Inverse Problems Workshop Series II

Total Variation Minimization and Applications

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Total variation minimization

- An algorithm for minimizing $TV(u) + \frac{1}{2\lambda}\|u - g\|^2$

- Applications:

$\longrightarrow$ Inverse problems in image processing (denoising, restoration, zooming),

$\longrightarrow$ Evolution of sets by the mean curvature flow.

Main approach

The idea is to minimize numerically $TV + L^2$ norm via the dual problem.

$$J(u) = |Du|(\Omega) =$$

$$\sup \left\{ \int_\Omega u \, \mathrm{div}\, \varphi \ : \ \varphi \in C_c^1(\Omega; \mathbb{R}^N), \|\varphi(x)\| \le 1 \ \forall x \right\}$$

Problem (primal): given $g \in L^2$,

$$\min_u J(u) + \frac{1}{2\lambda} \|u - g\|_{L^2}^2 \tag{1}$$

Dual problem

Several ways to derive the dual problem:

1) Problem is in the form (infconvolution)

$$F(g) \;=\; \min_{u+v=g} J(u) + H(v)$$

$F = J \triangle H$ is convex l.s.c., so that $F(g) = F^{**}(g)$ ($F^*(f) = \sup_g \langle f, g\rangle - F(g)$ is the Legendre-Fenchel transform).

Hence one has $F(g) = \sup_f \langle f, g\rangle - F^*(f)$ with

$$\begin{aligned} F^*(f) &= \sup_g \langle f, g\rangle - \min_{u+v=g} (J(u) + H(v)) \\ &= \sup_{u,v} g\langle f, u+v\rangle - J(u) - H(v) \\ &= J^*(f) + H^*(f) \end{aligned}$$

The dual problem is thus (changing the sign)

$$\min_f J^*(f) + H^*(f) - \langle f, g \rangle$$

Here, $H^*(f) = \lambda \|f\|^2/2$, hence the problem is

$$\text{(2)} \qquad \min_f J^*(f) + \frac{\lambda}{2}\|f - (g/\lambda)^2\|^2 \ - \frac{1}{2\lambda}\|g\|^2$$

Dual Problem (2)

2) A second way to derive the dual problem in this situation (Yosida regularization)

Euler equation: $\dfrac{u-g}{\lambda} + \partial J(u) \ni 0$

[$\quad p \in \partial J(u) \Leftrightarrow \forall v, J(v) \geq J(u) + \langle p, v-u \rangle \quad$]

That is, $\frac{g-u}{\lambda} \in \partial J(u)$.

We have Fenchel's identity:
$p \in \partial J(u) \Leftrightarrow u \in \partial J^*(p) \Leftrightarrow \langle u, p \rangle = J(u) + J^*(p)$
We deduce

$$u \in \partial J^* \left(\frac{g-u}{\lambda} \right)$$

Letting $w = g-u$ we get $\frac{w-g}{\lambda} + \frac{1}{\lambda} \partial J^* \left(\frac{w}{\lambda} \right) \ni 0$ which is the Euler equation for

(3) $$\min_w \frac{\|w-g\|^2}{2\lambda} + J^* \left(\frac{w}{\lambda} \right)$$

It is the same as (2) if we let $f = w/\lambda$.

What is J^*?

If J is the total variation one has

$$J(u) = \sup_{w \in K} \langle u, w \rangle$$

with K given by (the closure in L^2 of)

$$\left\{ \operatorname{div} \varphi \,:\, \varphi \in C_c^1(\Omega; \mathbb{R}^N),\, \|\varphi(x)\| \leq 1 \;\forall x \right\}.$$

Hence $J(u) = \sup_w \langle u, w \rangle - \delta_K(w)$,

$$\delta_K(w) \;= \begin{cases} 0 & \text{if } w \in K, \\ +\infty & \text{otherwise.} \end{cases}$$

We get $\delta_K^* = J$, yielding $J^* = \delta_K$. Therefore (3) (or (2)) is an orthogonal projection and we find:

(4) $$u \;=\; g - \Pi_{\lambda K}(g)$$

Discretization Total Variation

To solve the nonlinear projection problem (4) we have to discretize.

A discrete Total Variation is

$$J(u) \;=\; \sum_{i,j=1}^{N} |(\nabla u)_{i,j}| \quad \text{with}$$

$$(\nabla u)_{i,j} \;= \begin{pmatrix} u_{i+1,j} - u_{i,j} \\ u_{i,j+1} - u_{i,j} \end{pmatrix} \quad (+\ \text{B.C.}).$$

One has (as in the continuous setting):

$$\begin{aligned} J(u) &= \sup_{|\xi_{i,j}|\le 1} \sum_{i,j} \xi_{i,j} \cdot (\nabla u)_{i,j} \\ &= -\sup_{|\xi_{i,j}|\le 1} \sum_{i,j} (\operatorname{div} \xi)_{i,j} u_{i,j} \end{aligned}$$

with $(\operatorname{div} \xi) = \xi^1_{i,j} - \xi^1_{i-1,j} + \xi^2_{i,j} - \xi^2_{i,j-1}$ + B.C., i.e., $\operatorname{div} = -\nabla^*$.

We see that, again,

$$J(u) = \sup_{v \in K} \langle u, v \rangle = \sup_{v} \langle u, v \rangle - \delta_K(v)$$

with $K = \{\operatorname{div} \xi \,:\, |\xi_{i,j}| \le 1 \ \forall i,j\}$ and

$$\delta_K(v) = J^*(v) = \begin{cases} 0 & \text{if } v \in K \\ +\infty & \text{otherwise} \end{cases} .$$

Dual TV Problem

We find that the Dual of (1), for J the discrete Total Variation, is, again,

$$\min_{w} \frac{\|w-g\|^2}{2\lambda} + \delta_K\left(\frac{w}{\lambda}\right),$$

that is

$$\min_{w \in \lambda K} \|w-g\|^2$$

Hence w is the projection on λK of g and the solution of (1) is given by

(4) $$u = g - \Pi_{\lambda K}(g)$$

Algorithm(s)

- The problem is: $\min_{|\xi_{i,j}|\leq 1} \| \operatorname{div} \xi - g/\lambda\|^2$.

- Approach with Lagrange multipliers:

$$\min_{\xi} \| \operatorname{div} \xi - g/\lambda\|^2 + \sum_{i,j} \alpha_{i,j} |\xi_{i,j}|^2.$$

The Euler equation is

$$-(\nabla(\operatorname{div} \xi - g/\lambda))_{i,j} \ + \ \alpha_{i,j}\xi_{i,j} \ = \ 0 \ \forall i,j$$

with $\alpha_{i,j} \geq 0$ and $\alpha_{i,j} = 0$ whenever $|\xi_{i,j}| < 1$. Computing the norm $|\cdot|$, we find that

$$\alpha_{i,j} \ = \ |(\nabla(\operatorname{div} \xi - g/\lambda))_{i,j}|.$$

Gradient Descent

A straightforward descent scheme is the following

$$\xi_{i,j}^{n+1} \;=\; \xi_{i,j}^{n} + \tau(\nabla(\mathrm{div}\,\xi^{n} - g/\lambda))_{i,j} - \tau\alpha_{i,j}^{n}\xi_{i,j}^{n+1},$$

or

$$\xi_{i,j}^{n+1} \;=\; \frac{\xi_{i,j}^{n} + \tau(\nabla(\mathrm{div}\,\xi^{n} - g/\lambda))_{i,j}}{1 + \tau|(\nabla(\mathrm{div}\,\xi^{n} - g/\lambda))_{i,j}|}$$

Theorem. The iterations converge as soon as $\tau \le 1/\|\,\mathrm{div}\,\|^2$ (which is greater or equal to 1/8).

