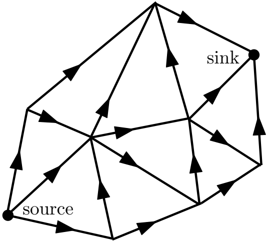
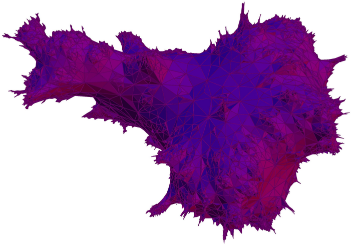
Directed distances in bipolar-oriented triangulations: exact exponent and scaling limits

Ewain Gwynne
(based on joint work with Jacopo Borga)

University of Chicago





Outline

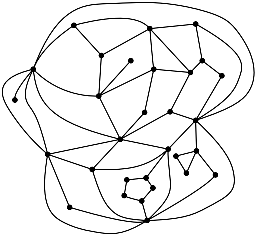
1 Introduction

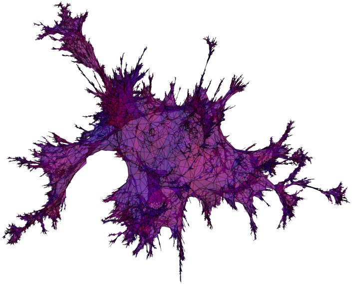
2 Busemann function and scaling limit

3 Proof ideas

4 Extensions and outlook

Random planar maps





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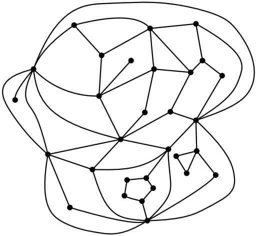

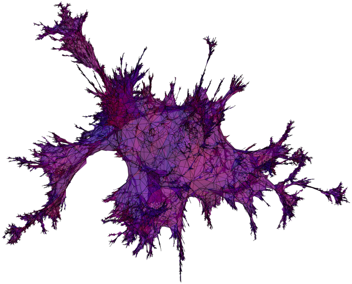

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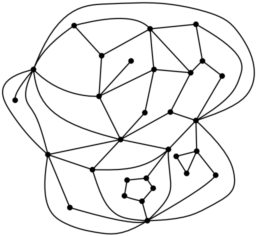

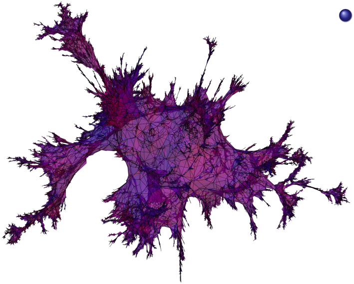

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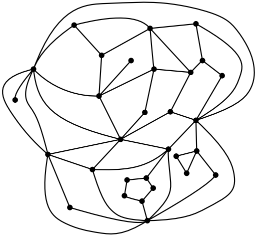

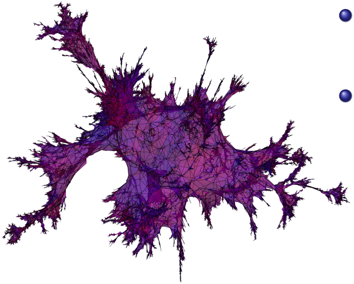

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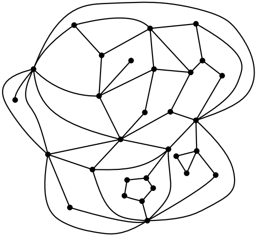

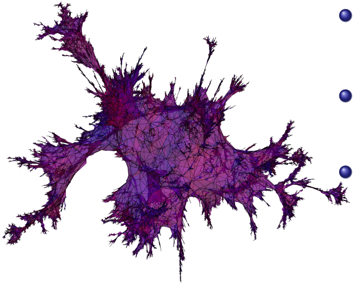

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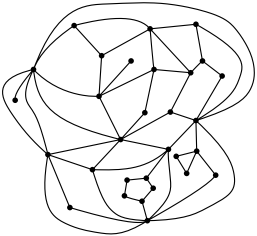

