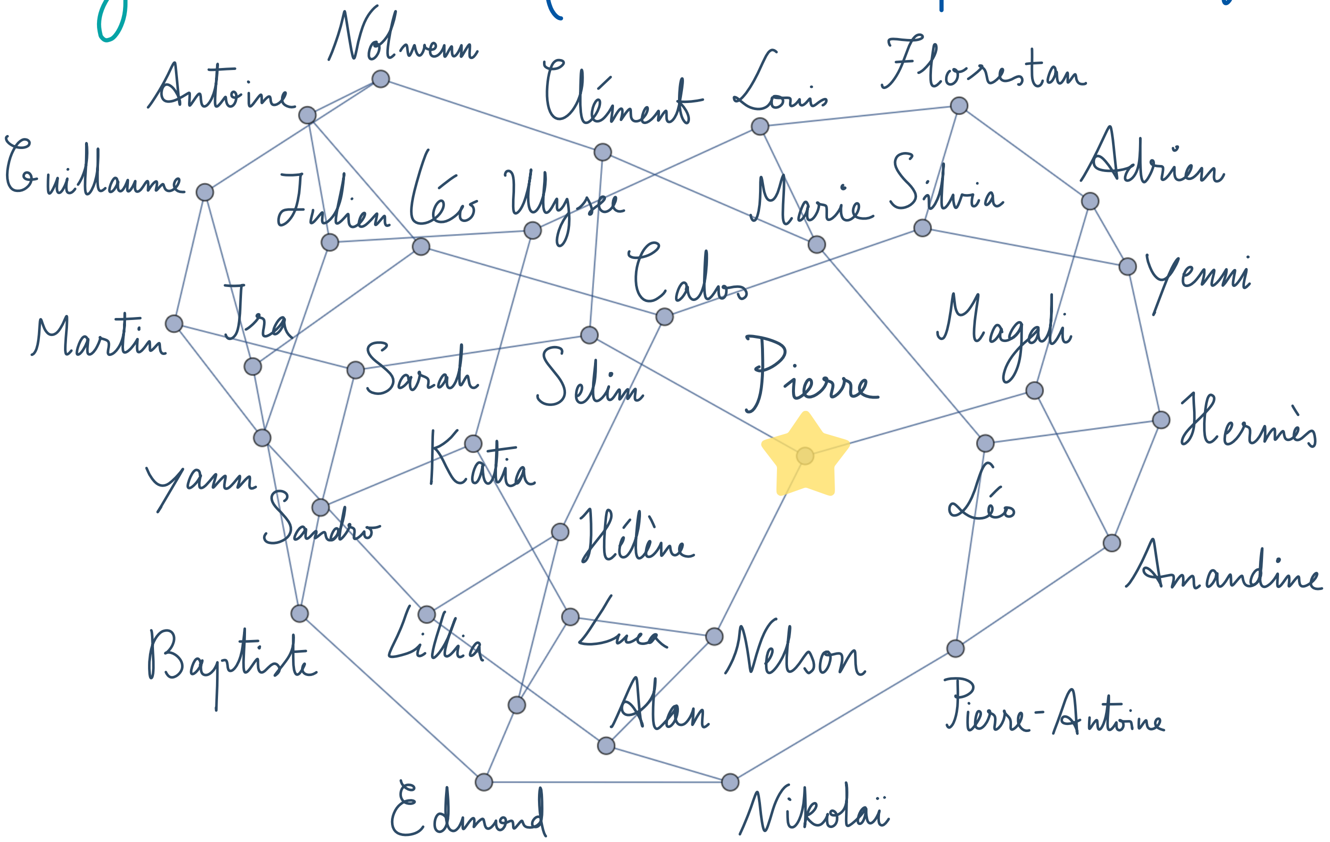
Fenchel–Nielsen coordinates



Teichmüller space : space of marked hyperbolic surfaces of genus g

Fix a topological surface Σ_g

$$\mathcal{T}_g := \left\{ (X,\varphi) \;\middle|\; \varphi : \Sigma_g \xrightarrow{\text{homeo}} X \right\} \Big/ \sim$$

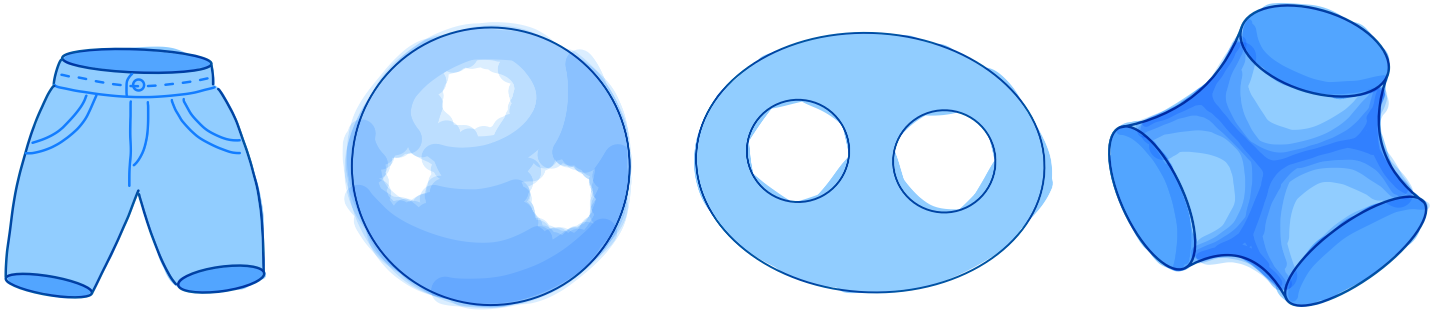


$(X,\varphi) \sim (X',\varphi')$
if $\varphi' \circ \varphi^{-1}$ isotopic to an isometry

pair of pants

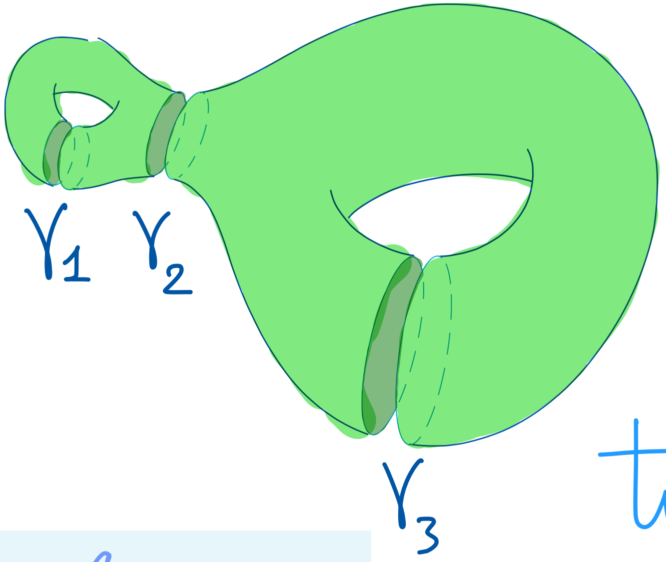
||

sphere with 3 holes



pants decomposition

$\gamma_1, \gamma_2, \dots, \gamma_{3g-3}$



γ_i simple

no self-intersection

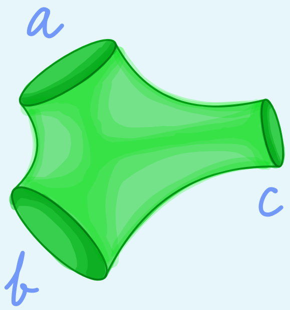
$\gamma_i \cap \gamma_j = \emptyset$

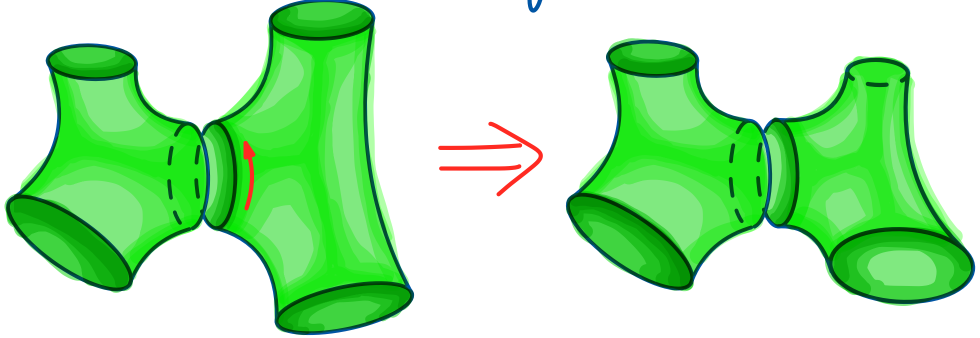
length of $\gamma_i = l_i$

twist along $\gamma_i = \tau_i$

Fact $\forall a, b, c \in \mathbb{R}_{>0}$

$\exists!$ hyperbolic



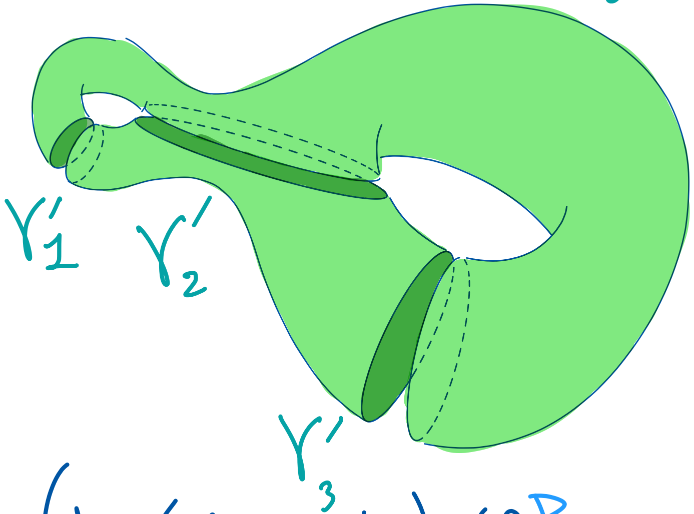


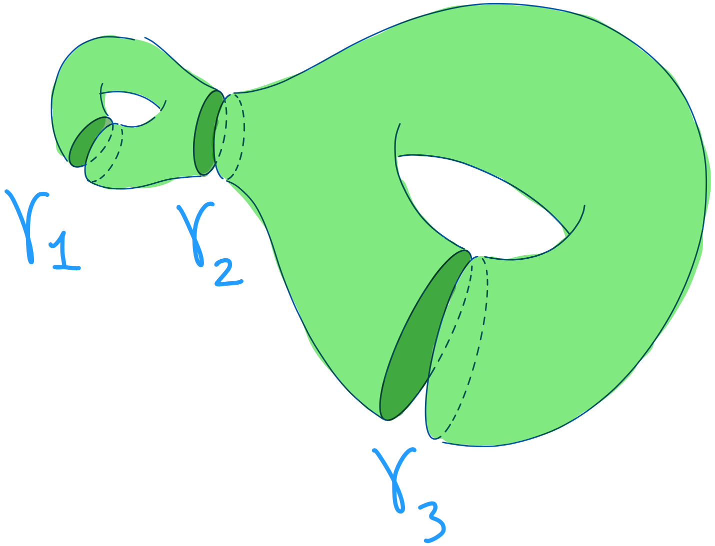
$$\left\{ (l_1, \tau_1, \dots, l_{3g-3}, \tau_{3g-3}) \,\middle|\, \begin{matrix} l_i \in \mathbb{R}_{>0} \\ \tau_i \in \mathbb{R} \end{matrix} \right\} \xrightarrow[\text{Fenchel-Nielsen}]{} \mathcal{T}_g$$