Proof (simple). One just shows that

$$\|\,\mathrm{div}\,\xi^{n+1} - g/\lambda\|^2 \le \|\,\mathrm{div}\,\xi^{n} - g/\lambda\|^2$$

with $<$ as long as ξ^n is not a solution of the problem.

Remark: Same convergence result for the (more natural) variant

$$\xi_{i,j}^{n+1} = \Pi_{\{|\xi|\le 1\}}(\xi_{i,j}^{n} + \tau(\nabla(\mathrm{div}\,\xi^{n} - g/\lambda))_{i,j}),$$

however (for unknown reasons) it is much slower (even if one can prove the convergence up to $\tau = 1/4$, which also works in the previous algorithm).

→ See also [Carter] or [Chan-Golub-Mulet] for primal/dual approaches.

Applications: Image Denoising

- Classical Model:

$$g = u + n,$$

$g = (g_{i,j})_{i,j=1}^N$ observed image,
$u = (u_{i,j})$ *a priori* piecewise smooth image,
$n = (n_{i,j})$ Gaussian noise (average 0, variance σ^2 hence $\frac{1}{N^2}\sum_{i,j} n_{i,j}^2 \simeq \sigma^2$).

(Or: $g = Au + n$, $A =$ linear transformation.)

- Problem: recover u from g.

- Tichonov's Method:

(1) $$\min_u J(u) \;+\; \frac{1}{2\lambda}\|u - g\|^2$$

or

(1′) $$\min_u J(u) \;\text{ subject to }\; \|u - g\|^2 = N^2\sigma^2$$

- Choice of J: H^1 norm ($\sum |\nabla u|^2$), TV (Rudin-Osher-Fatemi), Mumford-Shah...

(1) with varying λ

Denoising by Constrained TV Minimization

The problem proposed by Rudin-Osher-Fatemi is

$$(1') \qquad \min_u J(u) \ \text{ subject to } \ \|u-g\|^2 = N^2\sigma^2$$

The constraint $\|u-g\| = N\sigma$ is satisfied if λ in (1) is chosen such that $\|\Pi_{\lambda K}(g)\| = \lambda\|\operatorname{div}\xi\| = N\sigma$ (where $\Pi_{\lambda K}(g) = \lambda \operatorname{div}\xi \in \lambda K$).

We propose the following algorithm for (1'): we fix an arbitrary value $\lambda_0 > 0$ and compute $v_0 = \Pi_{\lambda_0 K}(g)$. Then for every $n \geq 0$, we let $\lambda_{n+1} = (N\sigma/\|v_n\|)\lambda_n$, and $v_{n+1} = \Pi_{\lambda_{n+1}K}(g)$. We have the following theorem:

Theorem. As $n \to \infty$, $g - v_n$ converges to the unique solution of (1').

Resolution of (1') with $\sigma = 12$.

Other Applications: Zooming

The setting of the Zooming problem is the following: We have $u = (u_{i,j})_{i,j=1}^N \in X = \mathbb{R}^{N\times N}$, and g belongs to a "coarser" space $Z \subset X$ (for instance, $Z = \{u \in X \,:\, u_{i,j} = u_{i+1,j} = u_{i,j+1} = u_{i+1,j+1}$ for every even $i,j\}$), A is the orthogonal projection onto Z, and the problem to solve (as proposed for instance by [Guichard-Malgouyres])

$$\text{(5)} \qquad \min_u J(u) \;+\; \frac{1}{2\lambda}\|Au - g\|^2$$

(for some small value of λ). Since $Ag = g$, $Au - g = A(u-g)$ and $\|Au-g\|^2 = \min_{w\in Z^\perp}\|u-g-w\|^2$. Hence (5) is equivalent to

$$\text{(6)} \qquad \min_{w\in Z^\perp, u\in X} J(u) \;+\; \frac{1}{2\lambda}\|u-(g+w)\|^2$$

Hence, to solve the zooming problem, one readily sees that the following algorithm will work: letting $w_0 = 0$, we compute u_{n+1}, w_{n+1} as follows

$$\begin{aligned} u_{n+1} &= g + w_n - \Pi_{\lambda K}(g + w_n), \\ w_{n+1} &= \Pi_{Z^\perp}(u_{n+1} - g). \end{aligned}$$

Unfortunately, this method is not very fast. (*cf.* [Guichard-Malgouyres] for the original introduction of the problem and a different implementation.)

Any linear operator A can be implemented, with speed of convergence depending of the condition number (and quite slow if A non invertible, like in this example.)
[Aubert-Bect-Blanc-Féraud-AC]

Zooming

Image Decomposition

cf : Y. Meyer, Osher-Vese, Osher-Solé-Vese, AC + Aujol-Aubert-Blanc-Féraud

Meyer introduces the norm $\|\cdot\|_*$ which is dual of the Total Variation:

$$\|v\|_* \;=\; \sup_{J(u)\leq 1} \langle u,v\rangle \;=\; \min\{\lambda \geq 0, v \in \lambda K\}$$

(it is $+\infty$ if $\sum_{i,j} v_{i,j} \neq 0$).

He proposes to decompose an image f into a sum $u+v$ of a u with low Total Variation and a v containing the oscillations, by solving

$$\min_{f=u+v} J(u) \;+\; \mu\|v\|_*$$

The idea:

- $J(u)$ is low when the signal u is very regular (with edges);

- $\|v\|_*$ is low when the signal v is oscillating.

Method

- Osher-Vese: minimize (for λ large)

$$J(u) \; + \; \lambda \|f - u - v\|^2 \; + \; \mu \|v\|_*$$

that is approximated by

$$J(u) \; + \; \lambda \|f - u - \mathrm{div}\, \xi\|^2 \; + \; \mu \|\xi\|_{l^p}$$

for $p >> 1$.