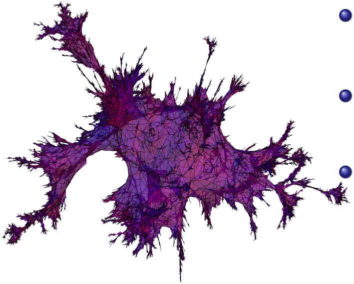

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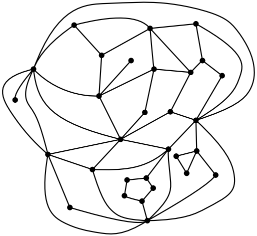

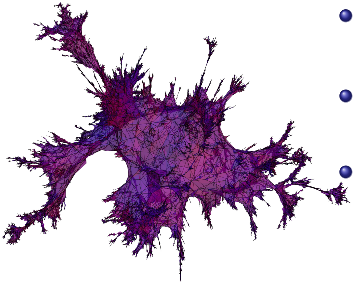

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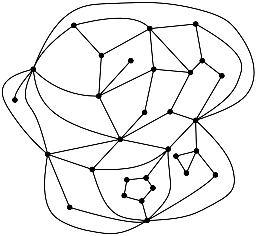

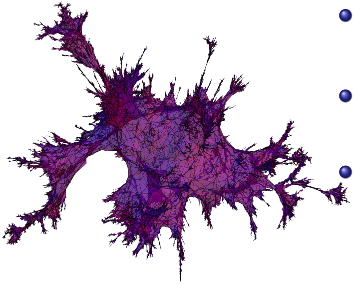

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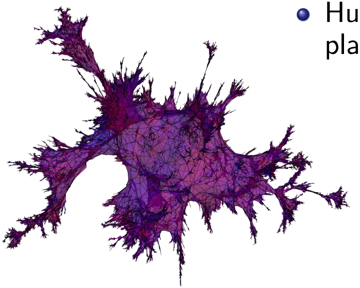
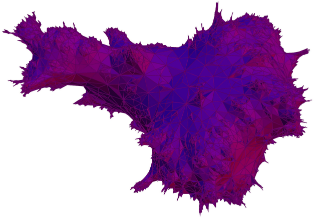
Undirected distances in random planar maps





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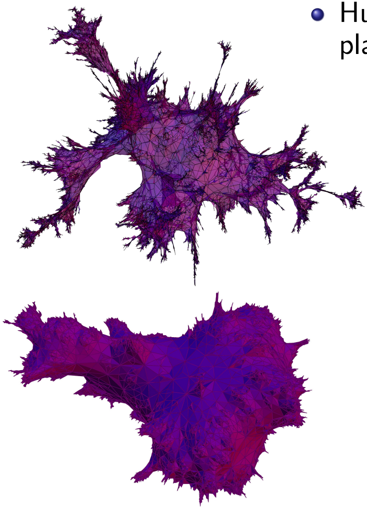

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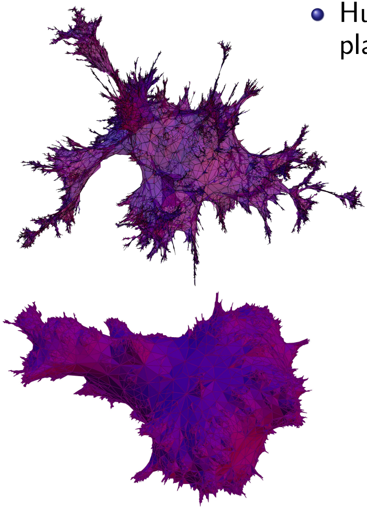