a pants decomposition $\mathcal{P}=\{\gamma_1,\dots,\gamma_{3g-3}\}\Rightarrow$

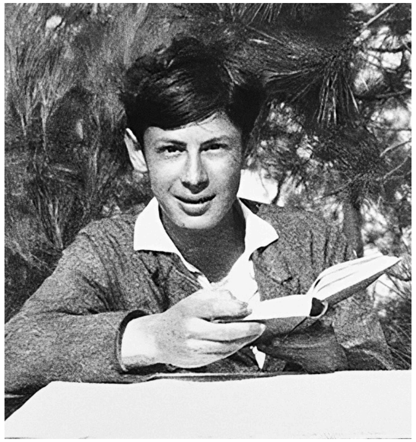
$$\begin{tikzcd} (\mathbb{R}_{>0}\times\mathbb{R})^{3g-3} \arrow[d, shift right=2, "\varphi^{\mathcal{P}}"'] \arrow[d, shift left=2, "\varphi^{\mathcal{P}'}"] \\ T_g \end{tikzcd}$$

$\mathcal{P}'=\{\gamma_1',\dots,\gamma_{3g-3}'\}$





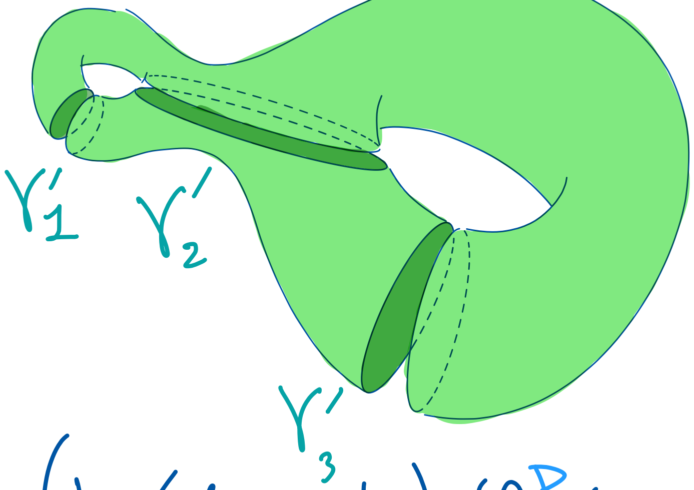
∘ (Wolpert) $\varphi^{\mathcal{P}}_*(\mu_{Leb})=\varphi^{\mathcal{P}'}_*(\mu_{Leb})$ = Weil–Petersson measure

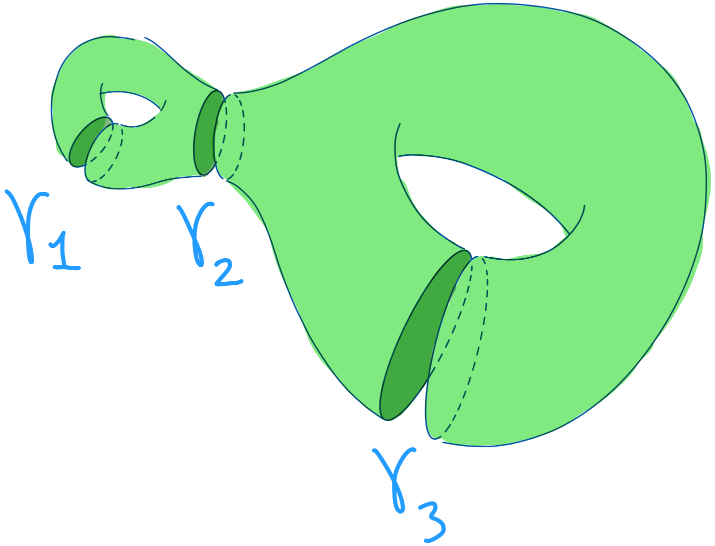


a pants decomposition $\mathcal{P} = \{\gamma_1, \dots, \gamma_{3g-3}\} \Rightarrow$

$$\begin{tikzcd}
(\mathbb{R}_{>0} \times \mathbb{R})^{3g-3} \arrow[d, shift right, "\varphi^{\mathcal{P}}"'] \arrow[d, shift left, "\varphi^{\mathcal{P}'}"] \\
T_g
\end{tikzcd}$$

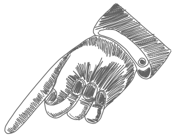
$\mathcal{P}' = \{\gamma_1', \dots, \gamma_{3g-3}'\}$





- (Wolpert) $\varphi^{\mathcal{P}}_*(\mu_{\text{Leb}}) = \varphi^{\mathcal{P}'}_*(\mu_{\text{Leb}})$ = Weil–Petersson measure
- μ_{WP} is invariant under $\pi_1(\mathcal{M}_g) \Rightarrow \mu_{WP}$ descends to $\mathcal{M}_g$
- $\mu_{WP}(\overset{\text{not compact}}{\mathcal{M}_g}) < \infty \Rightarrow$ WP model for random hyp surf

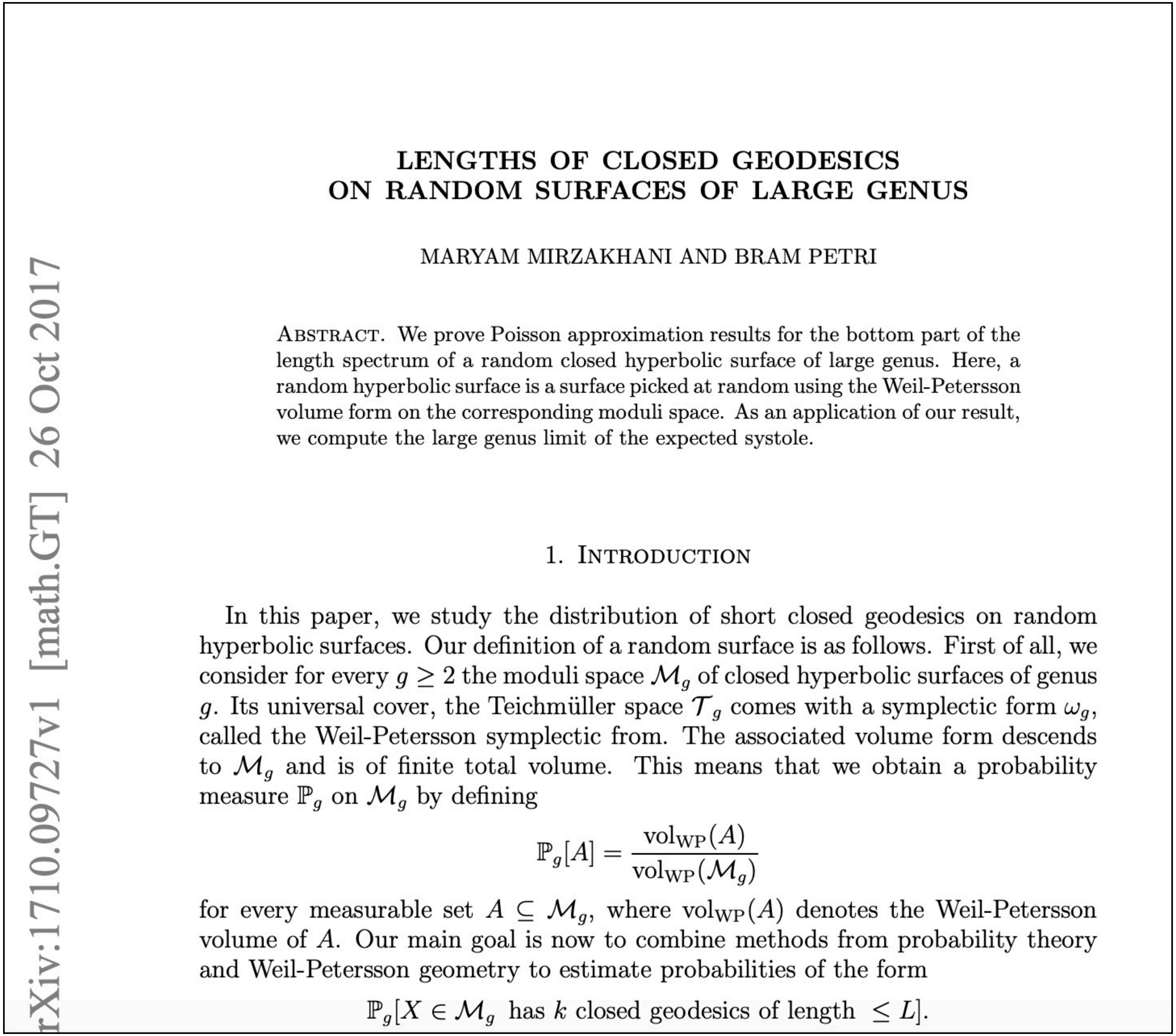
Theorem (Mirzakhani 2010) Weil-Petersson measure



$$\frac{1}{\mu_{WP}(\mathcal{M}_g)} \int_{\mathcal{M}_g} \mathbb{1}_{\{S \in \mathcal{M}_g \mid \mathrm{diam}(S) < 40 \log g\}}(X)\, dX \xrightarrow[g \to \infty]{} 1$$

Notation X_g a WP random hyperbolic surface of genus g

$$\mathbb{P}(\mathrm{diam}(X_g) < 40 \log g) \xrightarrow[g \to \infty]{} 1$$



LENGTHS OF CLOSED GEODESICS ON RANDOM SURFACES OF LARGE GENUS

MARYAM MIRZAKHANI AND BRAM PETRI

Abstract. We prove Poisson approximation results for the bottom part of the length spectrum of a random closed hyperbolic surface of large genus. Here, a random hyperbolic surface is a surface picked at random using the Weil-Petersson volume form on the corresponding moduli space. As an application of our result, we compute the large genus limit of the expected systole.

