Various approximations to the inverse problem of the Helmholtz equation.

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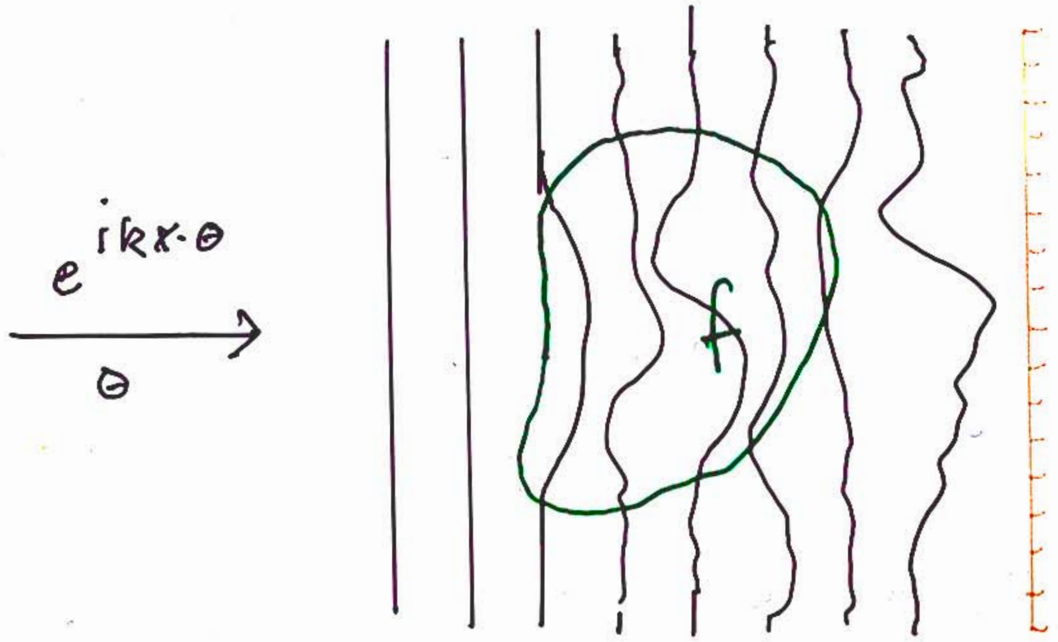


$$R_\theta(f) = g_\theta$$

$$\Delta u + k^2(1+f)u = 0$$

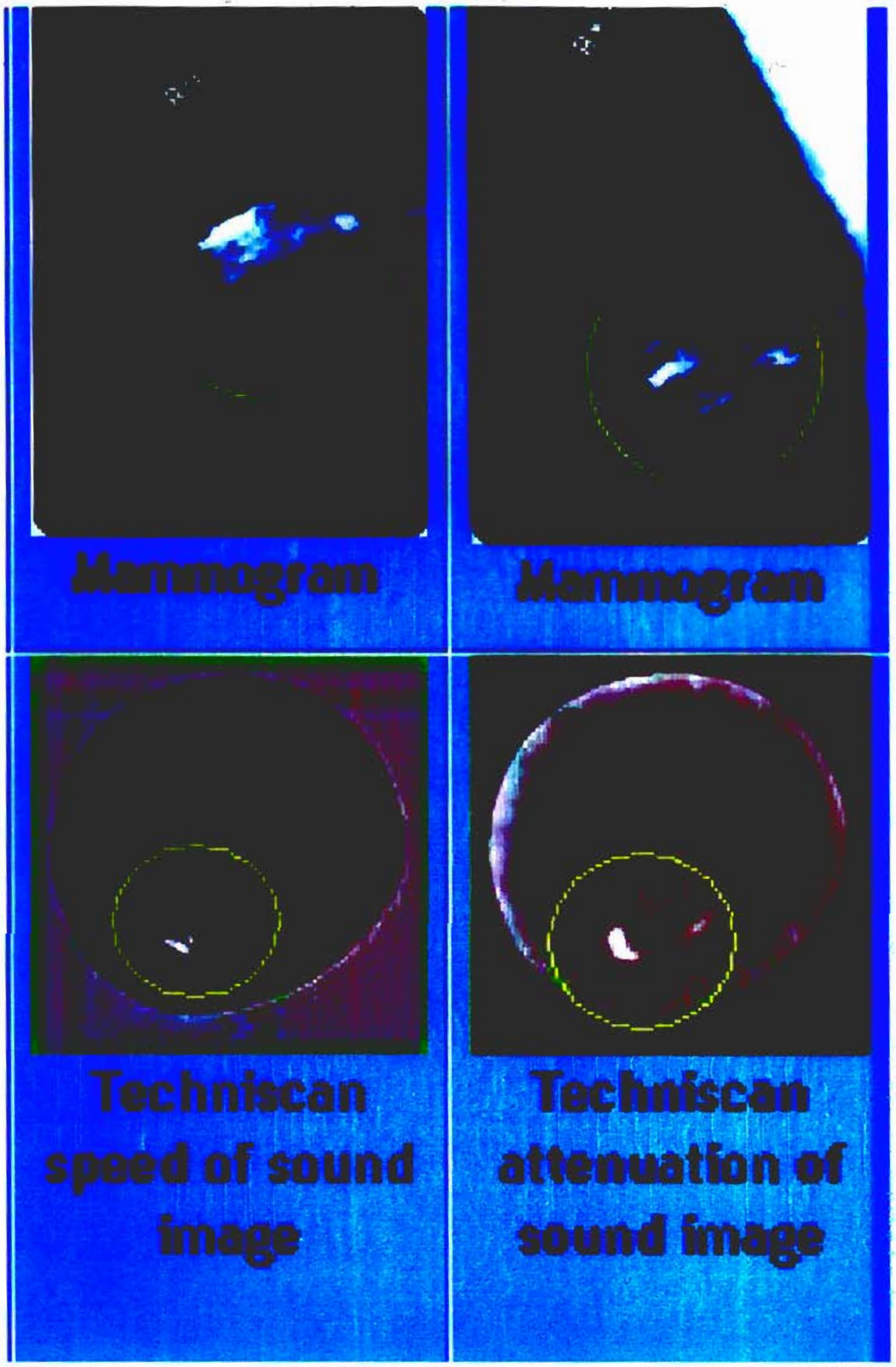
$$f = \frac{c_0^2}{c^2} - 1 - i\,\frac{2\alpha c_0}{kc}$$