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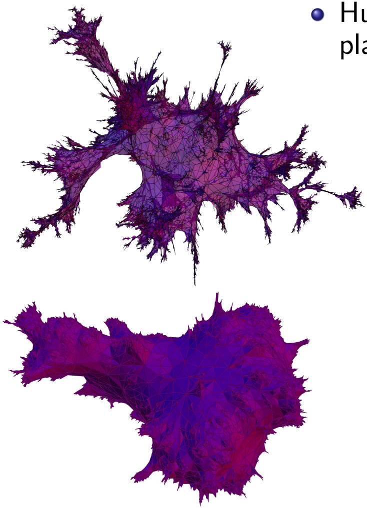

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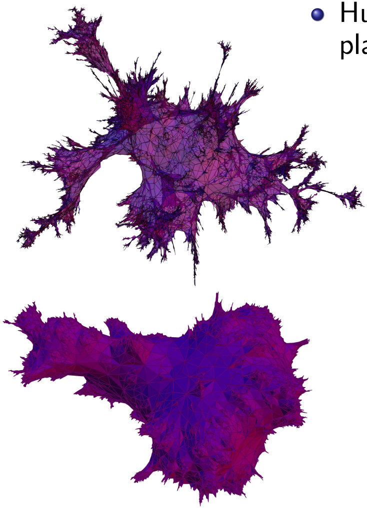

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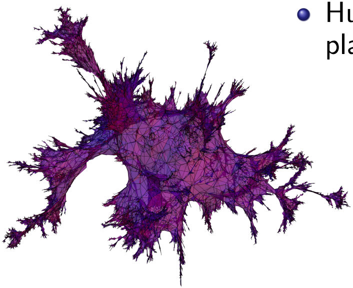

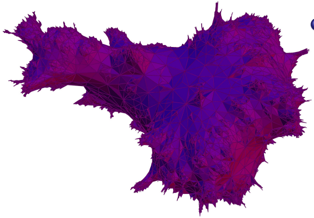

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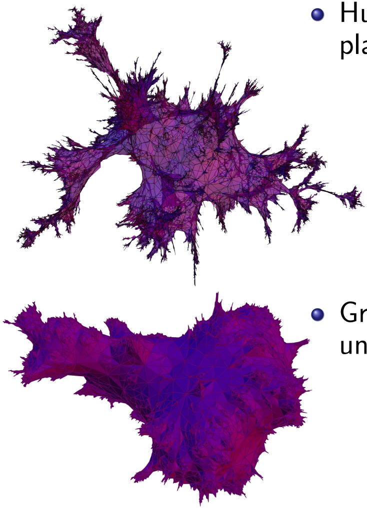

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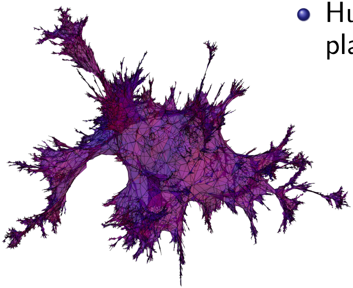

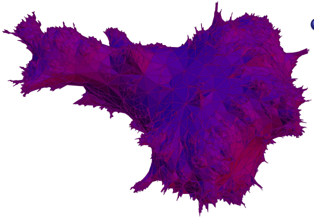

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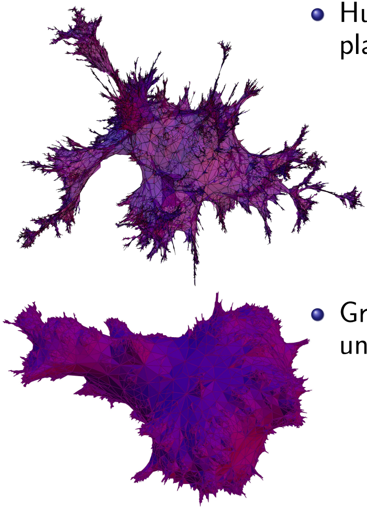

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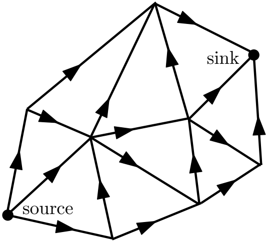
  - Proven in some cases up to $o(1)$ error in the exponent using LQG methods.
  - No known scaling limit results.