- We propose the variant (our λ must be small)

$$\min_{u,v} J(u) \; + \; \frac{1}{2\lambda} \|f - u - v\|^2 \; + \; J^* \left(\frac{v}{\mu} \right)$$

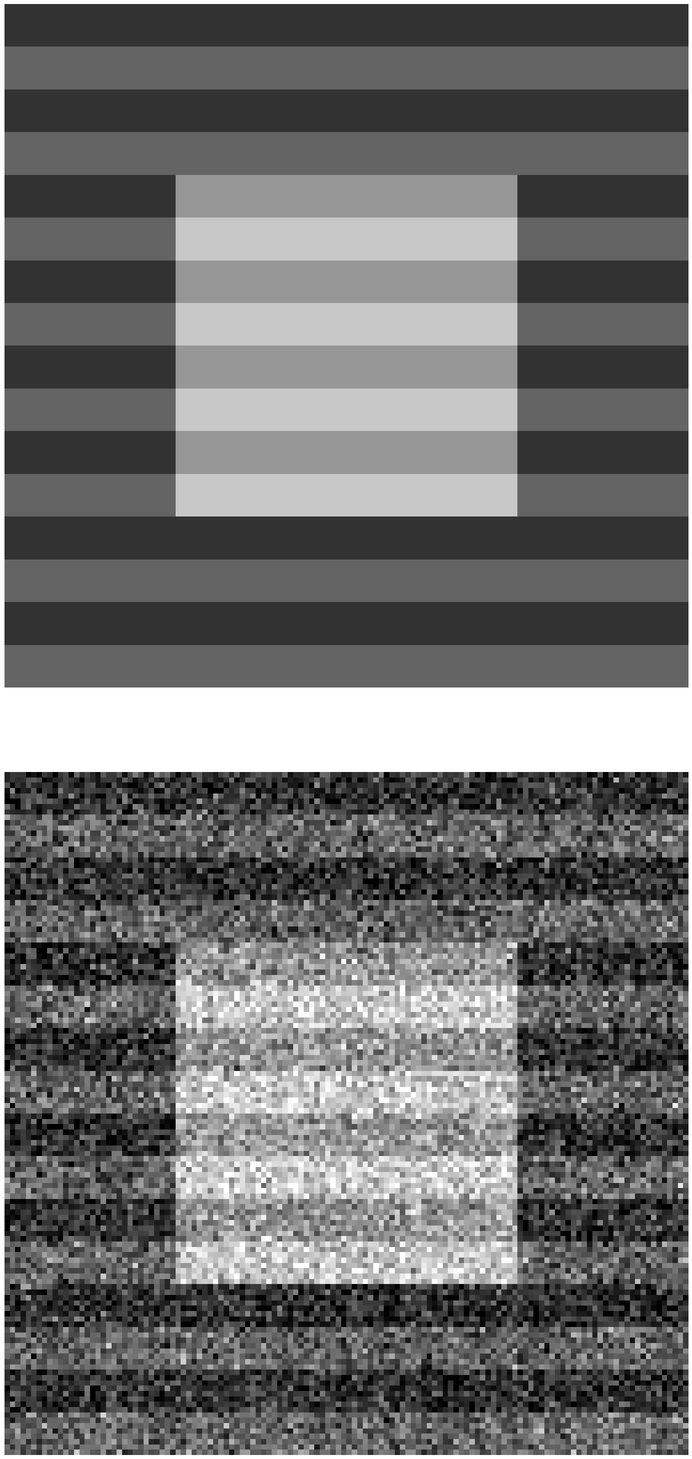
that corresponds to a constraint $\|v\|_* \le \mu$.

Algorithm

An advantage of our approach: straightforward algorithm. Let $u_0, v_0 = 0$, then alternate:

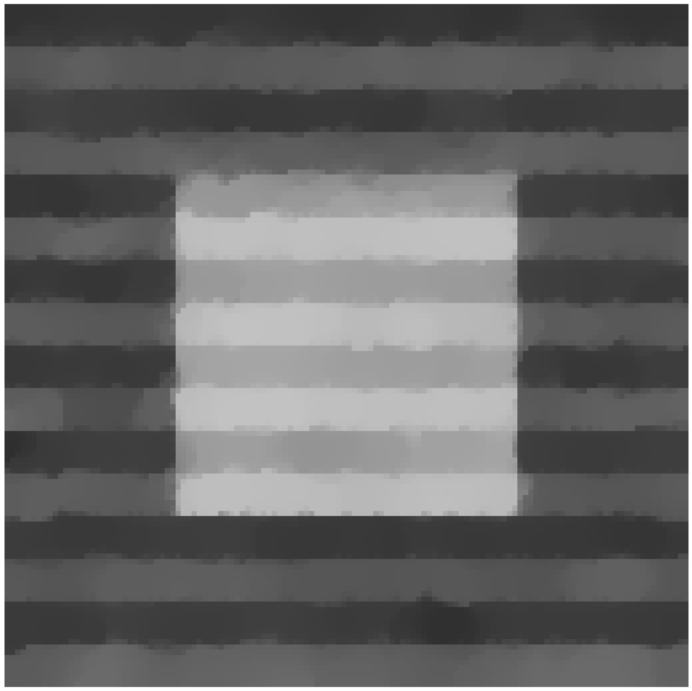
- $v_n \ = \ \Pi_{\mu K}(f - u_{n-1})$
- $u_n \ = \ (f - v_n) - \Pi_{\lambda K}(f - v_n)$

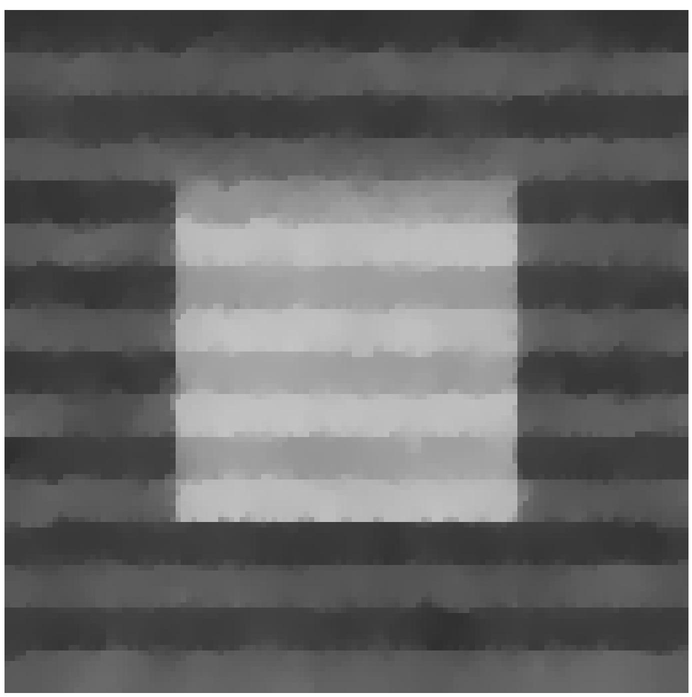
Examples



Original synthetic image and
same image with noise ($\sigma = 34$).