Problem: Given g_θ for p directions $\theta_j,\ j = 1, \dots, p$, find an approximation to f!

Applications: Mammography

Difficulties:
- Fully 3D
- Born / Rytov not valid
- Resolution $< 1mm$
- Imaginary part of f small
- Object diameter $\sim 100\lambda$

Image_2003-10-30 1647.12.320.jpg



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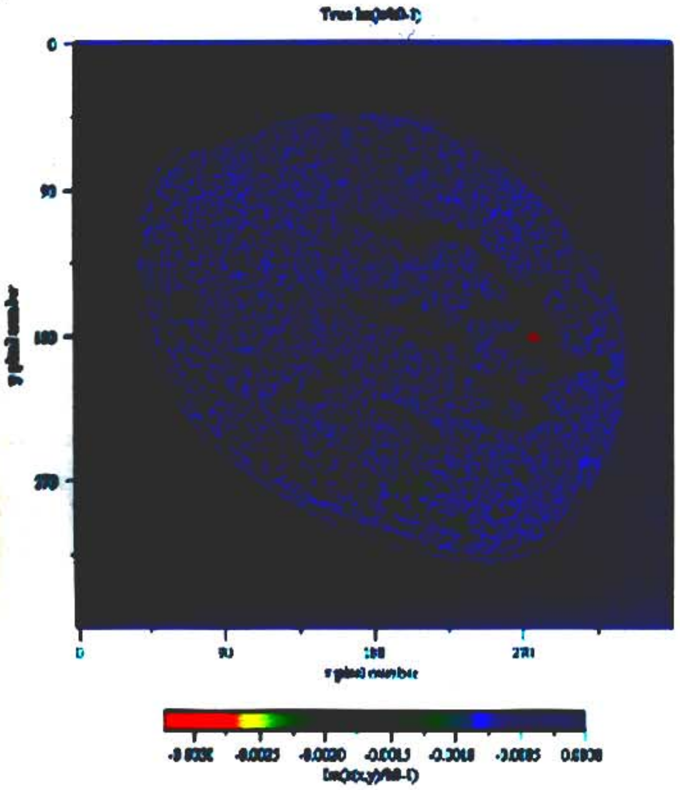
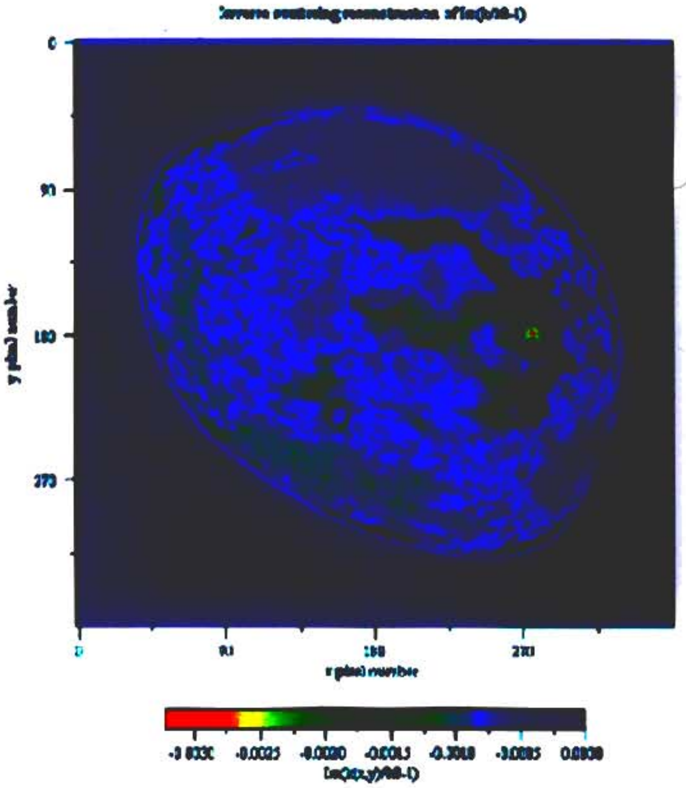
g4000_09b.jpg