Directed distances in bipolar-oriented triangulations



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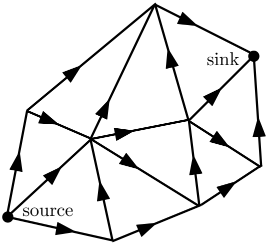

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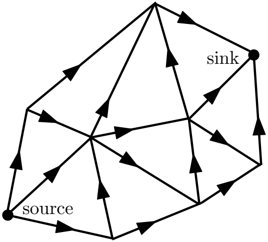

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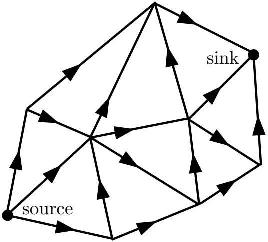

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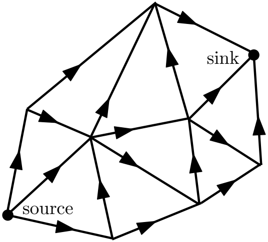

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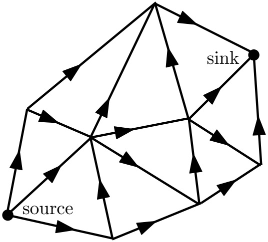

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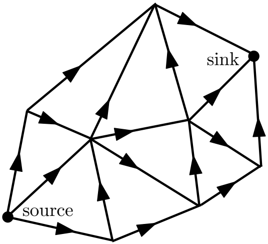

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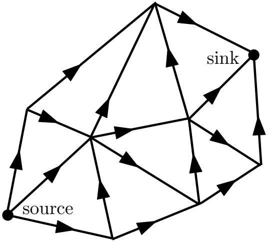

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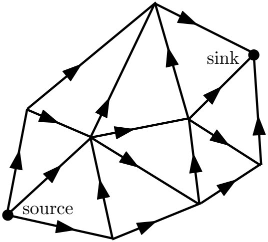

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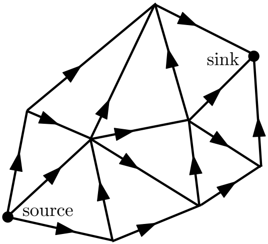

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- Technique also applies to other RPMs in other universality classes.

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Uniform infinite bipolar-oriented triangulation

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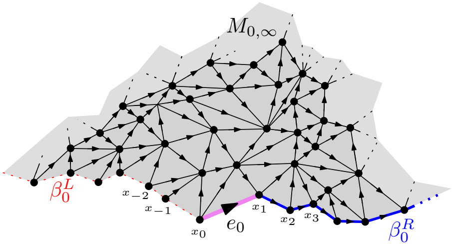
- Infinite directed random planar map with no source or sink (at ∞).



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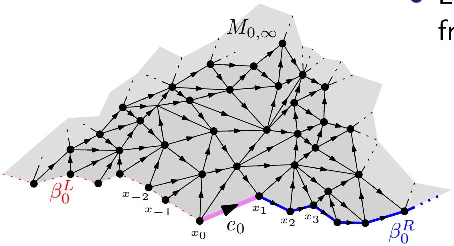

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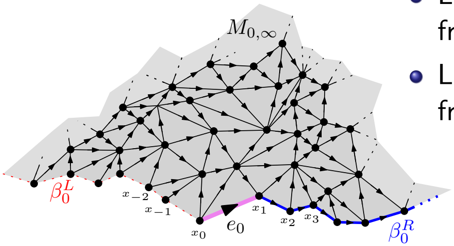

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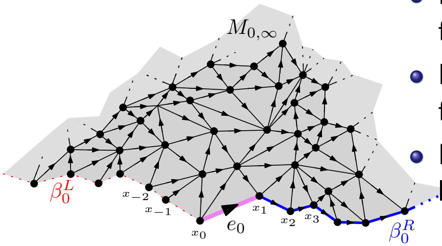