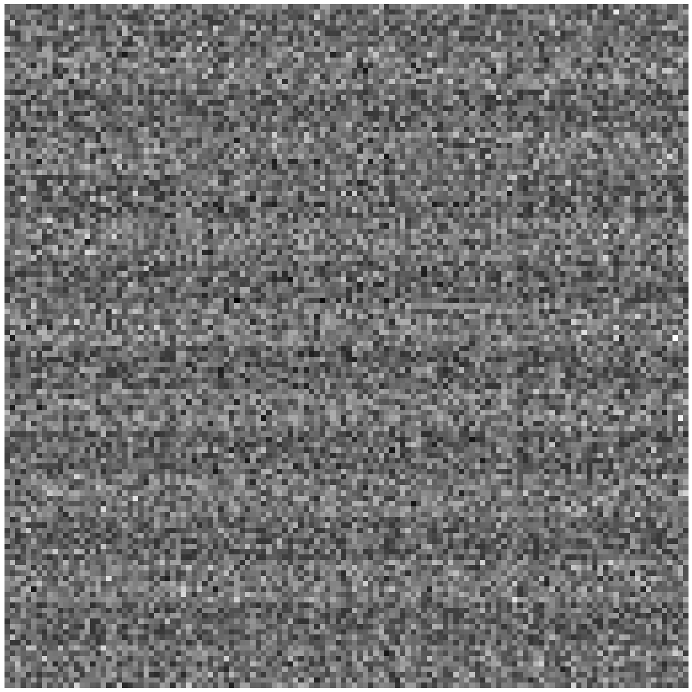
Reconstruction

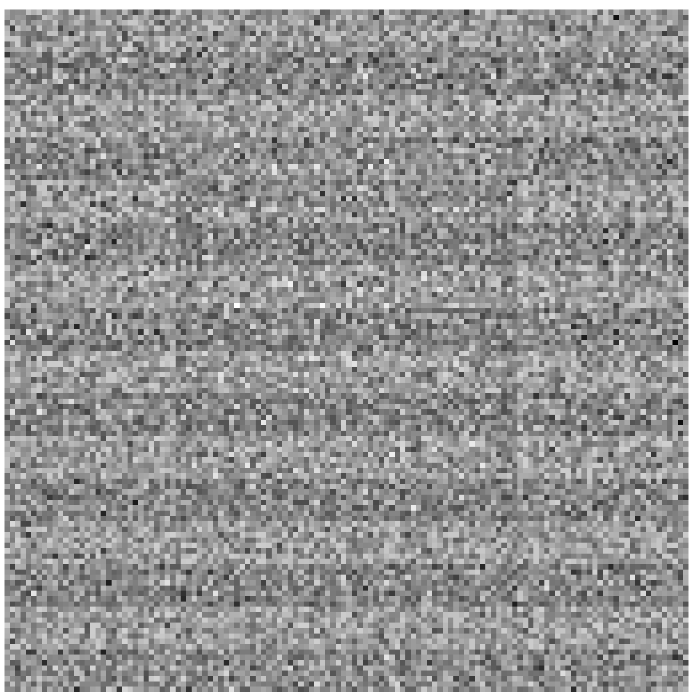




Reconstructed with Meyer's problem
and with ROF's method ($\mu = 55$, $\sigma = 34$).

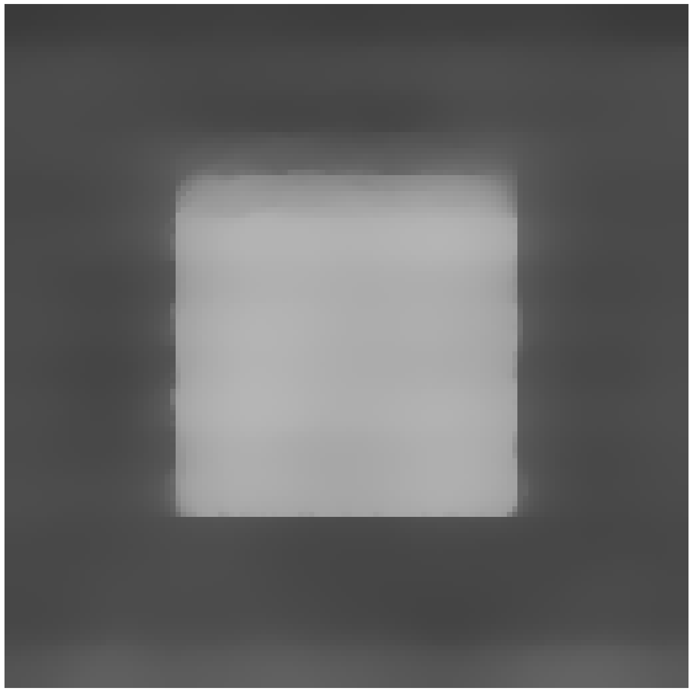
Difference

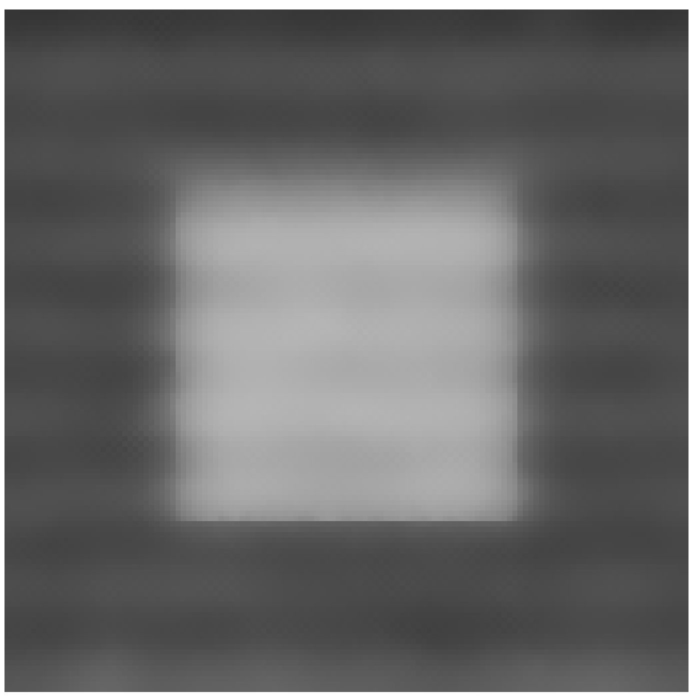




Difference image with Meyer's problem and with ROF's method.

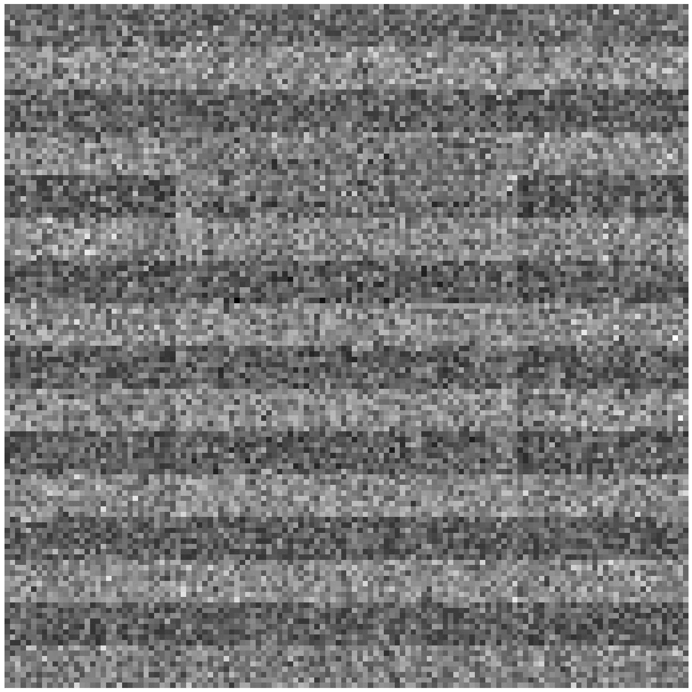
Removing more...