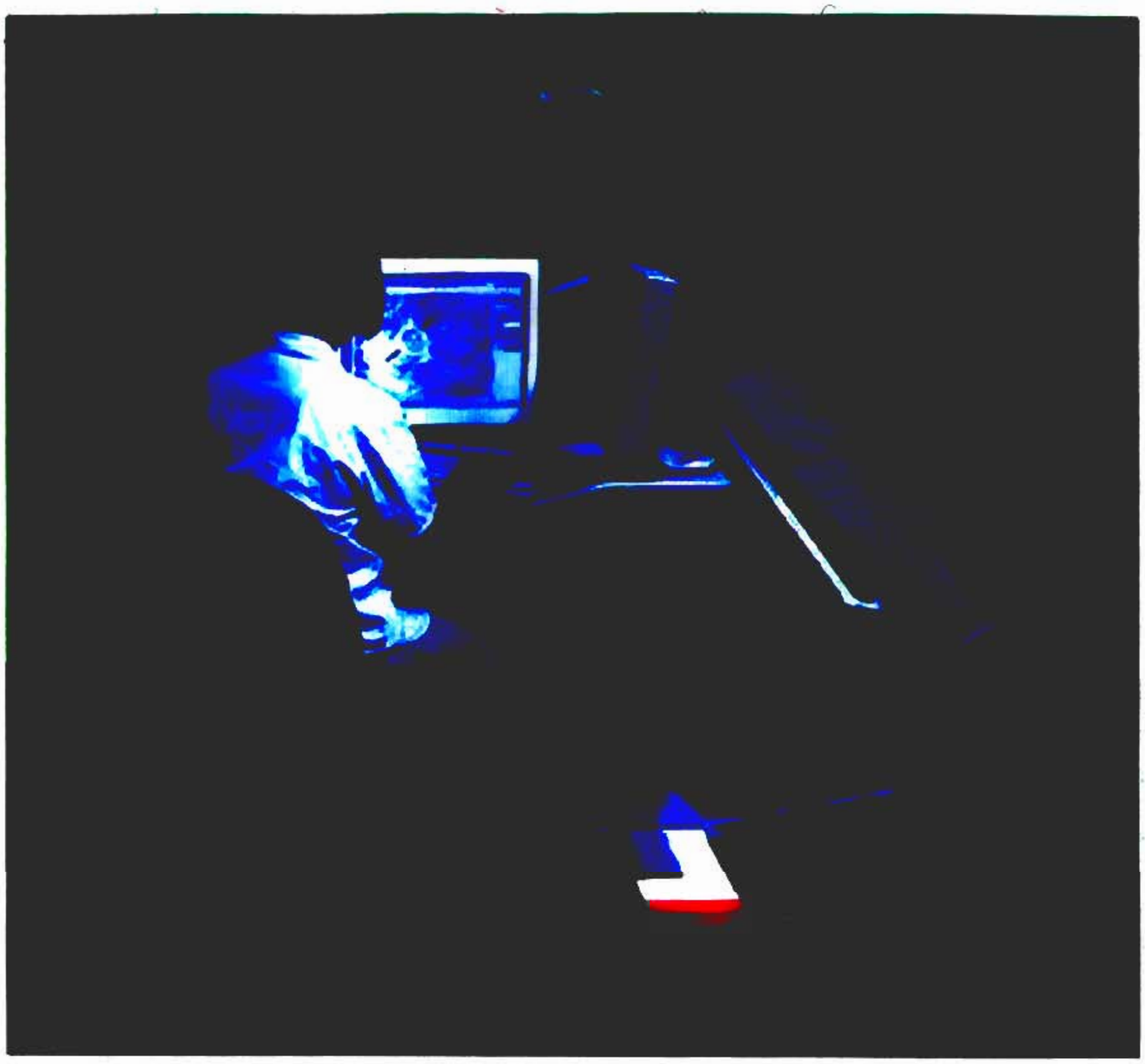


True breast image of acoustic absorption generated from MRI image of breast anatomy, from which synthetic data were made.