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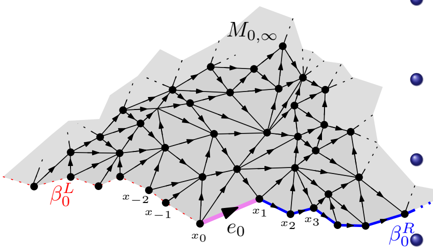

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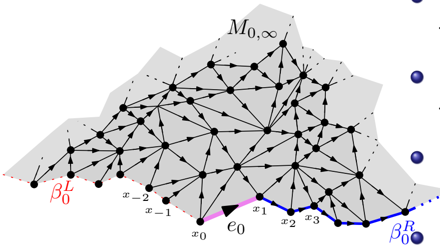

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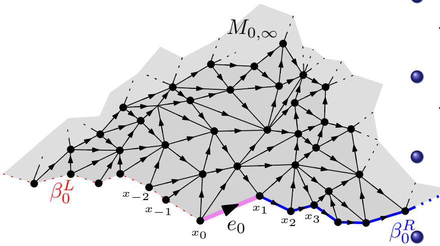

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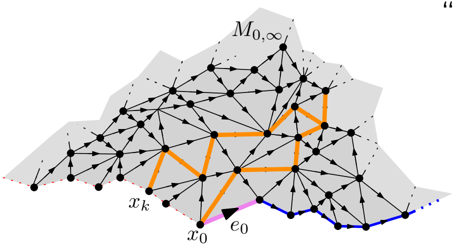
- $\{x_k\}_{k\in\mathbb{Z}}$ boundary vertices, from left to right.

Busemann function



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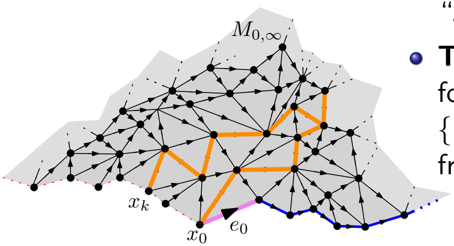

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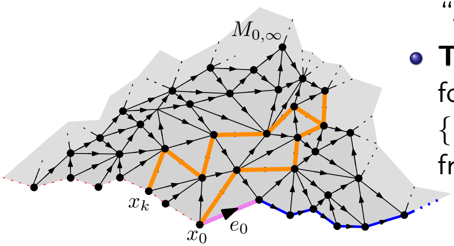

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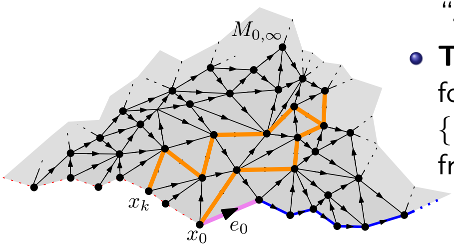

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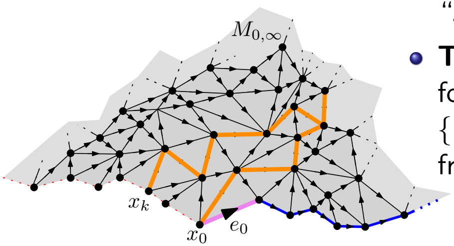

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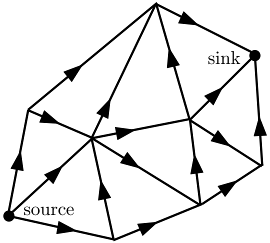
- I.e., directed distances along a single interface converge.