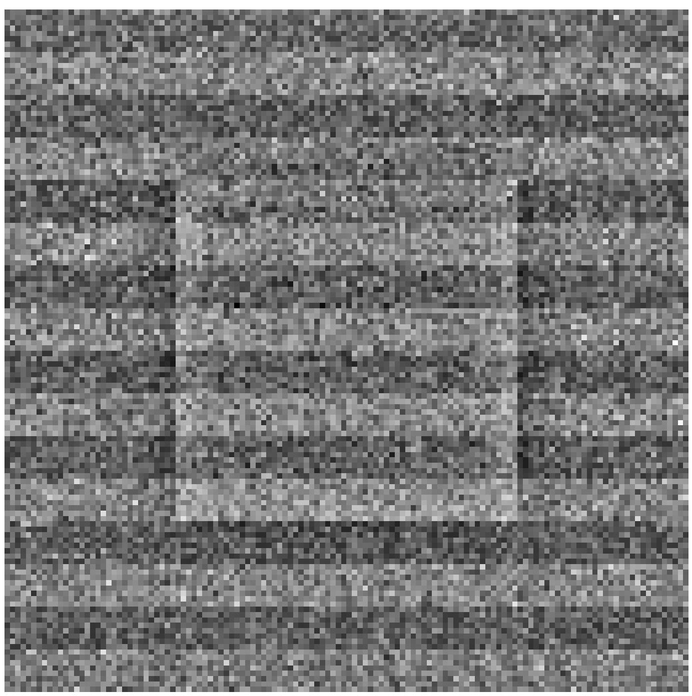




"Texture removal" with Meyer's problem
and with ROF's method ($\mu = 200$, $\sigma = 40.8$).

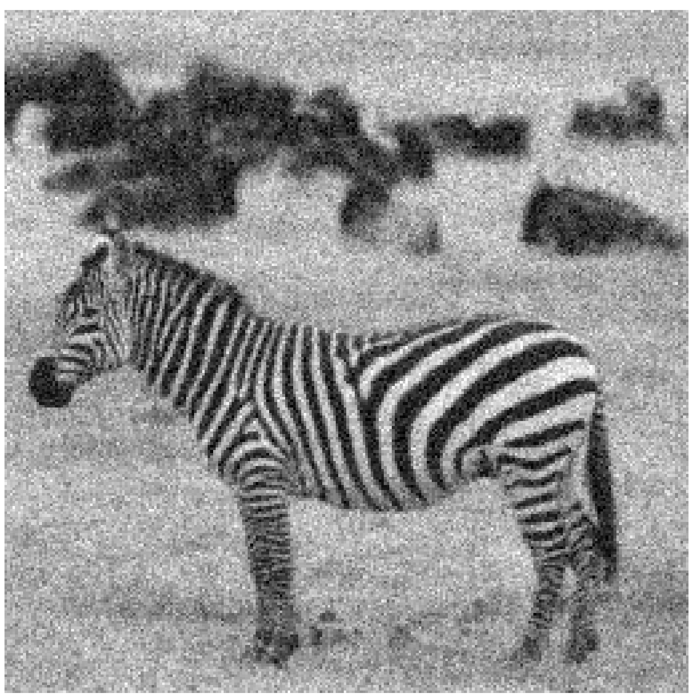
Difference





Difference image with Meyer's problem and with ROF's method.

Another example



A noisy zebra

Reconstruction

Reconstructed with Meyer's problem
and with ROF's method ($\mu = 20$, $\sigma = 19$).

and more...

“Texture removal” with Meyer’s problem
and with ROF’s method ($\mu = 200$, $\sigma = 32.6$).

Differences

Difference image with Meyer's problem
and with ROF's method.

Osher-Solé-Vese

$$\min_u J(u) \;+\; \frac{1}{2\lambda}\|f-u\|^2_{H^{-1}}$$

- Dual (cf first derivation of the dual problem)

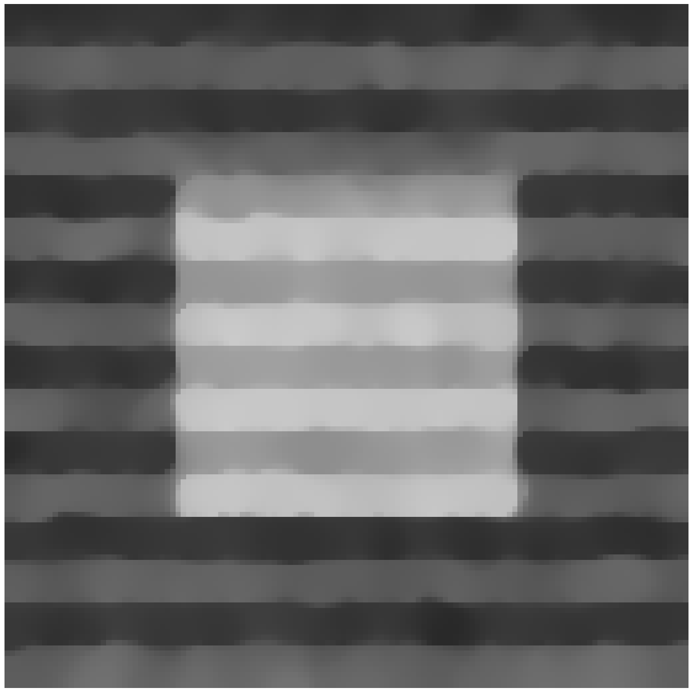
$$\min_w J^*(w) \;+\; \frac{\lambda}{2}\|\nabla w\|^2 \;-\; \langle f, w\rangle$$