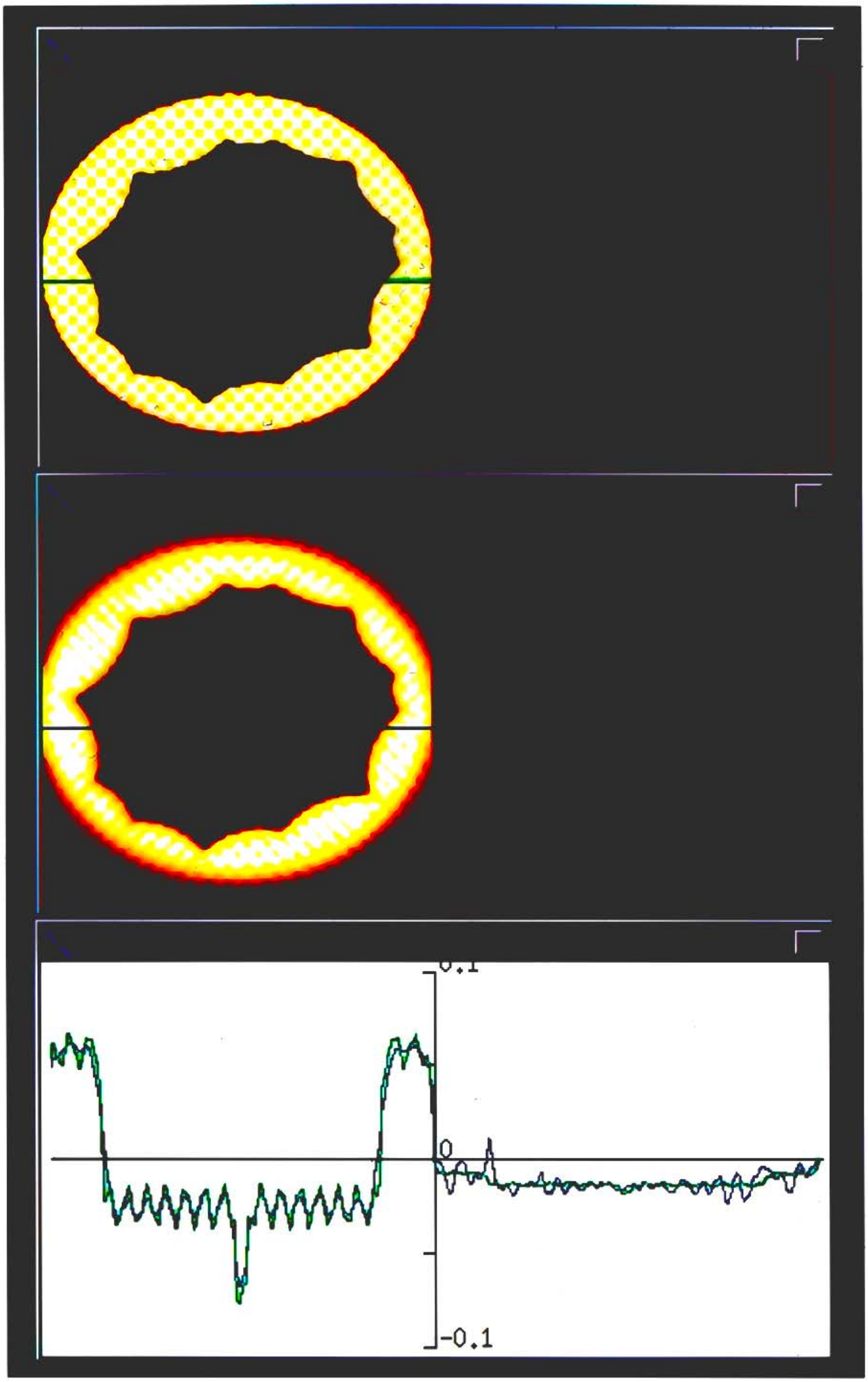


TMI image of acoustic absorption made by inverse scattering from synthetic data generated by scattering from image of absorption.

g4000_02.jpg



Re Im



Born approximation.

$$u = u_i + u_s \,, \quad u_i = e^{ik x\cdot\theta}$$

$$\Delta u_s + k^2 u_s = -k^2 f\,(u_i + u_s)$$

$$\Delta u_s + k^2 u_s = -k^2 f_B\, u_i$$

Heuristics: Born accurate if $2\rho k \|f\|_\infty << \pi$

($f = 0$ outside $|x| < \rho$)

Theorem: Let $q = \gamma_n \rho k \|f\|_\infty < 1$. Then

$$\|f_B - f_{2k}\|_{L_2(|x|<\rho)} \le c_n \frac{(\rho k)^{1+n/2}}{1-q} \|f\|_\infty^2$$

Proof: 1) G Green's function

$$\|G\|_{L_2(|x|<\rho)} = \gamma_n(\rho k)\,\frac{\rho}{k}$$

2) Explicit formula for f_B.

Rytov approximation.

$$u = u_i \, e^{v/u_i}$$

$$\Delta v + k^2 v = -k^2 \left(f + k^{-2} \left|\nabla \frac{v}{u_i}\right|^2 \right) u_i$$

$$\Delta v + k^2 v = -k^2 f_R \, u_i$$

Heuristics: Rytov accurate if

$$|f| \ll 1, \quad \rho^2 |\nabla f|^2 \ll |f|$$

Theorem?

Eikonal approximation:

$$u_E = A\, e^{ik\Phi}$$

$$|\nabla\Phi|^2 = 1 + f_1 , \qquad 2\nabla\Phi\cdot\nabla A + A\,\Delta\Phi + f_2 = 0$$

$$\Delta u_E + k^2\left(1 + f + \frac{r}{k^2}\right) u_E = 0 , \qquad r = -\frac{\Delta A}{A}$$

Theorem: $f = f_0 + \delta f$. Then, if no caustics,

$$\frac{1}{2}\int_{\Gamma_\Theta(x)} \delta f \, ds = \Phi(x) - \Phi_0(x) + O(\rho|\delta f|^2) + O\left(\frac{1}{\rho k^2}\right) + O\left(\frac{\delta f}{k}\right)$$

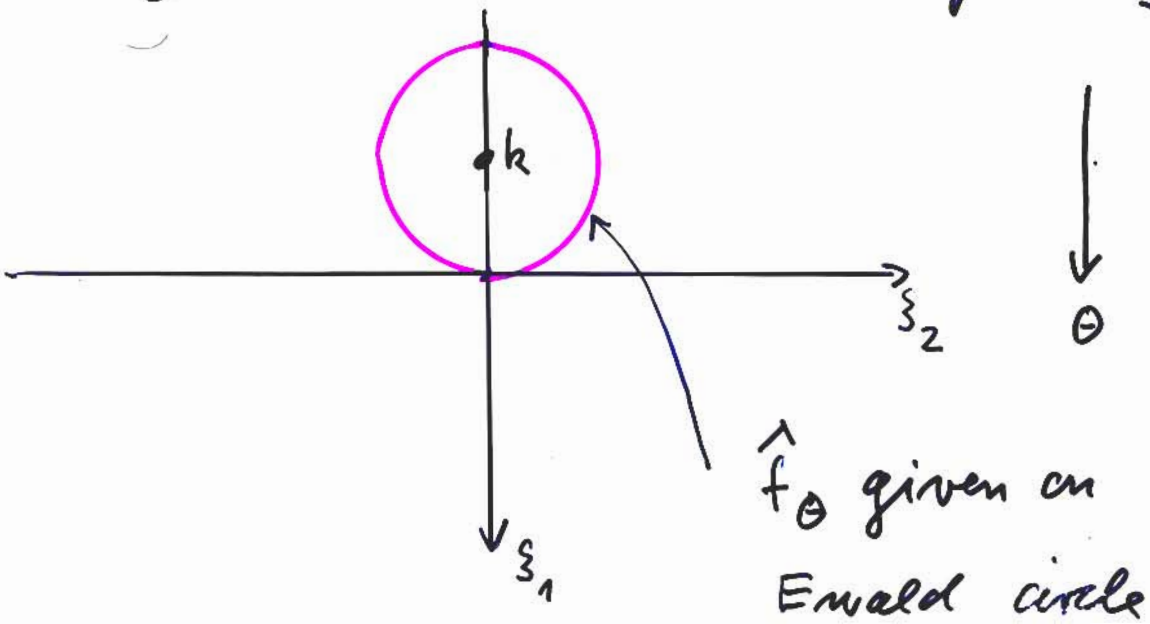
Degression: Find support of f from 1 direction.