1. Introduction

In this paper, we study the distribution of short closed geodesics on random hyperbolic surfaces. Our definition of a random surface is as follows. First of all, we consider for every $g \geq 2$ the moduli space $\mathcal{M}_g$ of closed hyperbolic surfaces of genus g. Its universal cover, the Teichmüller space $\mathcal{T}_g$ comes with a symplectic form ω_g, called the Weil-Petersson symplectic from. The associated volume form descends to $\mathcal{M}_g$ and is of finite total volume. This means that we obtain a probability measure $\mathbb{P}_g$ on $\mathcal{M}_g$ by defining

$$\mathbb{P}_g[A] = \frac{\mathrm{vol}_{\mathrm{WP}}(A)}{\mathrm{vol}_{\mathrm{WP}}(\mathcal{M}_g)}$$

for every measurable set $A \subseteq \mathcal{M}_g$, where $\mathrm{vol}_{\mathrm{WP}}(A)$ denotes the Weil-Petersson volume of A. Our main goal is now to combine methods from probability theory and Weil-Petersson geometry to estimate probabilities of the form

$$\mathbb{P}_g[X \in \mathcal{M}_g \text{ has } k \text{ closed geodesics of length } \leq L].$$

Fix $0 \leqslant a < b$

$$N_{[a,b)}(X) := \#\left\{ \gamma \begin{array}{l}\text{primitive closed}\\ \text{geodesic on } X\end{array} \,\middle|\, a \leqslant \ell(\gamma) < b \right\}$$

Theorem (Mirzakhani - Petri, 2017)

Fix $0 \leqslant a < b$

$Y \sim \mathrm{Poi}(\lambda)$ if $\mathbb{P}(Y=k) = e^{-\lambda}\frac{\lambda^k}{k!}, \forall k \in \mathbb{Z}_{\geqslant 0}$

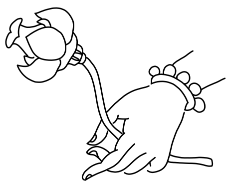
$$N_{[a,b)}(X) := \#\left\{\gamma \begin{array}{l}\text{primitive closed}\\ \text{geodesi on } X\end{array} \middle| \, a \leqslant \ell(\gamma) < b \right\}$$

Theorem (Mirzakhani–Petri, 2017)

X_g a WP random surface of genus g

$$N_{[a,b)}(X_g) \xrightarrow[g \to \infty]{(d)} \mathrm{Poi}\left(\int_a^b \overset{\lambda(x)}{\overset{\shortparallel}{\frac{\cosh(x)-1}{x}}}\, dx\right)$$

Length Spectrum



a multiset

$$\Lambda(X) := \left\{ \ell(\gamma) \in \mathbb{R}_{>0} \;\middle|\; \gamma \begin{array}{l} \text{primitive closed} \\ \text{geodesic on } X \end{array} \right\}$$

Theorem (Mirzakhani-Petri, 2017)

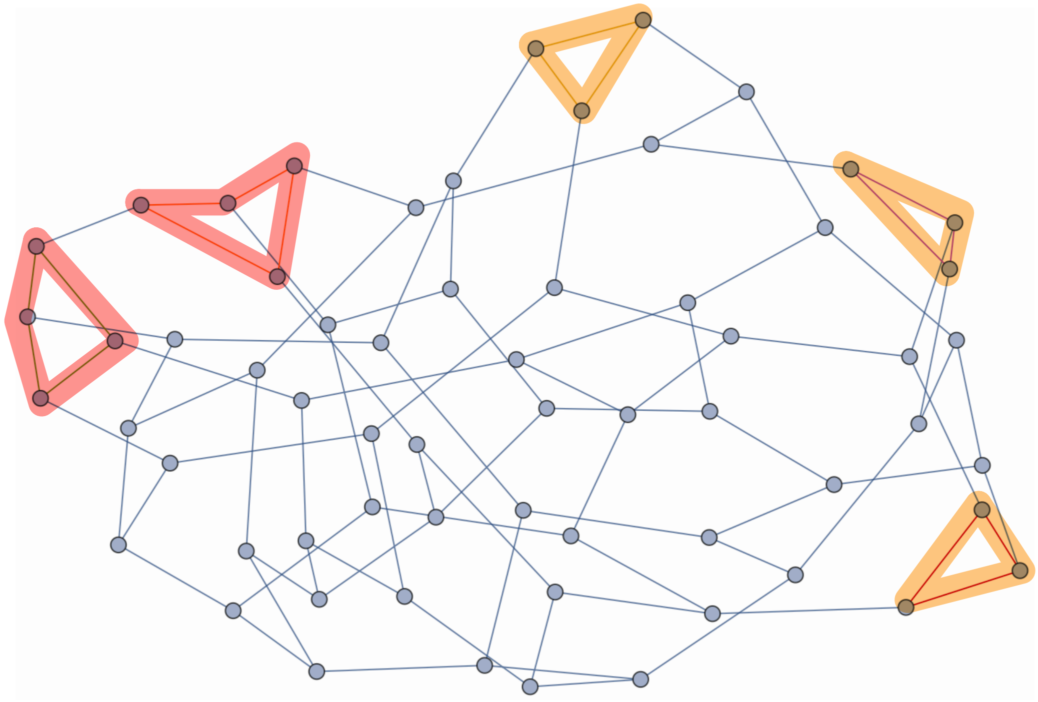
X_g a WP random surface of genus g.

Regarded as a point process on $\mathbb{R}_{>0}$,

$\Lambda(X_g) \xrightarrow[g \to \infty]{(d)}$ Poisson point process with intensity $\overset{\frac{\cosh(x)-1}{x}}{\overset{\|}{\lambda}}$.

Remark G a graph, $k \geqslant 1$ an integer

$$\mathcal{N}_k(G) := \{ \gamma \text{ cycle in } G \mid \ell(\gamma) = k \}$$



$k = 3, 4$

Remark G a graph, $k \geq 1$ an integer

$$N_k(G) := \{\gamma \text{ cycle in } G \mid \ell(\gamma) = k\}$$

Theorem (Bollobás, Wormald, ≈ 1980)

G_v a uni

aph with v vertices

Remark G a graph, $k \geq 1$ an integer

$$N_k(G) := \{ \gamma \text{ cycle in } G \mid \ell(\gamma) = k \}$$

Theorem (Bollobás, Wormald, ≈ 1980)

$\mathbf{G}_v$ a uniform random 3-regular graph with v vertices

For any integer $k \geq 3$,

$$N_k(\mathbf{G}_v) \xrightarrow[v \to \infty]{(d)} \mathrm{Poi}\left(\frac{2^k}{2k}\right)$$

Remark

$$\mathbb{E}\left(N_{[0,L]}(X_g)\right) \xrightarrow[g\to\infty]{} \int_0^L \lambda(x)\,dx \underset{L\to\infty}{\sim} \frac{e^L}{2L}$$

≈1960

Theorem (Huber, Selberg, Margulis, ...)

Remark

$$\mathbb{E}\left(N_{[0,L]}(X_g)\right) \xrightarrow[g\to\infty]{} \int_0^L \lambda(x)\,dx \underset{L\to\infty}{\sim} \frac{e^L}{2L}$$

Theorem (Huber, Selberg, Margulis, ...)

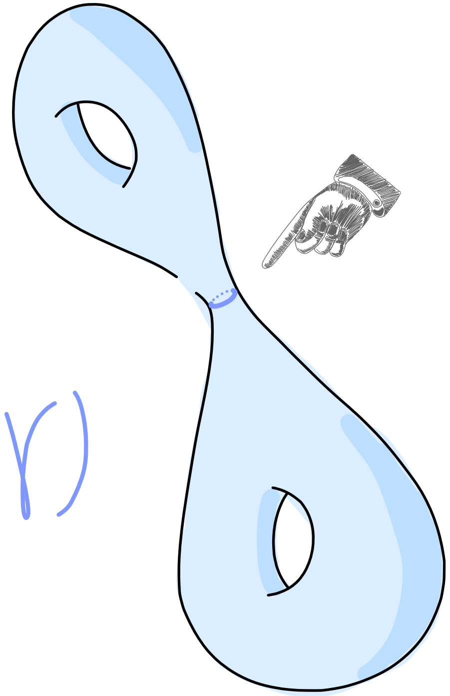
≈ 1960

For any hyperbolic surface X

$$N_{[0,L]}(X) \underset{L\to\infty}{\sim} \frac{e^L}{2L}$$

Corollary

$$\mathrm{sys}(X) := \min_{\gamma \text{ geod on } X} \ell(\gamma)$$

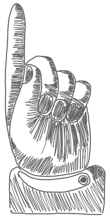


$$\mathbb{E}\left(\mathrm{sys}(X_g)\right) \xrightarrow[g\to\infty]{} 1.615\ldots$$

What is a map ?

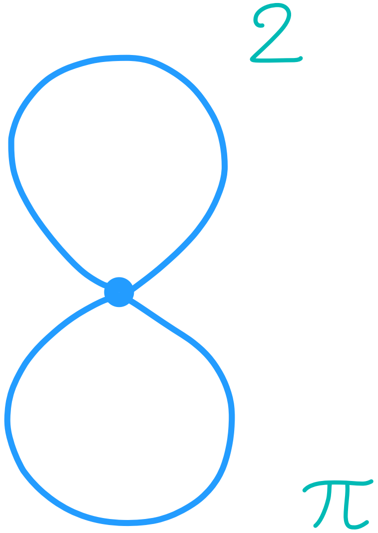
What is a map ?