Finite bipolar-oriented triangulations



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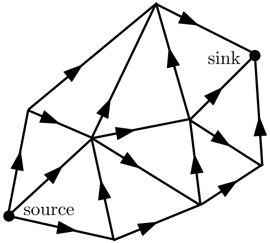

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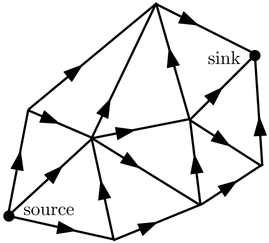

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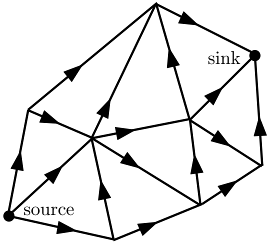

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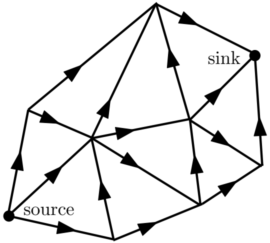

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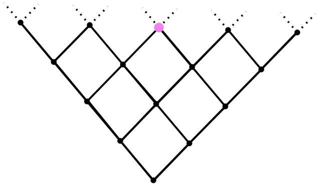
- Exact limiting law?

Analogy: last passage percolation on $\mathbb{Z}^2$



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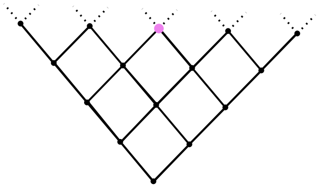

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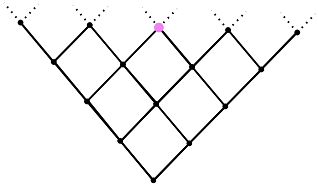

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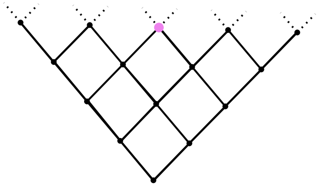

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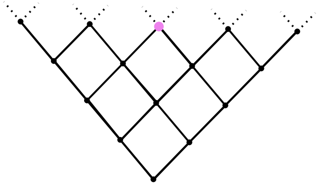

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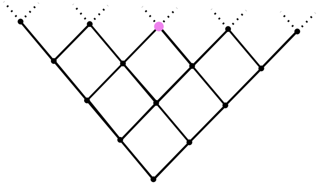

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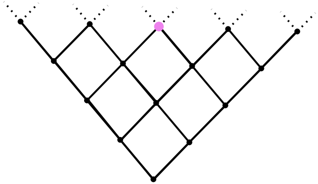

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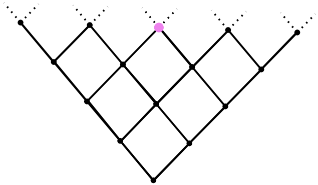

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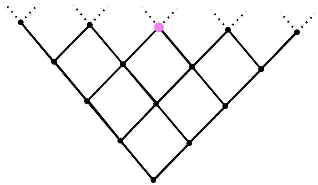

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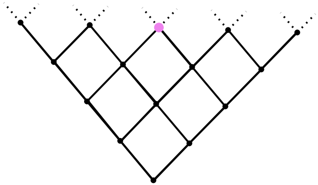

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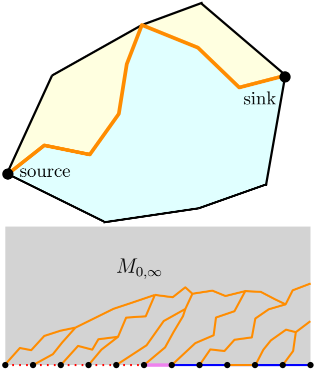
- Time evolution of Busemann function on $\mathbb{R} \times \{t\}$ governed by **KPZ equation**. Don't know analog of this for $\sqrt{4/3}$-LQG.