$$\Delta u_s + k^2 u_s = -k^2 f_\Theta u_i \, ,$$

$$f_\Theta = f\,(1 + u_s / u_i)$$

Note: 1) $\operatorname{supp} f_\Theta = \operatorname{supp} f$

2) f_Θ can be computed as easily as f_B



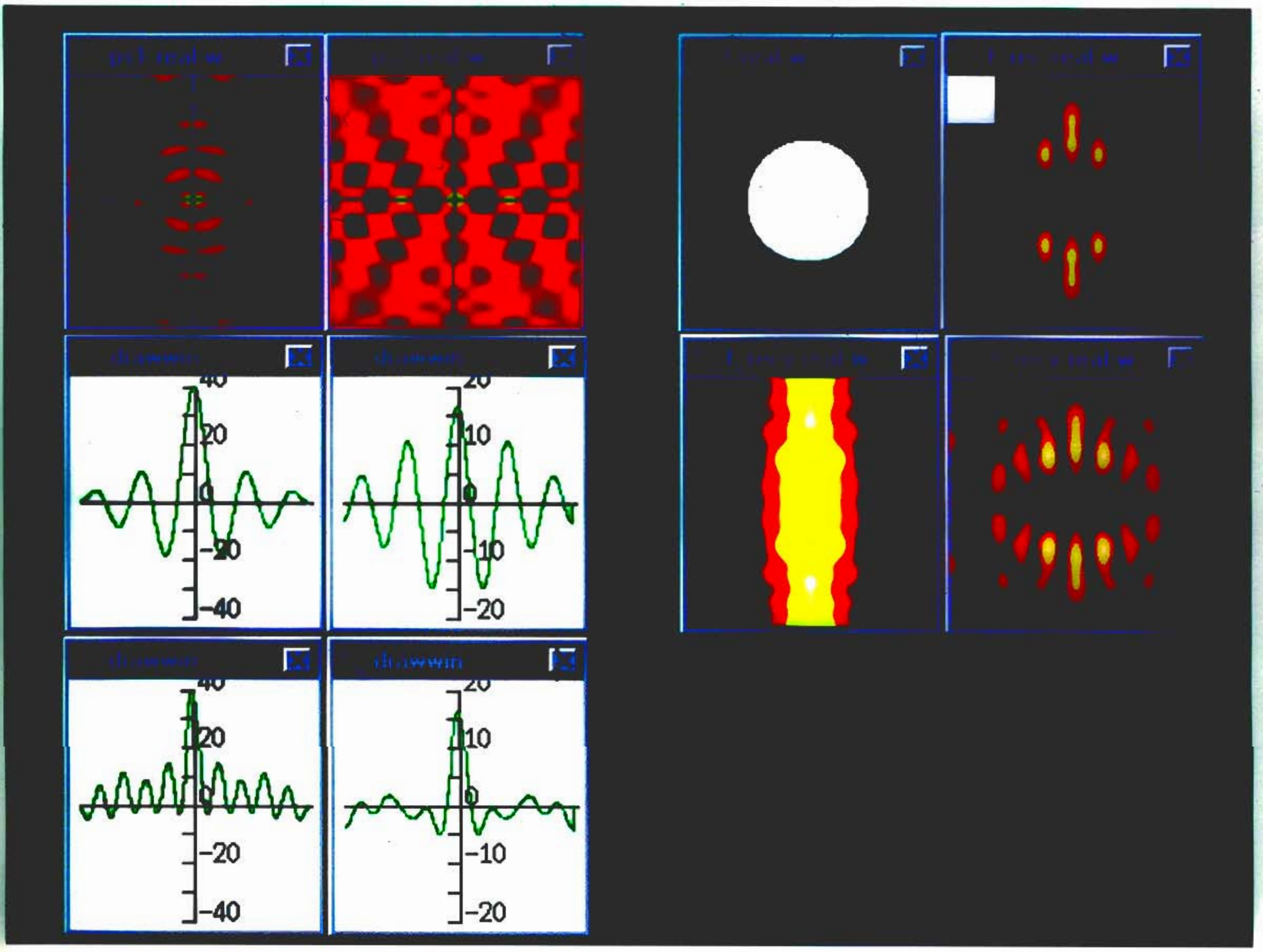
$$f_{Rec} = K * f$$

$$K(x) = e^{ikx_1} \, J_0(k|x|)$$

This K yields good resolution in x_2 direction

Good resolution in x_1 direction:

$$\text{Minimize} \int x_1^2 \, |K(x_1, 0)|^2 \, dx_1$$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Iterative methods.