A map is a graph G drawn on a surface S such that $S \setminus G$ is a disjoint union of polygons

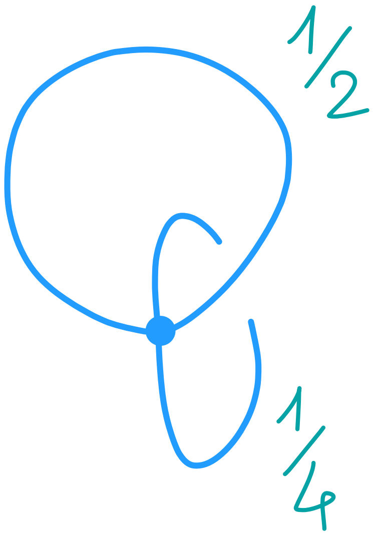


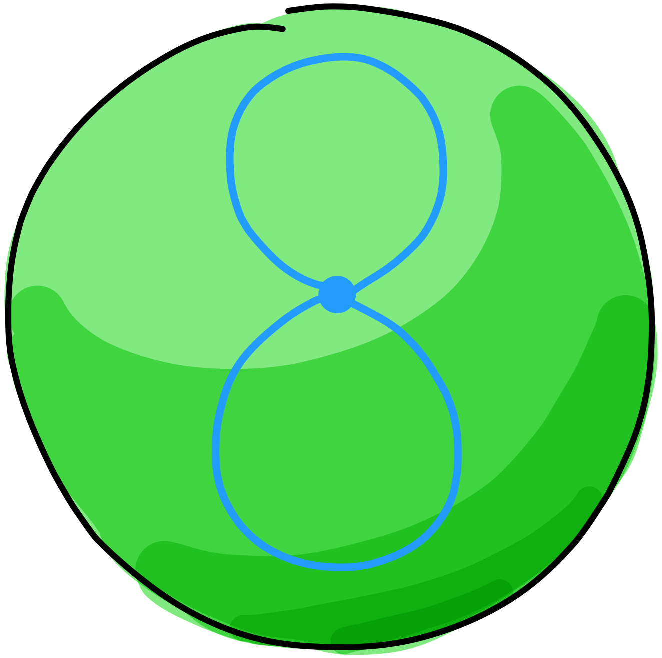
faces

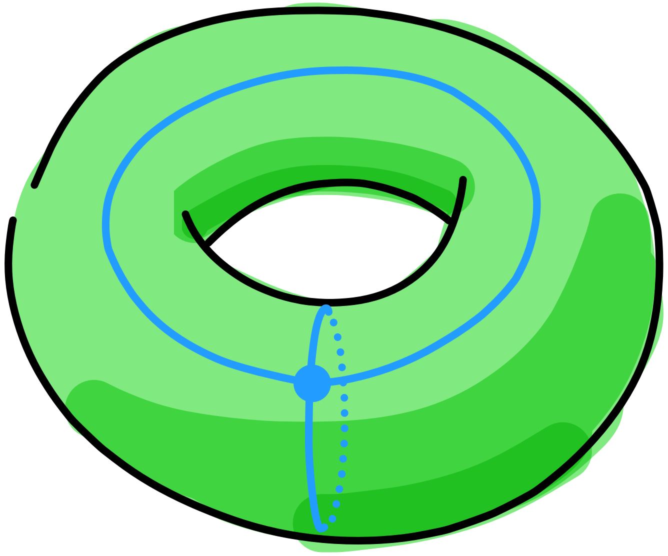
Example



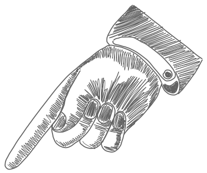
metric map





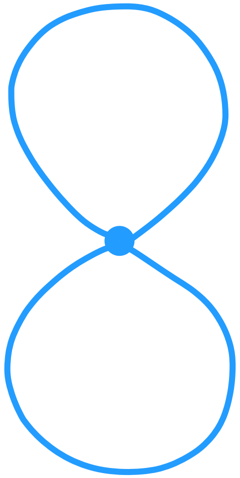


n° of faces



$g = 0, n = 3$ $g = 1, n = 1$

Example



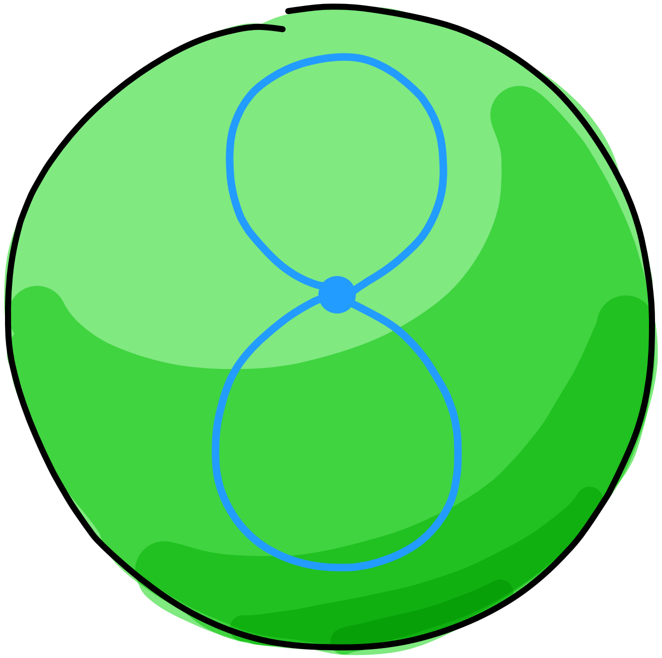
Def

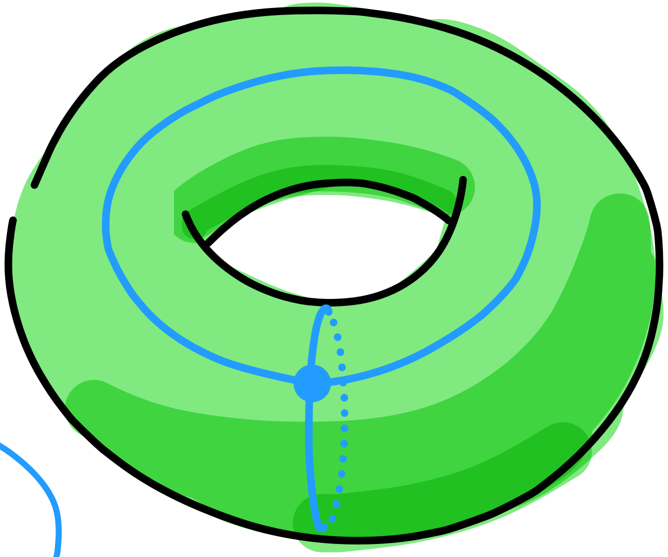
Unicellular if $n = 1$

$g = 0, \; n = 1$

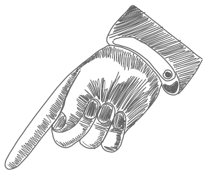
$\Downarrow$

plane trees

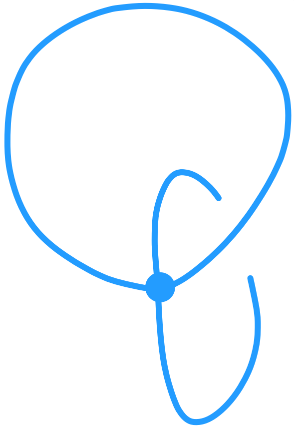




n° of faces



$g = 0, \; n = 3$



$g = 1, \; n = 1$

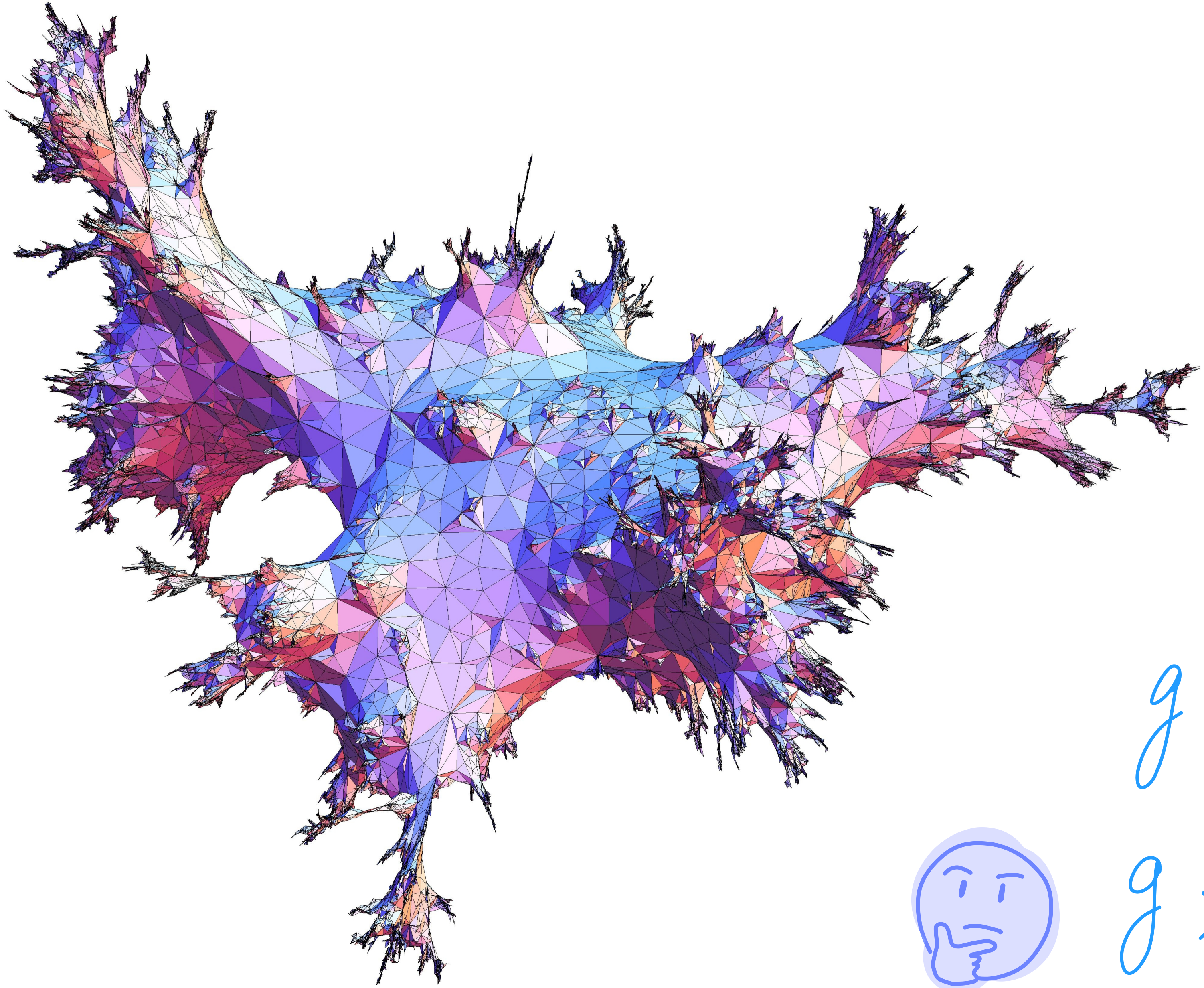
Tutte

(1960s)