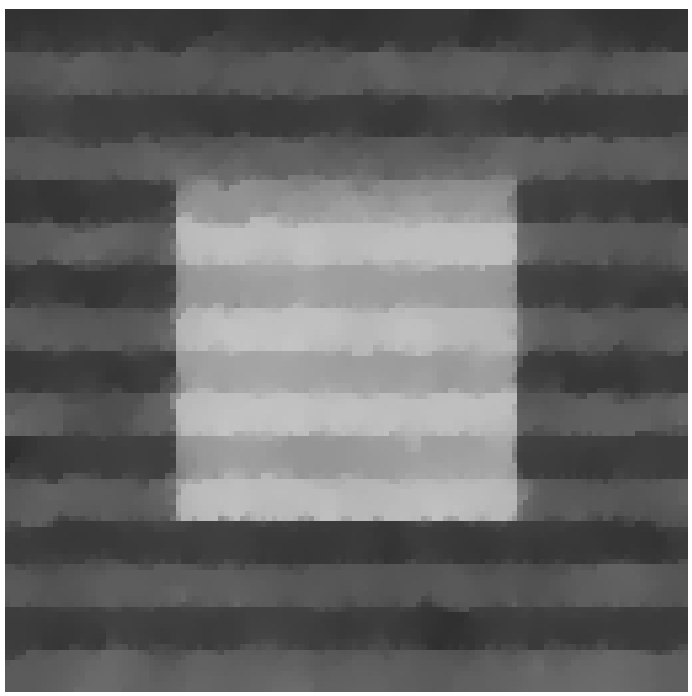
- Algorithm: variant of the TV algorithm (not extremely efficient, τ must be quite small)

$$\xi^{n+1}_{i,j} \;=\; \frac{\xi^n_{i,j} - \tau(\nabla(\Delta\,\mathrm{div}\,\xi^n - f/\lambda))_{i,j}}{1+\tau|(\nabla(\Delta\,\mathrm{div}\,\xi^n - f/\lambda))_{i,j}|}$$

($\Delta = \mathrm{div}\,\nabla$). Then $u = f - \lambda\Delta\,\mathrm{div}\,\xi^\infty$.

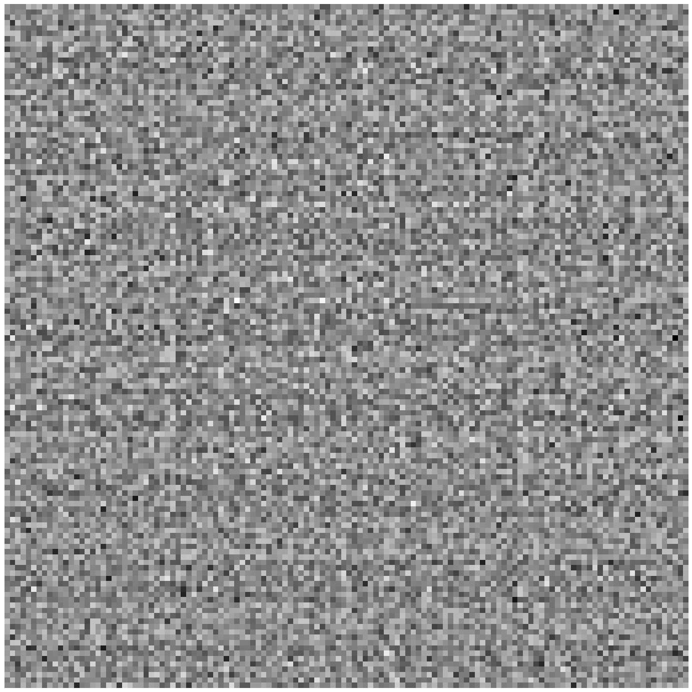
Denoising with OVS

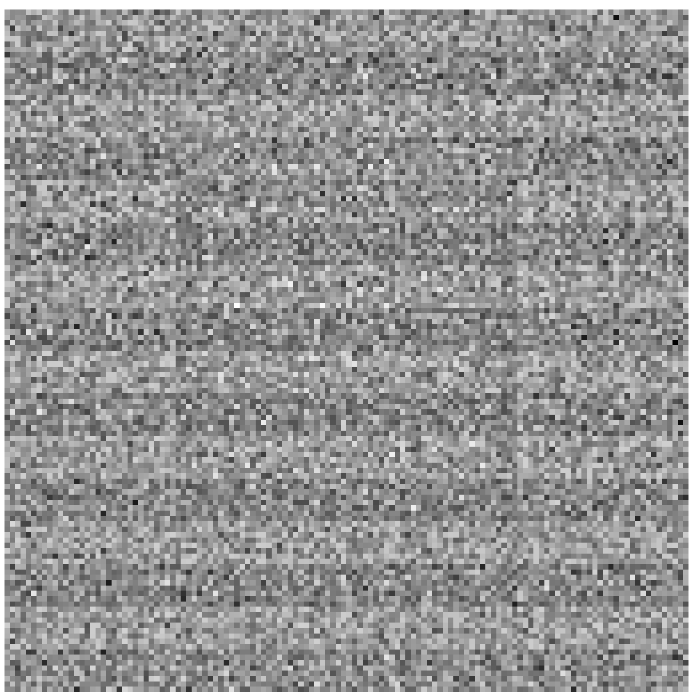




Reconstructed with OVS's method
and with ROF's method ($\lambda = 100$, $\sigma = 33.7$).

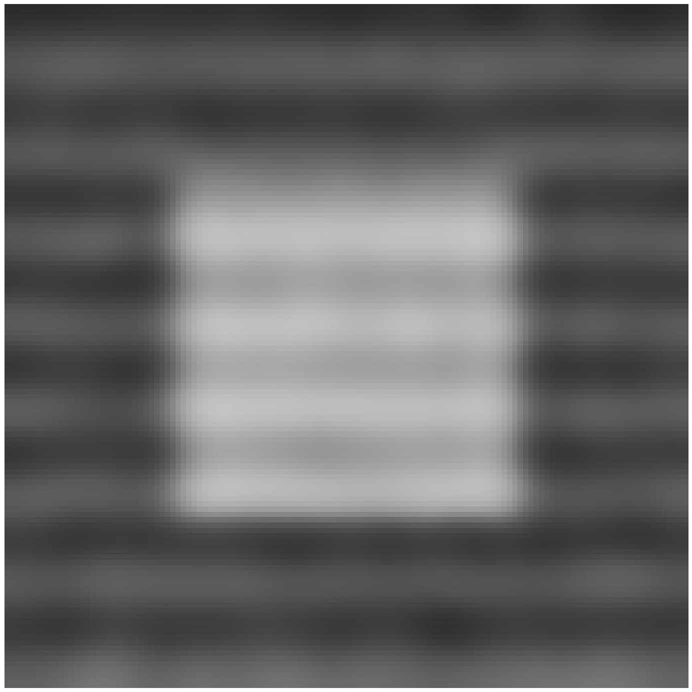
Difference

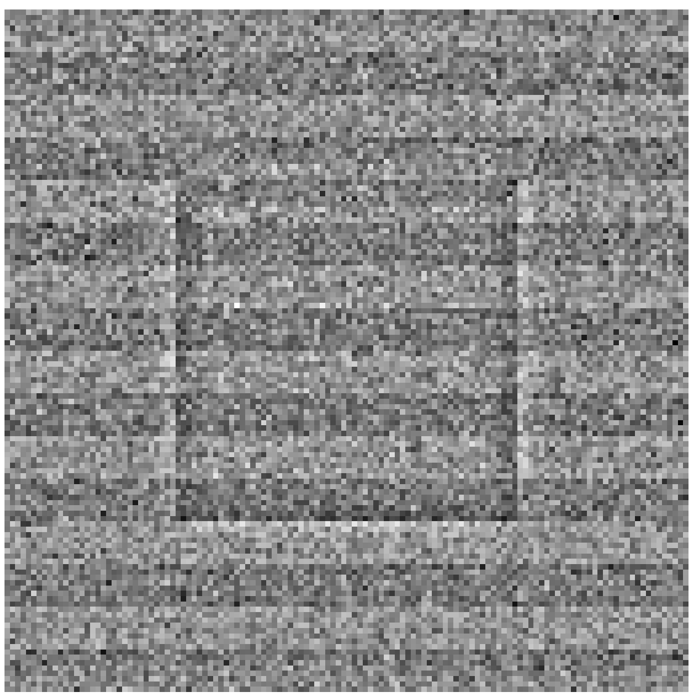




Difference image with OVS's approach and with ROF's method.