$R_j(f)$ = scattered field at detector for j-th incoming wave

$$R_j(f) = g_j \,, \quad j = 1, \dots, p \,.$$

Nonlinear Kaczmarz:

$$f \leftarrow f + \omega R_j'(f)^* C_j^{-1} (g_j - R_j(f))$$

1) $f \to R_j(f)$

parabolic approximation

Marching on eikonal equation

Marching on Helmholtz equation

2) $r \to R_j'(f)^* r$

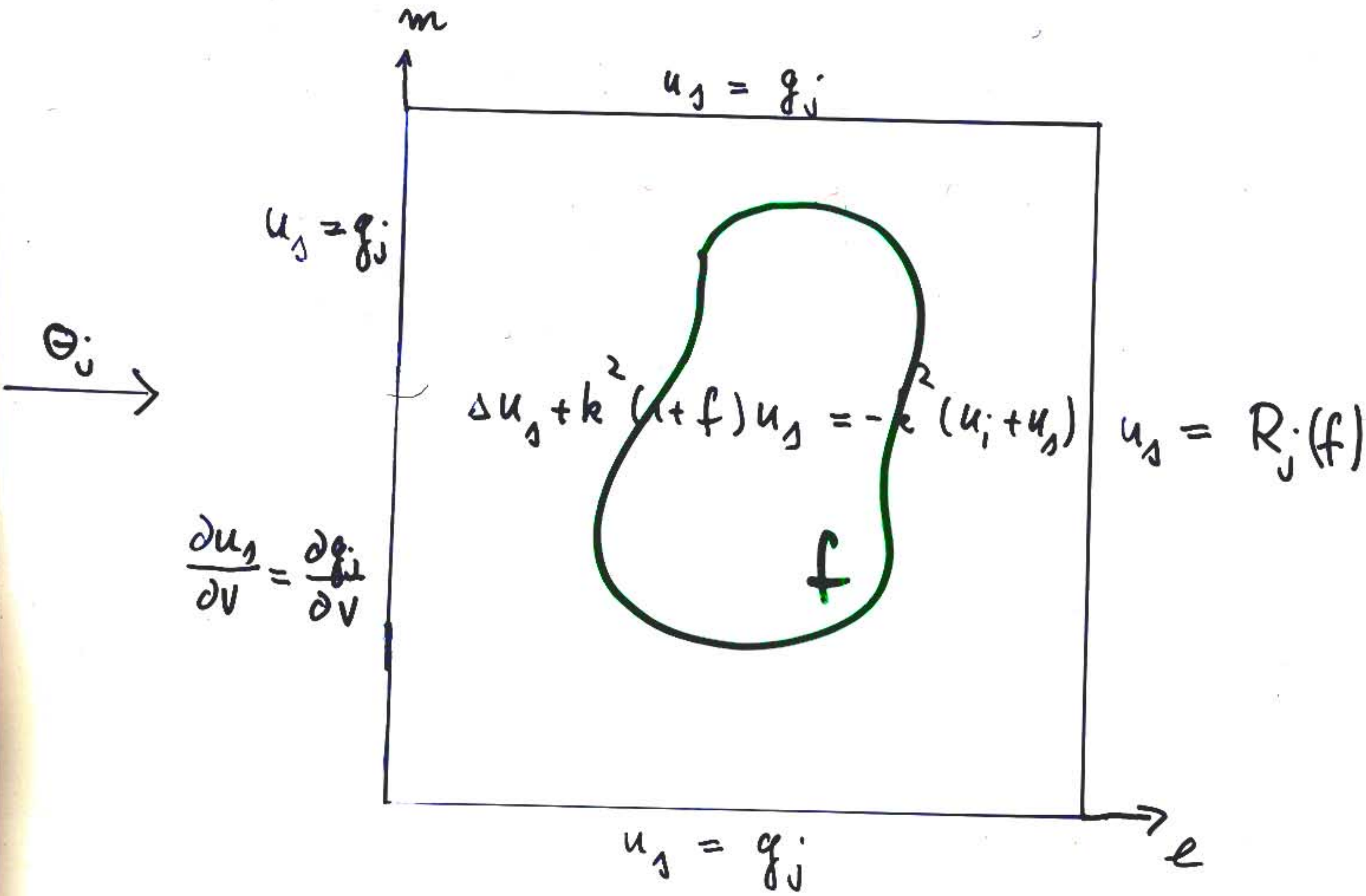
adjoint differentiation