Le Gall, Miermont

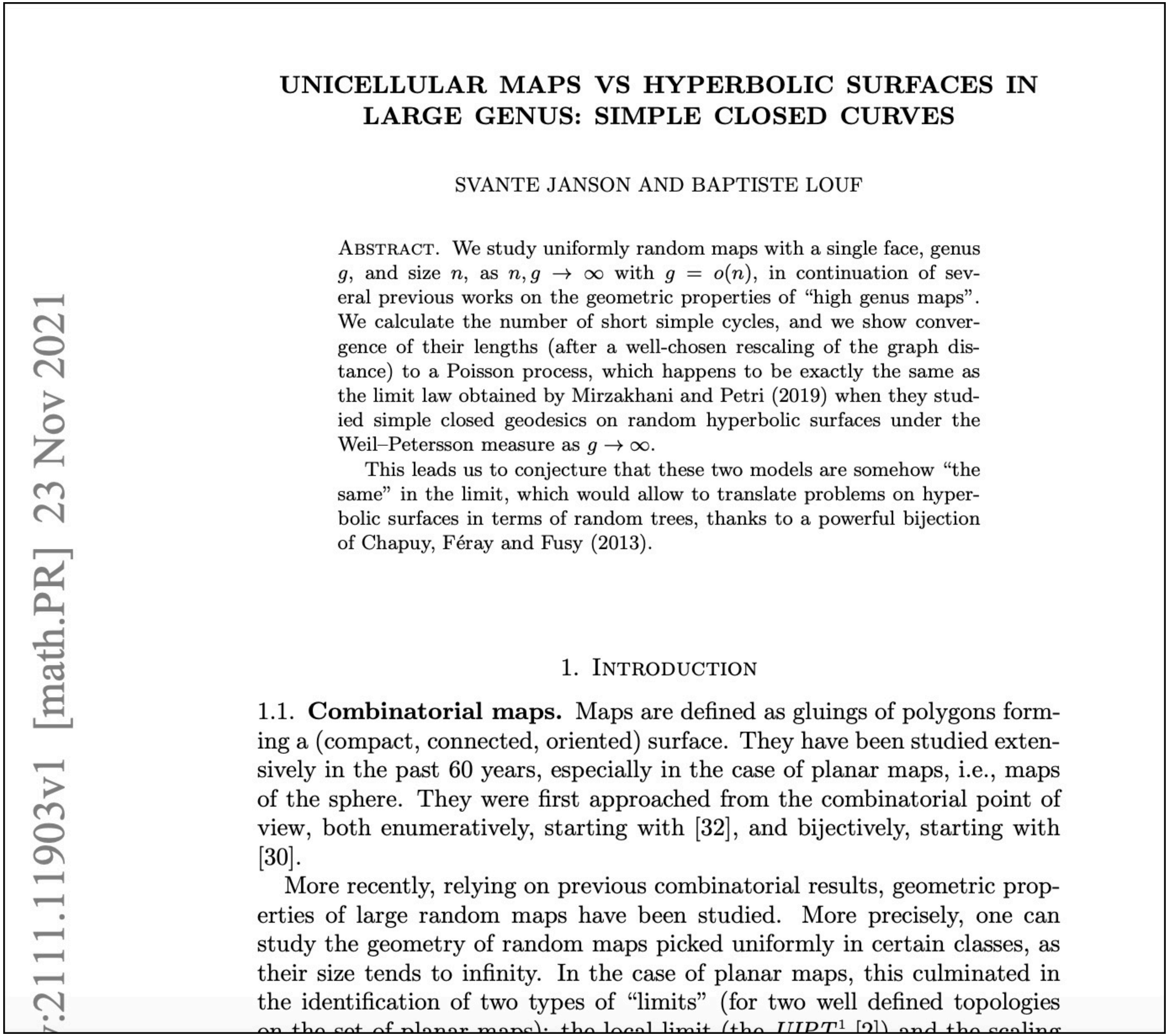
(2011)

Brownian sphere: scaling limit of uniform random plane triangulations / quadrangulations



$g = 0$

🤔 $g \gg 0$



UNICELLULAR MAPS VS HYPERBOLIC SURFACES IN LARGE GENUS: SIMPLE CLOSED CURVES

SVANTE JANSON AND BAPTISTE LOUF

Abstract. We study uniformly random maps with a single face, genus g, and size n, as $n, g \to \infty$ with $g = o(n)$, in continuation of several previous works on the geometric properties of "high genus maps". We calculate the number of short simple cycles, and we show convergence of their lengths (after a well-chosen rescaling of the graph distance) to a Poisson process, which happens to be exactly the same as the limit law obtained by Mirzakhani and Petri (2019) when they studied simple closed geodesics on random hyperbolic surfaces under the Weil–Petersson measure as $g \to \infty$.

This leads us to conjecture that these two models are somehow "the same" in the limit, which would allow to translate problems on hyperbolic surfaces in terms of random trees, thanks to a powerful bijection of Chapuy, Féray and Fusy (2013).

1. Introduction

1.1. Combinatorial maps.

Maps are defined as gluings of polygons forming a (compact, connected, oriented) surface. They have been studied extensively in the past 60 years, especially in the case of planar maps, i.e., maps of the sphere. They were first approached from the combinatorial point of view, both enumeratively, starting with [32], and bijectively, starting with [30].

More recently, relying on previous combinatorial results, geometric properties of large random maps have been studied. More precisely, one can study the geometry of random maps picked uniformly in certain classes, as their size tends to infinity. In the case of planar maps, this culminated in the identification of two types of "limits" (for two well defined topologies on the set of planar maps): the local limit (the $UIPT^1$ [2]) and the scaling

$\mathcal{U}$ a $\overset{n=1}{\text{unicellular}}$ map of genus g with v vertices

$$N_{[a,b)}(\rho \cdot \mathcal{U}) := \#\{\gamma_{\text{cycle in } \mathcal{U}} \mid a \le \rho \cdot \ell(\gamma) < b\}$$

$$\rho = \sqrt{12g/v}$$

Theorem (Janson – Louf, 2021)

$n=1$

$\mathcal{U}$ a unicellular map of genus g with v vertices

$$\mathcal{N}_{[a,b)}(\rho\cdot\mathcal{U}) := \#\left\{\gamma \text{ cycle in } \mathcal{U} \;\middle|\; a \leq \underset{\sqrt{12g/v}}{\underset{\shortparallel}{\rho}}\cdot\ell(\gamma) < b\right\}$$

<u>Theorem</u> (<u>Janson–Louf, 2021</u>)

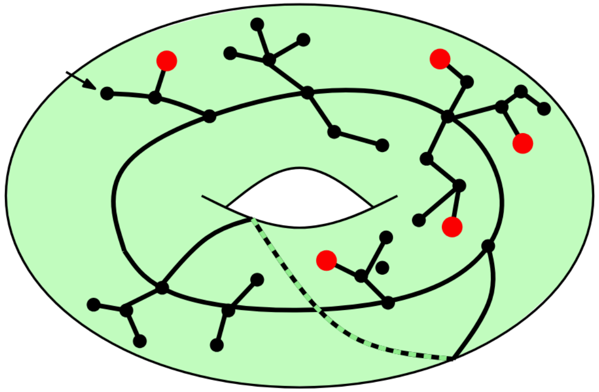
$v, g \to \infty$ with $g = o(v)$

$\mathcal{U}_{v,g}$ a uniform unicellular map of genus g with v vertices

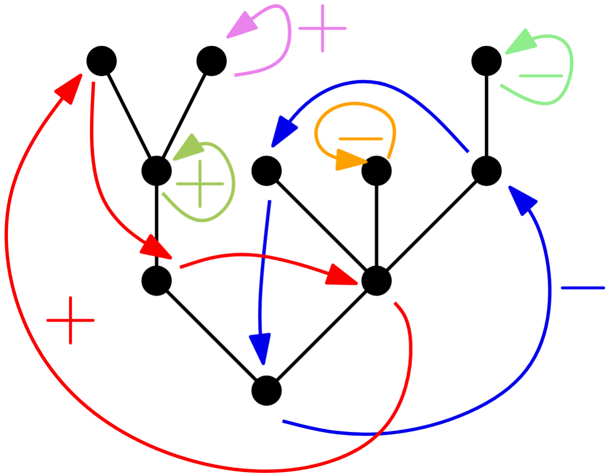
$$\mathcal{N}_{[a,b)}(\rho\cdot\mathcal{U}_{v,g}) \xrightarrow[g\to\infty]{(d)} \mathrm{Poi}\left(\int_a^b \lambda(x)\,dx\right)$$

One word about the proof

a magic bijection due to Chapuy-Féray-Fusy



$\xleftrightarrow{\text{bij}}$



unicellular map

plane tree + permutation

0517v1 [math.PR] 16 Dec 2023

LENGTH SPECTRUM OF LARGE GENUS RANDOM METRIC MAPS

SIMON BARAZER, ALESSANDRO GIACCHETTO, AND MINGKUN LIU

Abstract. We study the length of short cycles on uniformly random metric maps (also known as ribbon graphs) of large genus using a Teichmüller theory approach. We establish that, as the genus tends to infinity, the length spectrum converges to a Poisson point process with an explicit intensity. This result extends the work of Janson and Louf to the multi-faced case.