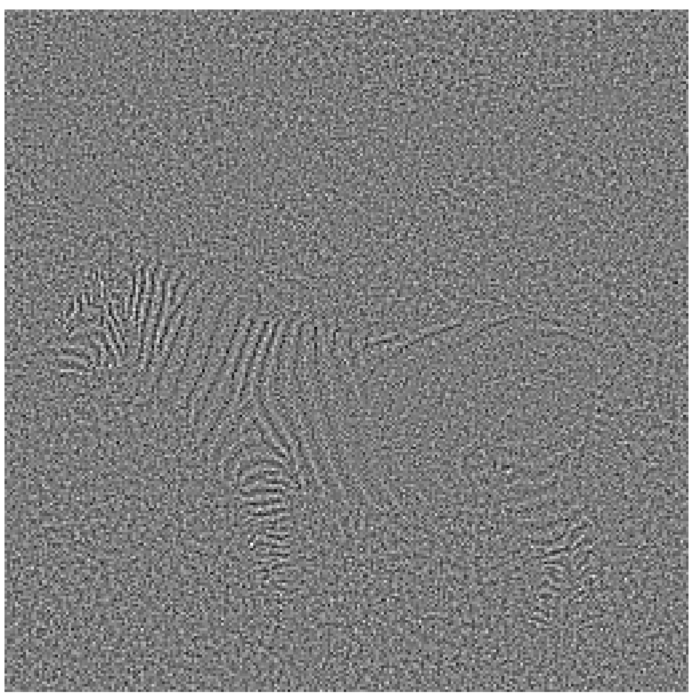
Removing more...



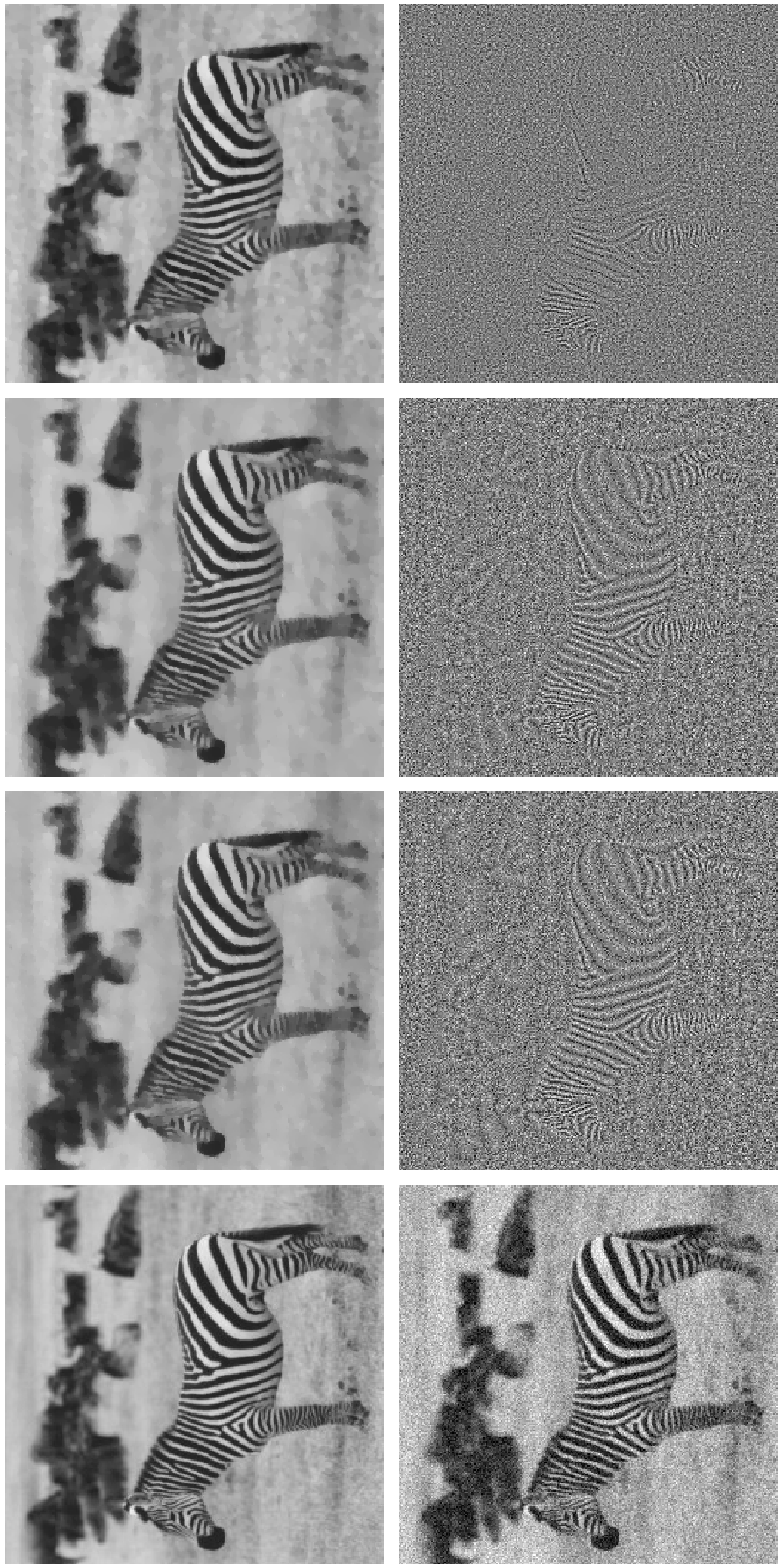


Try to remove the "texture" with OVS's approach ($\lambda = 1000$).

Denoising of the zebra



Zebra with OVS's approach ($\lambda = 10$),
and difference image.



Mean Curvature Motion

Let $\Omega \subset \mathbb{R}^N$ and $E \subset\subset \Omega$. Fix $h > 0$ a small time step. Let us solve

$$(7) \qquad \min_w J(w) \; + \; \frac{1}{2h}\int_\Omega |w - d_E|^2 \, dx$$

where $d_E(x) = \text{dist}\,(x, E) - \text{dist}\,(x, \mathbb{R}^N \setminus E)$. We let $T_h(E) = \{w < 0\}$.

