Quantum Field Theory Approach to Selberg-Dyson Integrals with large number of variables on Jordan curves

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(After discussions A. Etterer)

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Selberg 1941, Dyson 1961

1944 Selberg's paper "Remarks on a multiple integral" published in Norwegian

Atle Selberg: born in 1917, passed in Princeton on 2007, age 90,

$$\int_{[0,1]^N} \prod_{N\geq i>j\geq 1} |\xi_i - \xi_j|^{2\beta} = \prod_{j=0}^{N-1} \frac{\Gamma(\beta j)\,\Gamma(1+\beta j)\,\Gamma(1+\beta(1+j))}{\Gamma((N+j-1)\beta)\,\Gamma(1+\beta)}$$

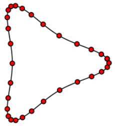
1961 Dyson's paper on random matrices

Freeman Dyson: born in 1923, passed in Princeton on 2020, age 96.

$$(2\pi)^{-N} \oint_{[S^1]^N} \prod_{N\geq i>j\geq 1} |\xi_i - \xi_j|^{2\beta} = \frac{\Gamma(1+N\beta)}{\Gamma^N(1+\beta)}$$

Primary interest

Dyson-Selberg integral on a Jordan curve at large number of variables $N \to \infty$



$$Z_{N,\beta}[\gamma] = \oint_{\gamma^N} \prod_{i>j\geq 1}^{N} |\xi_i - \xi_j|^{2\beta} = \exp\left(-F_0[\gamma,\beta]\right)\left(1 + \sum_{k>0} z_k[\gamma,\beta] N^{-k}\right)$$

- $\exp F_0[\gamma]$ is a spectral determinants of an elliptic operator (quantum geometry);
- $z_k[\gamma]$ are local functionals of curvature (complex geometry).

Coulomb gas

Statistical mechanics of particles with repulsive log (Coulomb) interaction

$$Z_{N,\beta}[\gamma] = \oint_{\gamma^N} \prod_{i>j} |\xi_i - \xi_j|^{2\beta} = \oint_{\gamma} e^{-\beta E(\xi_1,\dots,\xi_N)}, \quad E = -\sum_{i>j} \log|\xi_i - \xi_j|$$

Fekete points (1923) - finite-dimensional approximation of conformal maps

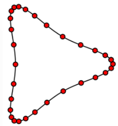
Riemann mapping $z \to w = f(z)$: an exterior of D to the exterior of a disk $\mathbb{D}$

$$\mathbb{C}\backslash D \to \mathbb{C}\backslash\mathbb{D}: \quad f(z) = \lim_{N\to\infty}\left(\prod_{i=1}^{N}(z-\xi_i)\right)^{1/N}$$

A set $\xi_1, \ldots, \xi_N$ are Fekete points, minimizing a Coulomb energy of a conductor

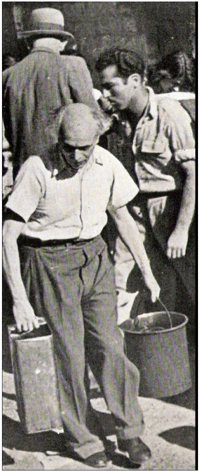
$$\min_{\xi\in\partial D} E(\xi_1, \ldots, \xi_N) = \min_{\xi\in\partial D} \sum_{i>j} -\log|\xi_i - \xi_j|$$

Dyson-Selberg integral $\oint_{\gamma^N} e^{-\beta E(\xi_1,\ldots,\xi_N)}$ as a quantization" of Fekete points.



Density of large number of Fekete points (equilibrium measure) is the *harmonic measure of γ*

$$\rho(\xi)d|\xi| = \frac{1}{2\pi}|f'(\xi)d\xi|$$



Mihaly Fekete (1886-1957), the Provost of the Hebrew University of Jerusalem, with his water quota, during the siege of Jerusalem

Related works

- Courteaut K., Johansson K. and Viklund F., 2025
  Planar Coulomb gas on a Jordan arc at any temperature

- Courteaut K., Johansson K. 2023
  Partition function for the 2d Coulomb gas on a Jordan curve

- Johansson K., 2022
  Strong Szegö Theorem on a Jordan Curve

- Wiegmann P., Zabrodin A. 2022
  Dyson gas on a curved contour

Conformal covarience

$$\beta = 1: \qquad Z_{N,\beta=1}[\gamma] := \oint_{\gamma^N} \prod_{i>j} |\xi_i - \xi_j|^2$$

Conformal covarience : $$\lim_{N\to\infty} \log\left(Z[\gamma + \delta\gamma]/Z[\gamma]\right) = \frac{1}{12\pi}\oint_{\gamma} \mathrm{Re}\left[\operatorname*{disc}_{\gamma}\{f, z\}\right]\delta n$$

"Quantum analog" of Hadamard formula

This property is emergent. It does not exist at finite N.

$$\lim_{N\to\infty} \log \frac{Z[\gamma]}{Z[S^1]} = \frac{1}{2}\,\mathrm{Det}\,\hat{\mathbf{N}}$$

Neumann jump operator : $$\hat{\mathbf{N}}\varphi := \partial_n^+ \varphi^H - \partial_n^- \varphi_H, \quad \varphi_H \in H_\varphi(D),\ \varphi^H \in H_\varphi(\mathbb{C}\setminus D)$$