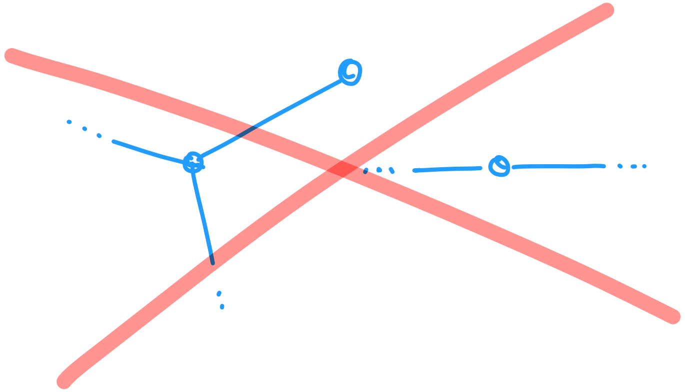
1. Introduction

A *map*, or a *ribbon graph*, is a graph with a cyclic ordering of the edges at each vertex. By substituting edges with ribbons and attaching them at each vertex in accordance with the given cyclic order, we create an oriented surface with boundaries on which the graph is drawn (see Figure 1). Since Tutte's pioneering work [Tut63], ribbon graphs have been extensively studied, partly due to the increased interest following the realisation of their importance in two-dimensional quantum gravity.

Much attention has been devoted to the study of *metric* maps, i.e. ribbon graphs with the assignment of a positive real number to each edge. Remarkably, the moduli space parametrising metric ribbon graphs of a fixed genus g and n faces of fixed lengths is naturally isomorphic to the moduli space of Riemann surfaces of genus g with n punctures [Har86; Pen87; BE88]. This fact was employed by Harer and Zagier to compute the Euler characteristic of the moduli space of Riemann surfaces [HZ86] and by Kontsevich in his proof of Witten's conjecture [Wit91; Kon92]. The latter is a formula that computes the "number" of metric ribbon graphs recursively on the Euler characteristic: a topological recursion. The same type of recursion applies to the "number" of hyperbolic surfaces, as discovered by Mirzakhani [Mir07].

$$\vec{L} = (L_1, \dots L_n) \in \mathbb{R}^n_{>0}$$

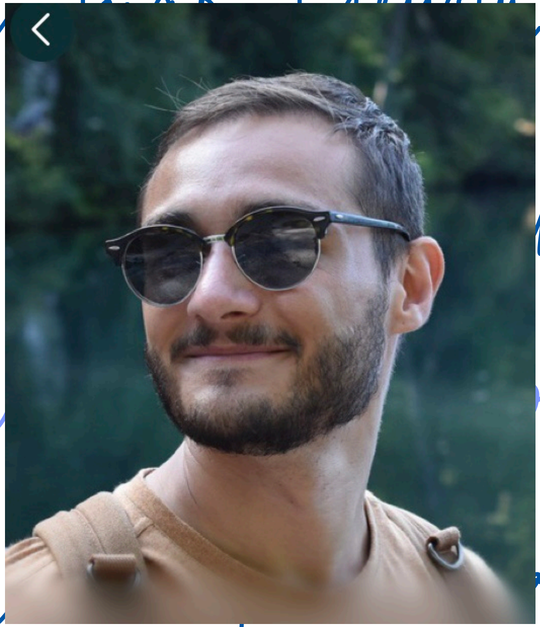
$$M^{comb}_{g,n}(\vec{L}) := \left\{ \begin{array}{c} G \text{ metric map} \\ \text{genus } g \text{ with } n \text{ faces} \end{array} \middle| \begin{array}{c} \text{length}(G\text{' i-th face}) = L_i \\ \text{valences} \geqslant 3 \end{array} \right\}$$



$\vec{L} = (L_1, \dots L_n) \in \mathbb{R}^n_{>0}$

$M^{\text{comb}}_{g,n}(\vec{L}$ $\{ G$ metri length$(G$'i-th face$) = L_i \}$

ith mes $\geqslant 3$



$\exists$ a natu n $M^{\text{comb}}_{g,n}(\vec{L})$

$G_{g,n}(\vec{L}$ p r.t. this measure

Theorem (Barazer-Giacchetto-L)

$$\vec{\mathcal{L}} = (\mathcal{L}_1, \ldots \mathcal{L}_n) \in \mathbb{R}^n_{>0}$$

$$\mathcal{M}^{\text{comb}}_{g,n}(\vec{\mathcal{L}}) := \left\{ \begin{array}{c} \mathcal{G} \text{ metric map} \\ \text{genus } g \text{ with } n \text{ faces} \end{array} \middle| \begin{array}{c} \text{length}(\mathcal{G}\text{'i-th face}) = \mathcal{L}_i \\ \text{valences} \geqslant 3 \end{array} \right\}$$

∃ a natural "uniform" proba measure on $\mathcal{M}^{\text{comb}}_{g,n}(\vec{\mathcal{L}})$

$\boldsymbol{\mathcal{G}}_{g,n}(\vec{\mathcal{L}})$ a random map sampled w.r.t. this measure

Theorem (Barazer-Giacchetto-L)

n fixed, $g \to \infty$, $\mathcal{L}_1 + \cdots + \mathcal{L}_n \sim 12g$.

length of each edge ≈ 1

$$N_{[a,b)}(\boldsymbol{\mathcal{G}}_{g,n}(\vec{\mathcal{L}})) \xrightarrow[g \to \infty]{(d)} \mathrm{Poi}\left(\int_a^b \lambda(x)\,dx\right)$$

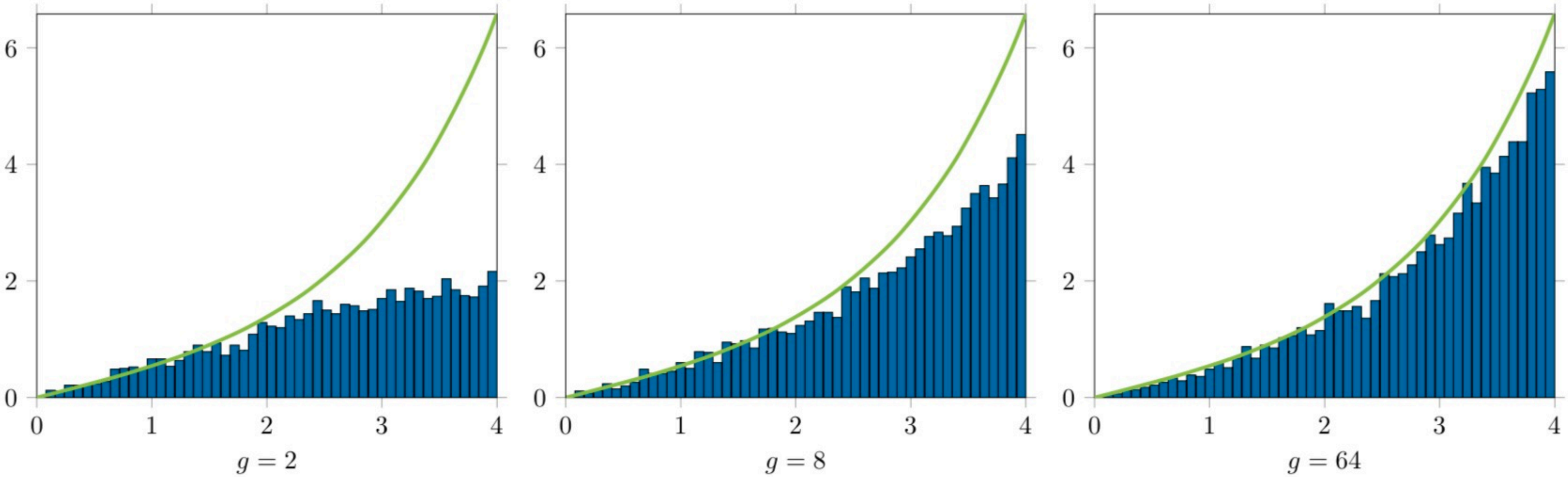


FIGURE 2. In blue, the cycle length statistics of random unicellular metric maps of genus $g = 2$, 8, and 64, sampled over 10^3 units and properly rescaled. The predicted intensity λ is depicted in lime.

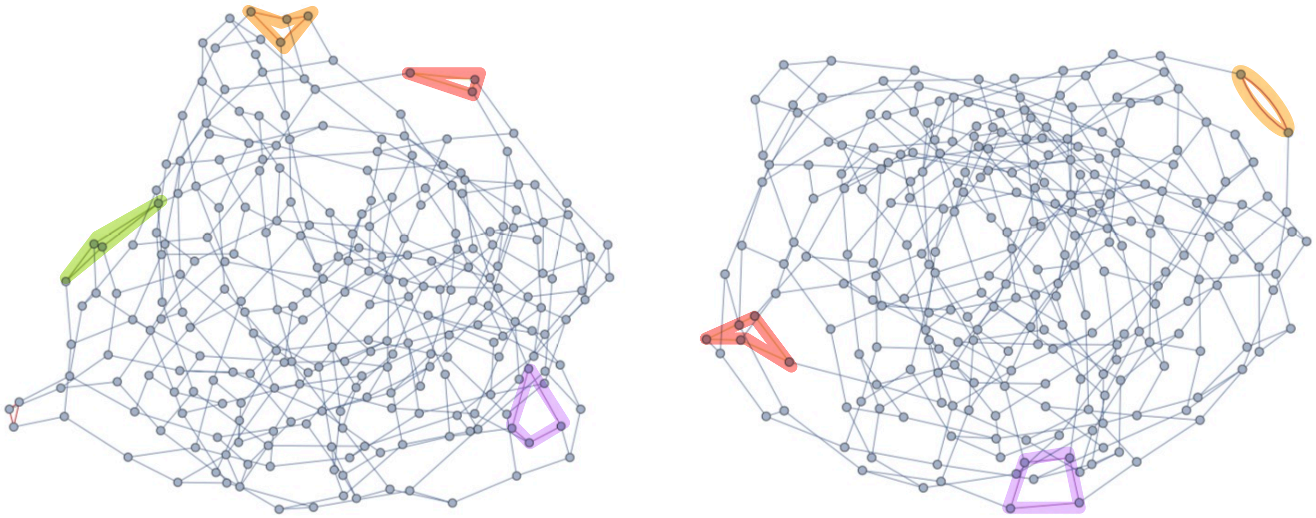


FIGURE 9. The graphs underlying two random unicellular maps of genus 64. The highlighted cycles include all cycles with at most 4 edges.