backpropagation

time reversal

3) $C_j \sim R_j'(f) R_j'(f)^*$ Jacobian

Marching on Helmholtz equation

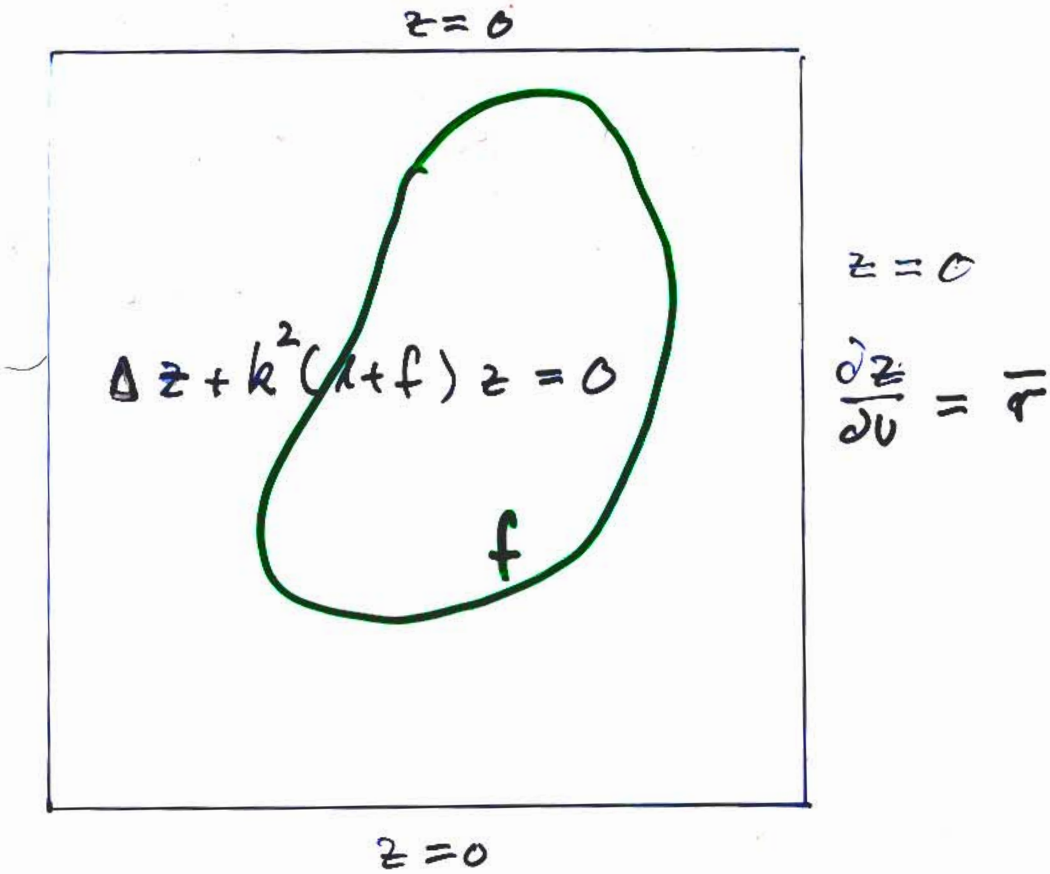


$$-4u_s^{l,m} + u_s^{l-1,m} + u_s^{l+1,m} + u_s^{l,m-1} + u_s^{l,m+1} + h^2k^2 f^{l,m}(u_i^{l,m} + u_s^{l,m}) = 0$$

preliminary value

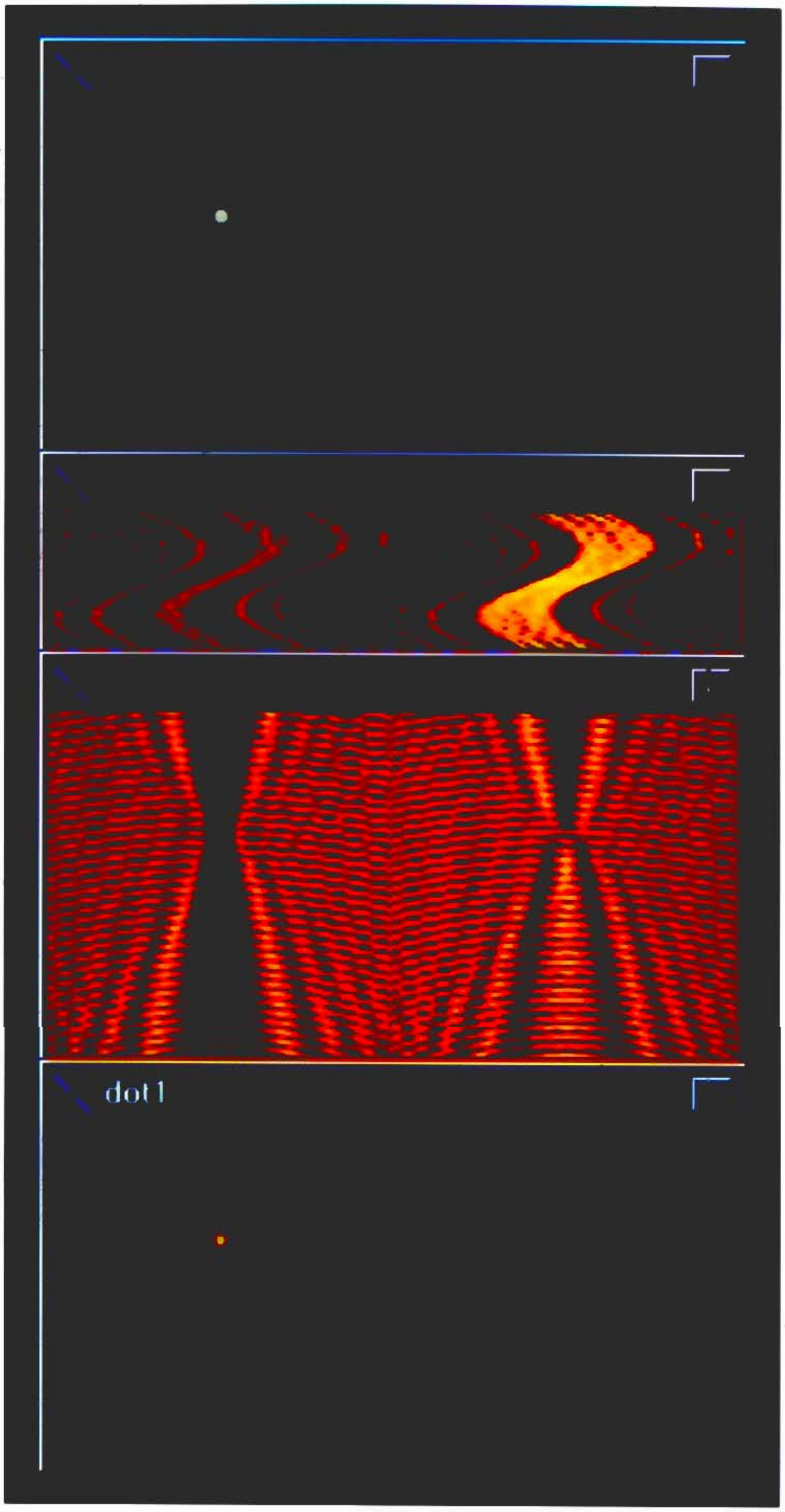
Final value: Low-pass filter $u_s^{l+1,\cdot}$