Outline

1. Introduction
2. Busemann function and scaling limit
3. Proof ideas
4. Extensions and outlook

Independence of geodesic slices



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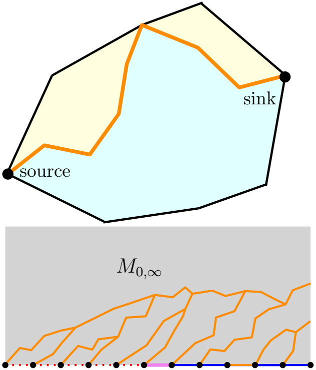

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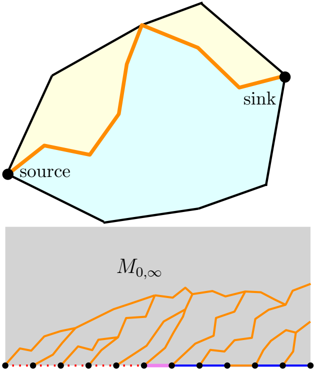

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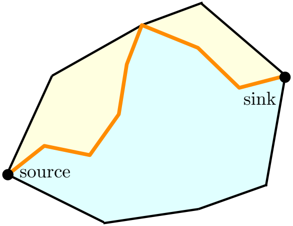

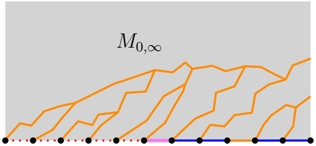

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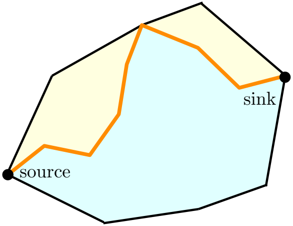

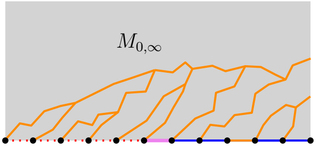

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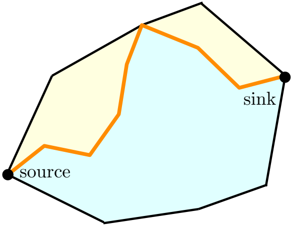

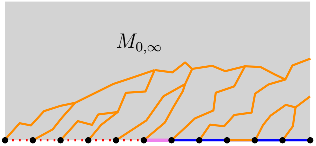

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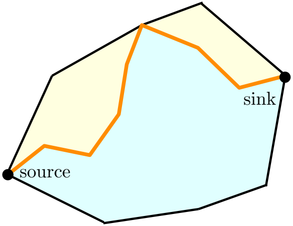

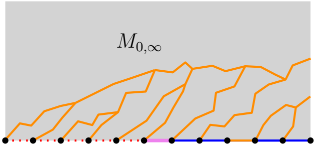

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- Can also show that $\{\mathcal{X}(k) - \mathcal{X}(k-1)\}_{k \geq 1}$ all have same law, and $\{\mathcal{X}(k) - \mathcal{X}(k-1)\}_{k \leq 0}$ all have same law.

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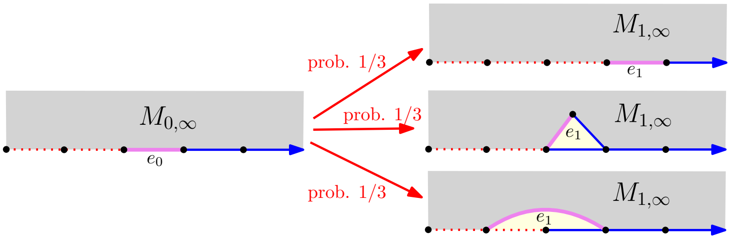

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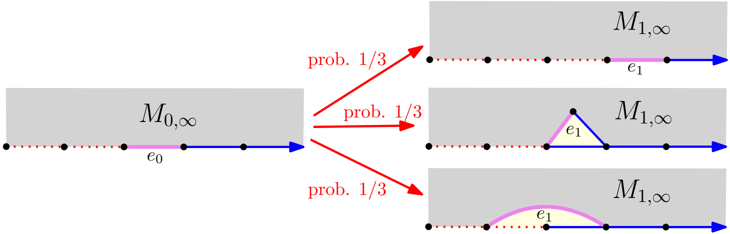