One word about the proof

$$M^{comb}_{g,n}(\vec{L}) \xrightarrow[\text{homeo}]{\sim} M_{g,n}(\vec{L})$$

Bowditch, Epstein, Luo, Penner

$$\| \qquad\qquad\qquad\qquad \|$$

$\{$ metric m

$\{$ hyperboli … of genus g
with n … boundaries
of lengths $L_1, \ldots, L_n$ $\}$

One word about the proof

and much more than that...

$$\begin{tikzcd}
\mathcal{M}^{\mathrm{comb}}_{g,n}(\vec{L}) \arrow[r, dash, "\sim", "\text{homeo}"'] \arrow[d, equal] & \mathcal{M}_{g,n}(\vec{L}) \arrow[d, equal] \\
\{\text{metric maps}\} & \left\{\begin{array}{c}\text{hyperbolic surfaces of genus } g \\ \text{with } n \text{ geodesic boundaries} \\ \text{of lengths } L_1, \dots, L_n\end{array}\right\}
\end{tikzcd}$$

Bowditch, Epstein,
Luo, Penner

22 Oct 2020

ON THE KONTSEVICH GEOMETRY OF THE COMBINATORIAL TEICHMÜLLER SPACE

Jørgen Ellegaard Andersen[**], Gaëtan Borot[*†], Séverin Charbonnier[*],
Alessandro Giacchetto[*], Danilo Lewański[*‡§], Campbell Wheeler[*]

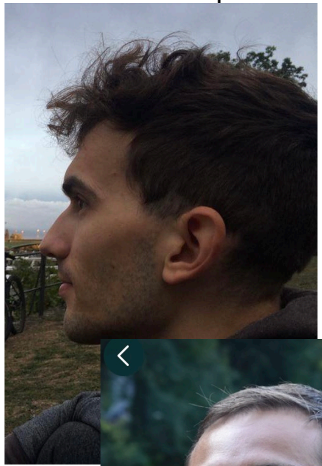
Abstract

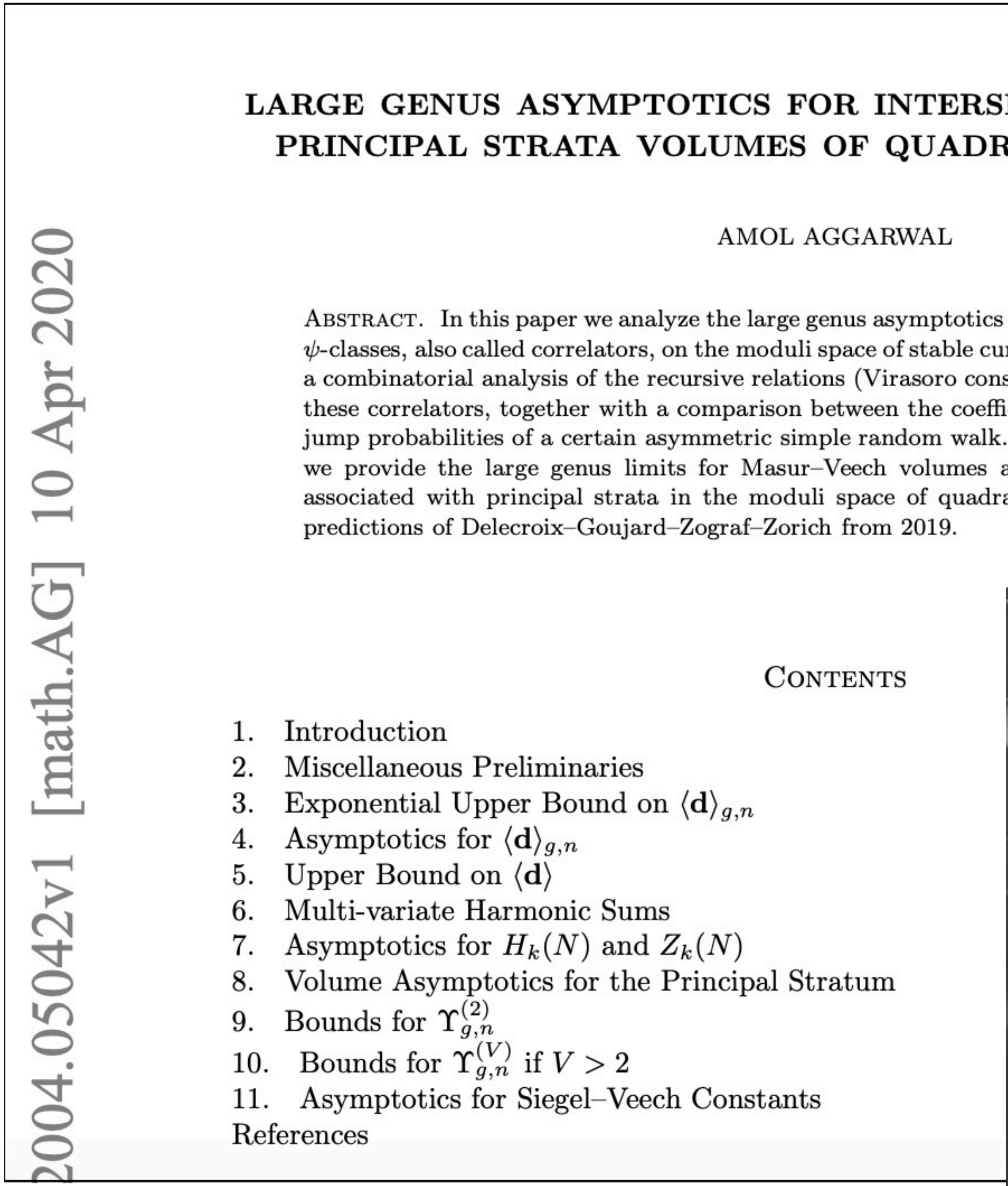
study the combinatorial Teichmüller space and construct on it global coordinates, anal-
o the Fenchel–Nielsen coordinates on the ordinary Teichmüller space. We prove that
ordinates form an atlas with piecewise linear transition functions, and constitute global Darboux coordinates for the Kontsevich symplectic structure on top-dimensional cells.

We then set up the geometric recursion in the sense of Andersen–Borot–Orantin adapted to the combinatorial setting, which naturally produces mapping class group invariant functions on the combinatorial Teichmüller spaces. We establish a combinatorial analogue of the Mirzakhani–McShane identity fitting this framework.

As applications, we obtain geometric proofs of Witten conjecture/Kontsevich theorem (Virasoro constraints for ψ-classes intersections) and of Norbury's topological recursion for the lattice point count in the combinatorial moduli spaces. These proofs arise now as part of a unified theory and proceed in perfect parallel to Mirzakhani's proof of topological recursion for the Weil–Petersson volumes.

We move on to the study of the spine construction and the associated rescaling flow on the Teichmüller space. We strengthen former results of Mondello and Do on the convergence of this flow. In particular, we prove convergence of hyperbolic Fenchel–Nielsen coordinates to the combinatorial ones with some uniformity. This allows us to effectively carry natural constructions on the Teichmüller space to their analogues in the combinatorial spaces. For instance, we