Neumann jump operator and Neumann-Poincaré operator

Simple layer potential=Inverse of the Neumann jump operator)

$$h_{\text{simple}}(z) = -\frac{1}{2\pi} \oint_{\gamma} \log|z-\xi| g_{\text{simple}}(\xi)|d\xi|$$

$$\hat{\mathbf{N}} h_{\text{simple}} = g_{\text{simple}}$$

Quantum Field Theoretical representation of the Selberg integral

[after Ferrari and Klevtsov (2D)]

Representation: $\overbrace{\prod_{i>j=1}^{N}|\xi_i-\xi_j|^{2\beta}}^{\text{Selberg-Dyson integral}} = \overbrace{\mathbb{E}[V(\xi_1)\dots V(\xi_N)]_{S_\gamma}}^{\text{average of a string of verticies over GFF}}$

Averaging: $\mathbb{E}[\mathscr{O}] = Z_G^{-1}\int \mathscr{O}[X]\,e^{-S_\gamma[X]}\mathscr{D}X$

Partition function: $Z_G[\gamma] := \int e^{-S_\gamma[X]}\mathscr{D}X$

Vertex operator: $V = \exp i\sqrt{\beta}X$

Gaussian action: $S[X] = \oint\left(\frac{1}{4\pi}X\hat{\mathbf{N}}X + i\alpha_0\kappa X\right)$

$2\alpha_0 = \sqrt{\beta} - \frac{1}{\sqrt{\beta}}$ –background charge, κ-curvature, $\hat{\mathbf{N}} = \partial_n^+ - \partial_n^-$–Neumann jump

Main approximation, Gaussian integral

$$Z_{N,\beta}[\gamma] := \oint_{\gamma^N} \prod_{i>j}^{N} |\xi_i - \xi_j|^{2\beta} =$$

$$Z_G^{-1} \underbrace{\int \left(\oint e^{i\sqrt{\beta} X(\xi)} \right)^N e^{-S_\gamma[X]} \mathscr{D}X}_{1+\mathscr{O}(1/N)} \approx \mathscr{Z}_G^{-1}$$

$$\lim_{N\to\infty} Z_{N,\beta}^{-1} \to Z_G = \int e^{-\oint \left(\frac{1}{4\pi} X \hat{\mathbf{N}} X + i\alpha_0 \kappa X \right)} \mathscr{D}X$$

$$Z_G = \frac{1}{\sqrt{\mathrm{Det}' \hat{\mathbf{N}}}} \qquad \overbrace{e^{\alpha_0^2 \oint\!\!\oint \kappa \hat{\mathbf{N}}^{-1} \kappa}}^{\text{Fekete energy} = \frac{1}{2\pi} \alpha_0^2 \oint \phi \hat{\mathbf{N}} \phi}$$

Corrections

$$\int \left(\oint e^{i\sqrt{\beta}X(\xi)} \right)^N e^{-\oint \left(\frac{1}{4\pi} X \hat{\mathbf{N}} X + i\alpha_0 \kappa X \right)} \mathscr{D}X =$$

$$\frac{1}{\Gamma(s)} \int dt\, t^{s-1} \int e^{-\oint \left(\frac{1}{4\pi} X \hat{\mathbf{N}} X + i\alpha_0 \kappa X + t e^{i\sqrt{\beta}X(\xi)} \right)} \mathscr{D}X \Big|_{s=-N} = 1 + \frac{a(\beta)}{N} \oint_\gamma \kappa^2 + \dots$$

Final result

$$\lim_{N\to\infty} \log \frac{Z[\gamma]}{Z[S^1]} = \frac{1}{2} \log \mathrm{Det}' \,\hat{\mathbf{N}} + \frac{1}{2\pi} \alpha_0^2 \oint \phi \hat{\mathbf{N}} \phi \qquad \phi = -\log |f'|$$

Another form of the result

Double layer potential and Neumann-Poincare operator

$$h_{\text{double}} = \text{p.v.} \frac{1}{\pi} \oint_{\gamma} \partial_{n_\xi} \log|z - \xi| g_{\text{double}}(\xi) |d\xi|$$

$$h_{\text{double}} = \hat{\mathbf{K}}^* g_{\text{double}}$$

The determinant of the Neumann jump operator is the inverse of the Fredholm determinant

$$\log \det{}' \hat{\mathbf{N}} + \log \det(\mathbf{I} + \hat{\mathbf{K}}) = -\log[\text{Perimeter}] + \text{const}.$$

Polyakov's formula

Surgery formula (S. Zelditch et al)

$$\log\det(-\Delta_{\text{int}}) + \log\det(-\Delta_{\text{ext}}) + \log\det{}'\hat{\mathbf{N}} = \log[\text{Perimeter}] + \text{const}$$

Determinants are expressed through the harmonic measure of the curve (Polyakov)

$$\phi_{\text{int/ext}} = -\log|f'|_{\text{int/ext}}$$

$$\log\det\left(-\Delta_{\text{int/ext}}\right) = \mp\frac{1}{12\pi}\oint_{|w|=1}\left(\phi_{\text{int/ext}}\partial_n\phi_{\text{int/ext}} + 2\phi_{\text{int/ext}}\right)|dw|,$$

Conclusion

- Emergent conformal covariance: knowing a deformation of the curve, we know the deformation of its harmonic measure it induced (by Hadamard), hence the deformation of the Selberg–Dyson integrals.

- Interested in large-N asymptotics? Methods of quantum field theory may help.