Backpropagation



$$R'(f)^* r = k^2 \bar{u} \bar{z}$$

Re Im



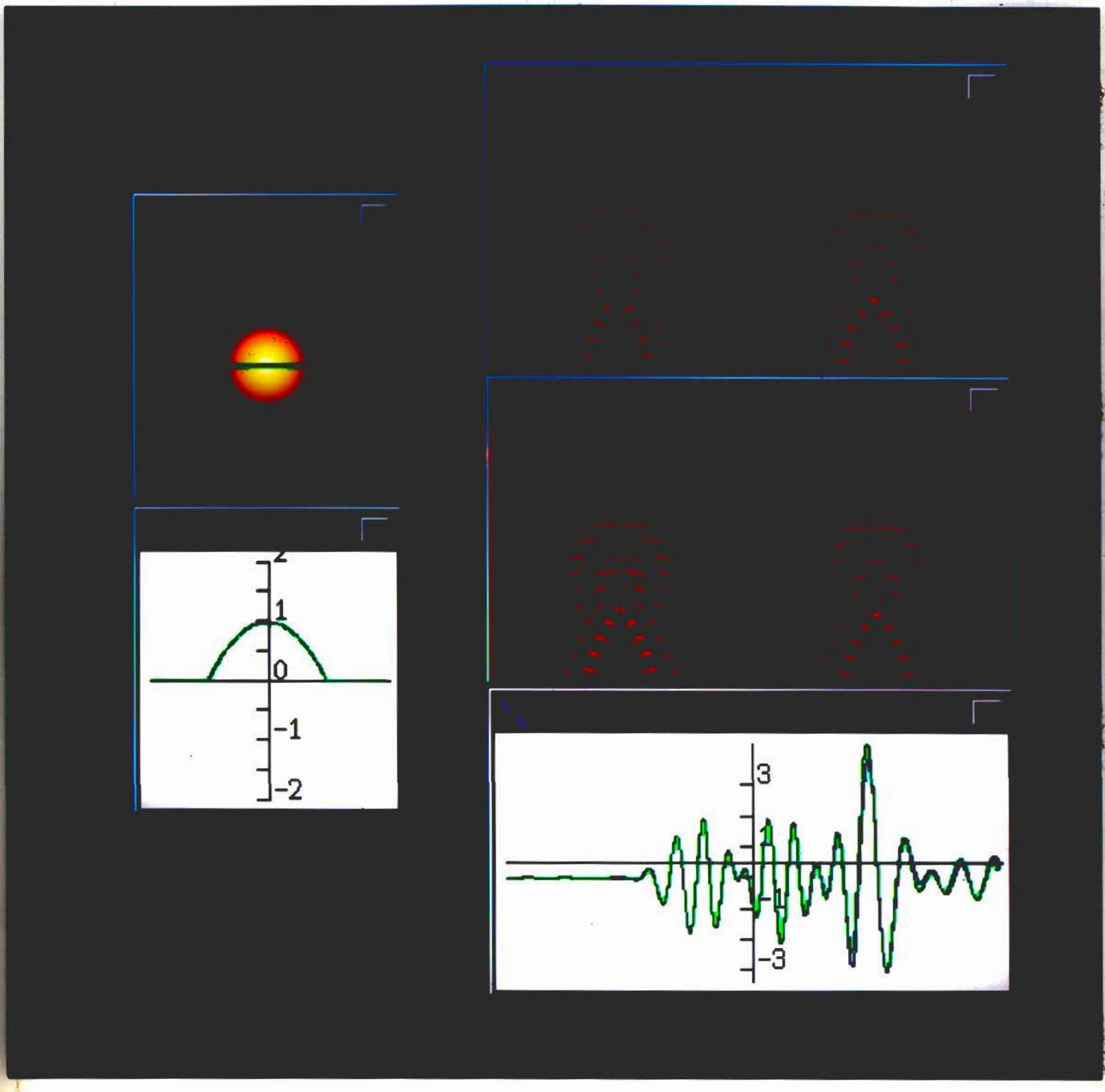
Object f

Data $g_j, \ j=1 \dots p$

Back-propagated field z

Superimposed back-propagated fields

Luneberg lense



The Maxwell equations:

$$\mathrm{curl}\, E(x) - ik\, H(x) = 0$$

$$\mathrm{curl}\, H(x) + ik\, n\, E(x) = 0$$

$$n = 1 + f$$

$$\Delta E(x) + \nabla\left(\frac{\nabla n(x)}{n(x)} \cdot E(x)\right) + k^2 n(x)\, E(x) = 0$$

M. Vögeler, Inverse Problems