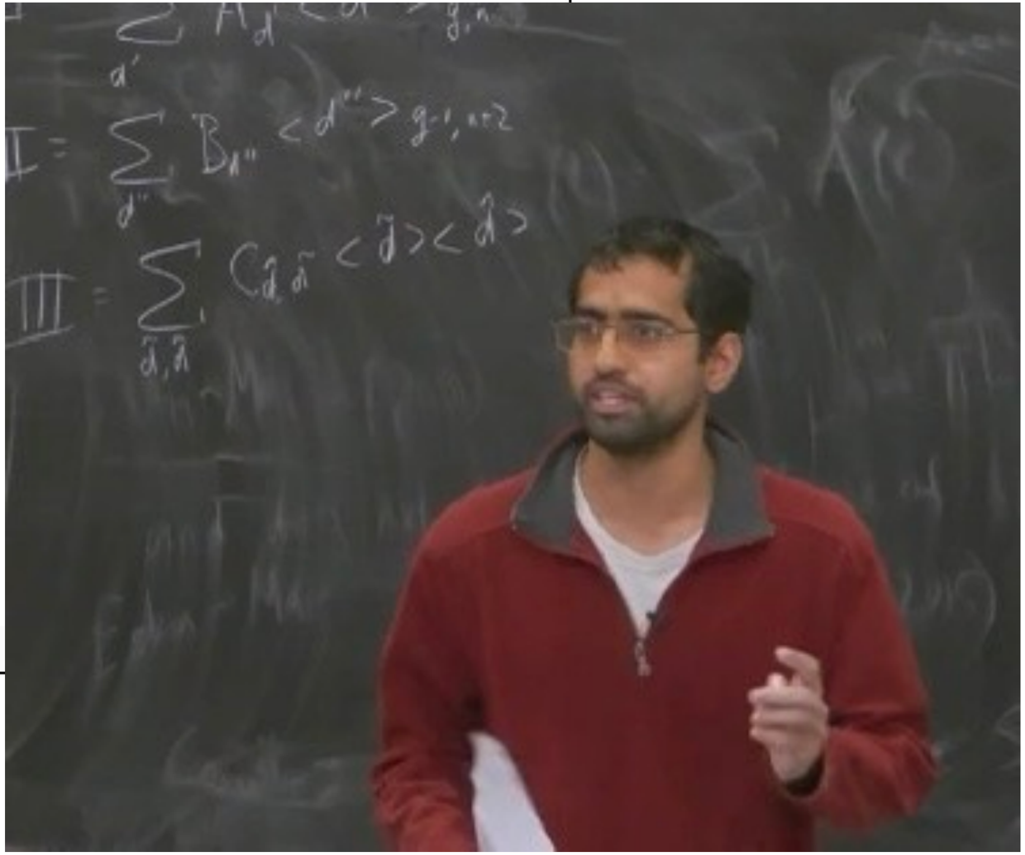
2004.05042v1 [math.AG] 10 Apr 2020

LARGE GENUS ASYMPTOTICS FOR INTERSECTION NUMBERS AND PRINCIPAL STRATA VOLUMES OF QUADRATIC DIFFERENTIALS

AMOL AGGARWAL

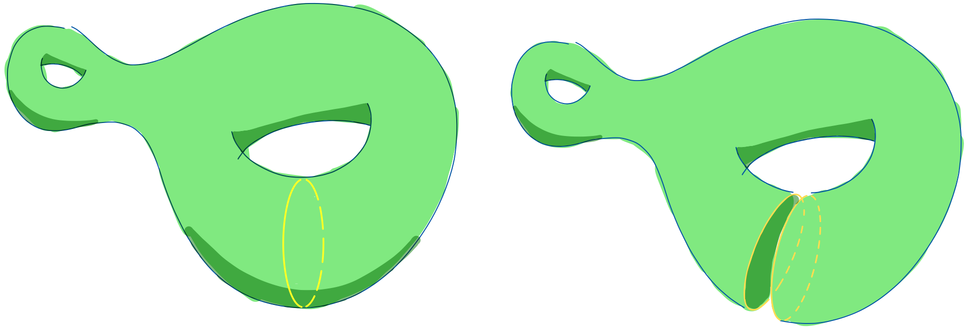
ABSTRACT. In this paper we analyze the large genus asymptotics for intersection numbers between ψ-classes, also called correlators, on the moduli space of stable curves. Our proofs proceed through a combinatorial analysis of the recursive relations (Virasoro constraints) that uniquely determine these correlators, together with a comparison between the coefficients in these relations with the jump probabilities of a certain asymmetric simple random walk. As an application of this result, we provide the large genus limits for Masur–Veech volumes and area Siegel–Veech constants associated with principal strata in the moduli space of quadratic differentials. These confirm predictions of Delecroix–Goujard–Zograf–Zorich from 2019.

CONTENTS



simple, non-separating

$$\mathbb{E}\left(N^{*}_{[a,b)}(X_g)\right)$$



simple, non-separating

$$\mathbb{E}\left(N^{*}_{[a,b)}(X_g)\right) = \frac{1}{\#\mathcal{M}_g} \sum_{X\in\mathcal{M}_g} N_{[a,b)}(X)$$

if everything was $\mathcal{M}_g$ discrete ...

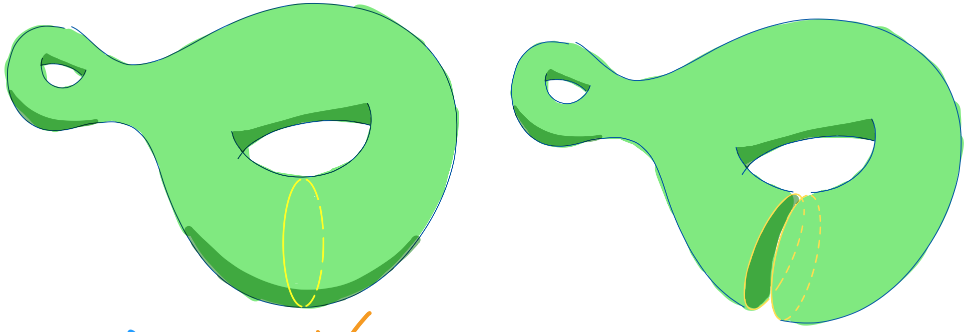
$$= \frac{1}{\#\mathcal{M}_g} \#\left\{(X,\gamma) \,\middle|\, \begin{array}{l} X\in\mathcal{M}_g, \gamma \text{ geod on } X \\ a\leq \ell(\gamma)<b \end{array}\right\}$$

$\Downarrow$

$$(X,\gamma) \longrightarrow X' \in \mathcal{M}_{g-1,2}(\ell,\ell)$$

$$\parallel$$

{ hyp surf of genus $g-1$ with 2 geod bd of length ℓ }



length γ

$\ell(\gamma)=\ell\in[a,b)$

simple, non-separating

$$\mathbb{E}\left(N^{*}_{[a,b)}(X_g)\right) = \frac{1}{\#\mathcal{M}_g} \sum_{X\in\mathcal{M}_g} N_{[a,b)}(X)$$

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$$= \frac{1}{\#\mathcal{M}_g} \#\left\{(X,\gamma) \,\middle|\, \begin{array}{l} X\in\mathcal{M}_g, \gamma \text{ geod on } X \\ a \leq \ell(\gamma) < b \end{array}\right\}$$

$$\|$$

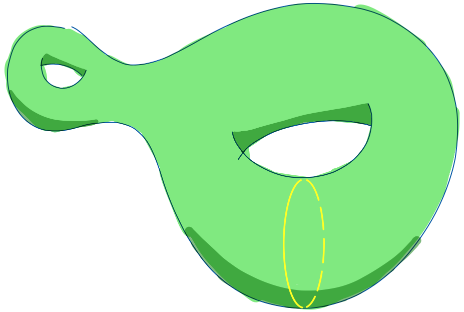
$$\frac{1}{\#\mathcal{M}_g} \sum_{\ell=a}^{b} \#\left\{(X',\tau) \,\middle|\, \begin{array}{l} X'\in\mathcal{M}_{g-1,2}(\ell,\ell) \\ 0\leq\tau<\ell \end{array}\right\}$$

$$= \frac{1}{\#\mathcal{M}_g} \sum_{\ell=a}^{b} \ell \cdot \#\mathcal{M}_{g-1,2}(\ell,\ell)$$

$\mathcal{M}_g$

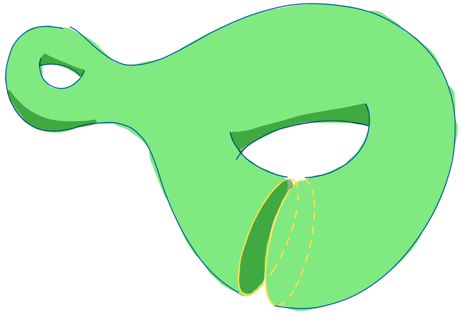
$$\Downarrow$$

$$(X,\gamma) \overset{1:1}{\longleftrightarrow} (X',\tau)$$



length γ

$\ell(\gamma) = \ell \in [a,b)$



twist

$0 \leq \tau < \ell$

simple, non-separating

$$\mathbb{E}\left(N^{*}_{[a,b)}(X_g)\right) = \frac{1}{\mathrm{vol}_{WP}(\mathcal{M}_g)} \int_a^b \ell \cdot \mathrm{vol}_{WP}(\mathcal{M}_{g-1,2}(\ell,\ell))\, d\ell$$

if everything was $\mathcal{M}_g$ discrete …

$$= \frac{1}{\#\mathcal{M}_g} \# \left\{ (X,\gamma) \,\middle|\, \begin{array}{l} X \in \mathcal{M}_g, \gamma \text{ geod on } X \\ a \leqslant \ell(\gamma) < b \end{array} \right\}$$

$$\|$$

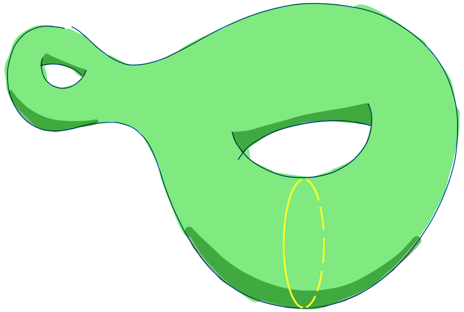
$$\frac{1}{\#\mathcal{M}_g} \sum_{\ell=a}^{b} \# \left\{ (X',\tau) \,\middle|\, \begin{array}{l} X' \in \mathcal{M}_{g-1,2}(\ell,\ell) \\ 0 \leqslant \tau < \ell \end{array} \right\}$$

$$= \frac{1}{\#\mathcal{M}_g} \sum_{\ell=a}^{b} \ell \cdot \#\mathcal{M}_{g-1,2}(\ell,\ell)$$

$\mathcal{M}_g$

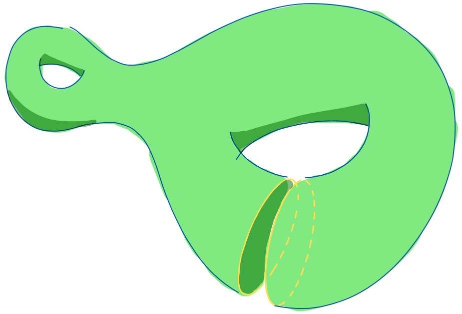
$\rotatebox{90}{$\in$}$

$$(X,\gamma) \overset{1:1}{\longleftrightarrow} (X',\tau)$$



length γ

$\ell(\gamma) = \ell \in [a,b)$



twist

$0 \leqslant \tau < \ell$

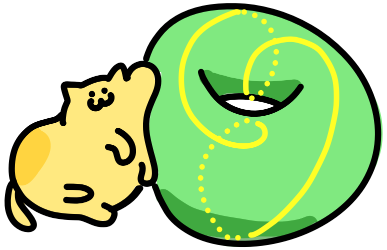
$$\mathbb{E}\left(\mathcal{N}^*_{[a,b)}(X_g)\right) = \int_a^b \frac{\ell \cdot \mathrm{vol}_{wp}(\mathcal{M}_{g-1,2}(\ell,\ell))}{\mathrm{vol}_{wp}(\mathcal{M}_g)}\, d\ell$$

Theorem (Mirzakhani–Petri)

$$\frac{\ell \cdot \mathrm{vol}_{wp}(\mathcal{M}_{g-1,2}(\ell,\ell))}{\mathrm{vol}_{wp}(\mathcal{M}_g)} \xrightarrow[g\to\infty]{} \lambda(\ell) = \frac{\cosh(\ell)-1}{\ell}$$

Theorem (BGL)

$$\frac{\ell \cdot \mathrm{vol}_{Kon}\left(\mathcal{M}^{comb}_{g-1,n+2}(\vec{L},\ell,\ell)\right)}{\mathrm{vol}_{Kon}\left(\mathcal{M}^{comb}_{g,n}(\vec{L})\right)} \xrightarrow[g\to\infty]{|\vec{L}|\sim 12g} \lambda(\ell)$$

Thank yu!