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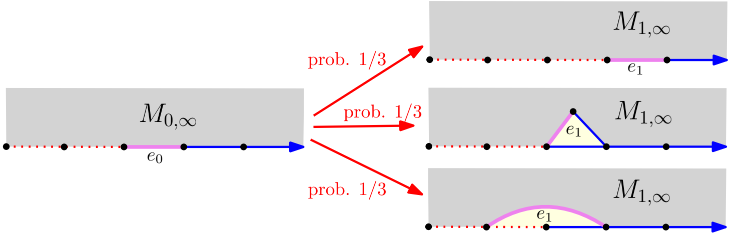

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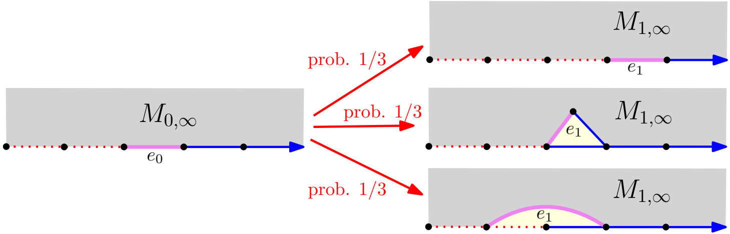

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3. Proof ideas
4. Extensions and outlook

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- **Conjecture:** the LIS for a uniform Baxter permutation of size n grows like $n^{3/4}$.

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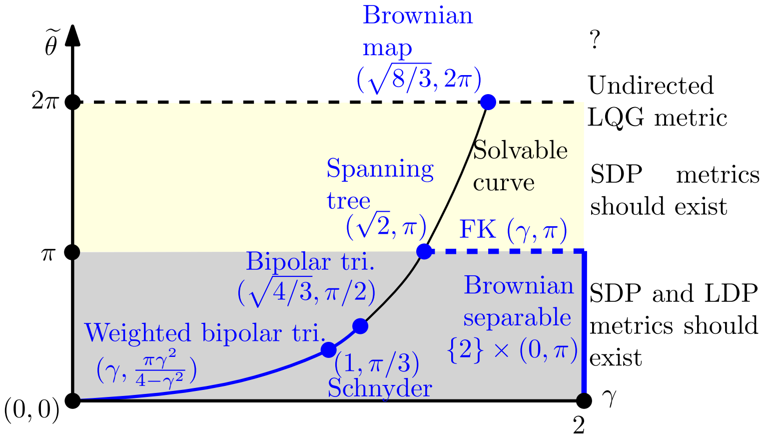
- More generally, should be two-parameter family of directed LQG metrics indexed by $\gamma \in (0,2)$, $\widetilde{\theta} \in (0,2\pi]$.

Directed LQG metrics



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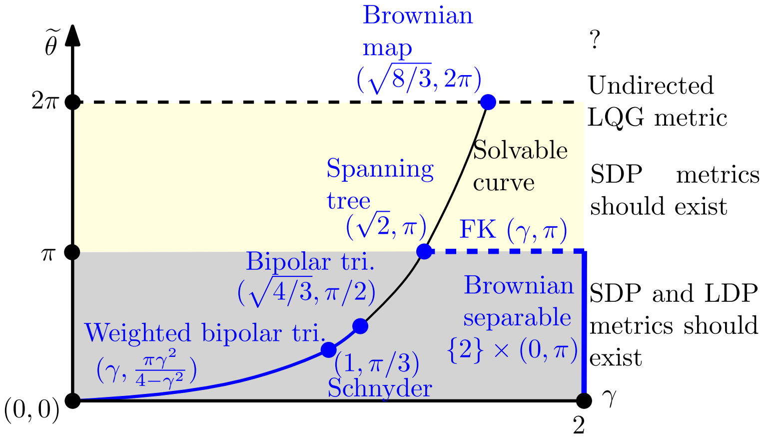

- Is there an interesting limit as $\gamma \to 0$ (appropriately renormalized)?