Given E, we can define $E_h(t) = T_h^{[t/h]}(E)$, a discrete evolution of the set E.

Anisotropic variant

Let φ be a convex one–homogeneous function in $\mathbb{R}^N$ (a distance, with $c|x| \le \varphi(x) \le c'|x|$ for all x).

Let $\varphi^\circ(\xi) = \sup_{\varphi(\eta)\le 1} \langle \xi, \eta \rangle$ be the polar function. We introduce the anisotropic TV:

$$J_\varphi(w) = \int_\Omega \varphi^\circ(\nabla w) =$$
$$\sup \left\{ \int_\Omega u \, \text{div } \psi : \psi \in C^1_c(\Omega; \mathbb{R}^N), \varphi(\psi(x)) \le 1 \ \forall x \right\}$$

$d^\varphi_E(x) = d^\varphi(x, E) - d^\varphi(x, \mathbb{R}^N \setminus E)$ is the anisotropic signed distance to E, with $d^\varphi(x, E) = \inf_{y \in E} \varphi(x - y)$.

We solve

$$(7') \qquad \min_w J_\varphi(w) + \frac{1}{2h} \int_\Omega |w - d^\varphi_E|^2 \, dx$$

and let again $T_h(E) = \{w < 0\}$.

What does it do?

The (formal) Euler Lagrange equation for (7) is

$$-h\,\mathrm{div}\,\frac{\nabla w}{|\nabla w|} \;+\; w \;-\; d_E \;=\; 0.$$

At the boundary of $T_h E$, $w = 0$ and we get

$$d_E(x) \;=\; -h\kappa_{\{w=0\}}(x)$$

which is an implicit discretization of the Mean Curvature Motion.

→ Is it related to [Almgen-Taylor-Wang] or [Luckhaus-Sturzenecker]? Answer is Yes.

(ATW) $$\min_{F\subset\mathbb{R}^N} \mathrm{Per}\,(F) \;+\; \frac{1}{h}\int_{F\triangle E} |d_E(x)|\,dx$$

$$\kappa_F(x) \;+\; \frac{1}{h} d_E(x) \;=\; 0$$

→ same Euler equation.

Theorem:

$T_h(E) = \{w < 0\}$ is a solution of (ATW).

Convergence

We deduce (from (ATW)): smoothness of $\partial T_h E$, Hölder-like continuity in time of $E_h(t)$, convergence (up to subsequences) of $E_h(t)$ to some movement $E(t)$ (in L^1). But we also have an important monotonicity property:

Lemma:

$$E \subset E' \;\Rightarrow\; T_h(E) \subset T_h(E')$$

[obvious, $d_E > d_{E'} \Rightarrow w > w' \Rightarrow T_h E \subset T_h E'$]

From which we deduce

Theorem: (Convergence to the generalized Mean Curvature Motion) Consider E and f such that $E = \{f < 0\}$, and $u(t)$ the (unique) viscosity solution of the MCM equation

$$\frac{\partial u}{\partial t} = |\nabla u| \,\mathrm{div}\, \frac{\nabla u}{|\nabla u|}$$

with initial condition $u(t = 0) = f$. Assume at any time, $\Gamma(t) = \partial\{u(t){<}0\} = \partial\{u(t){\leq}0\}$ (no fattening, Γ is the *unique* generalized evolution starting from ∂E and is independent of f). Then

$$E_h(t) = T_h^{[t/h]} E \longrightarrow E(t)$$

as $h \to 0$.

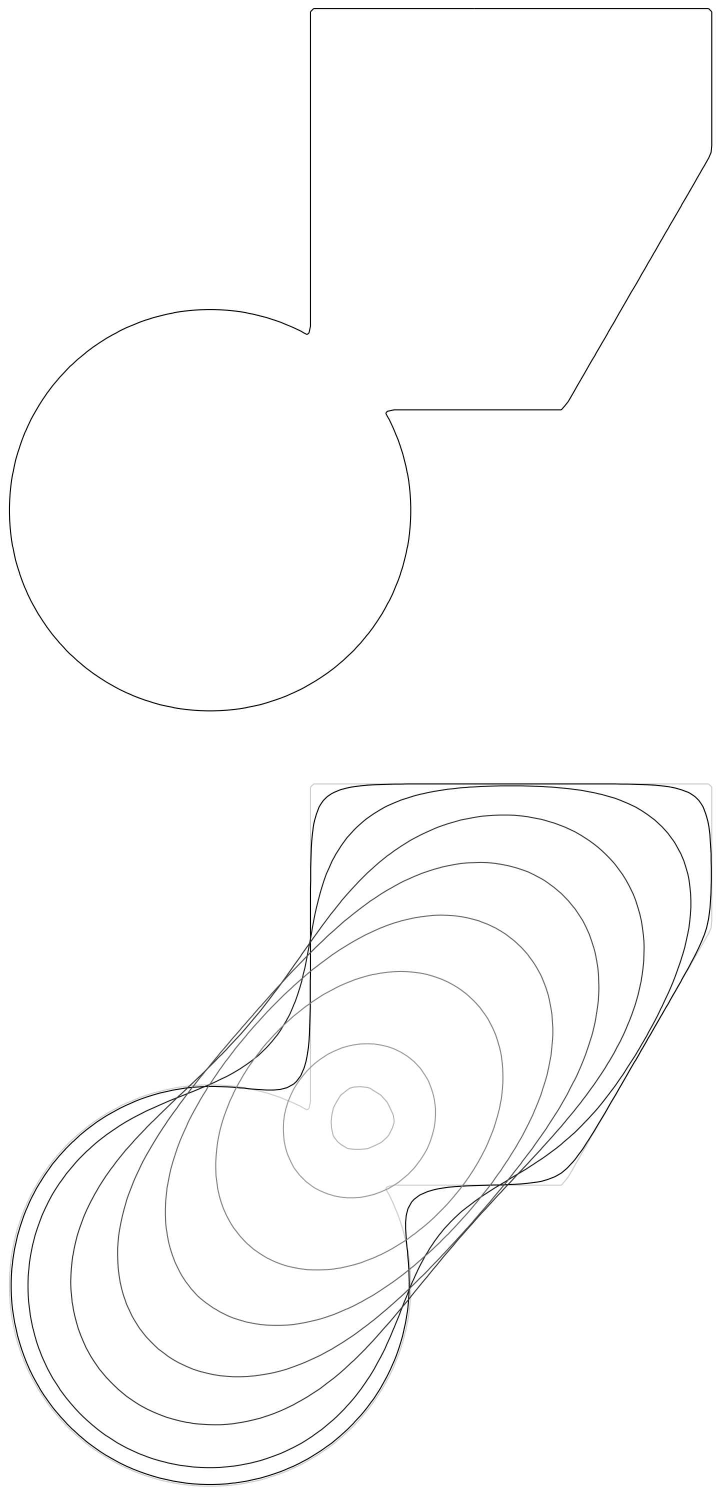
(Also for a smooth, elliptic anisotropy φ.)

The Convex case

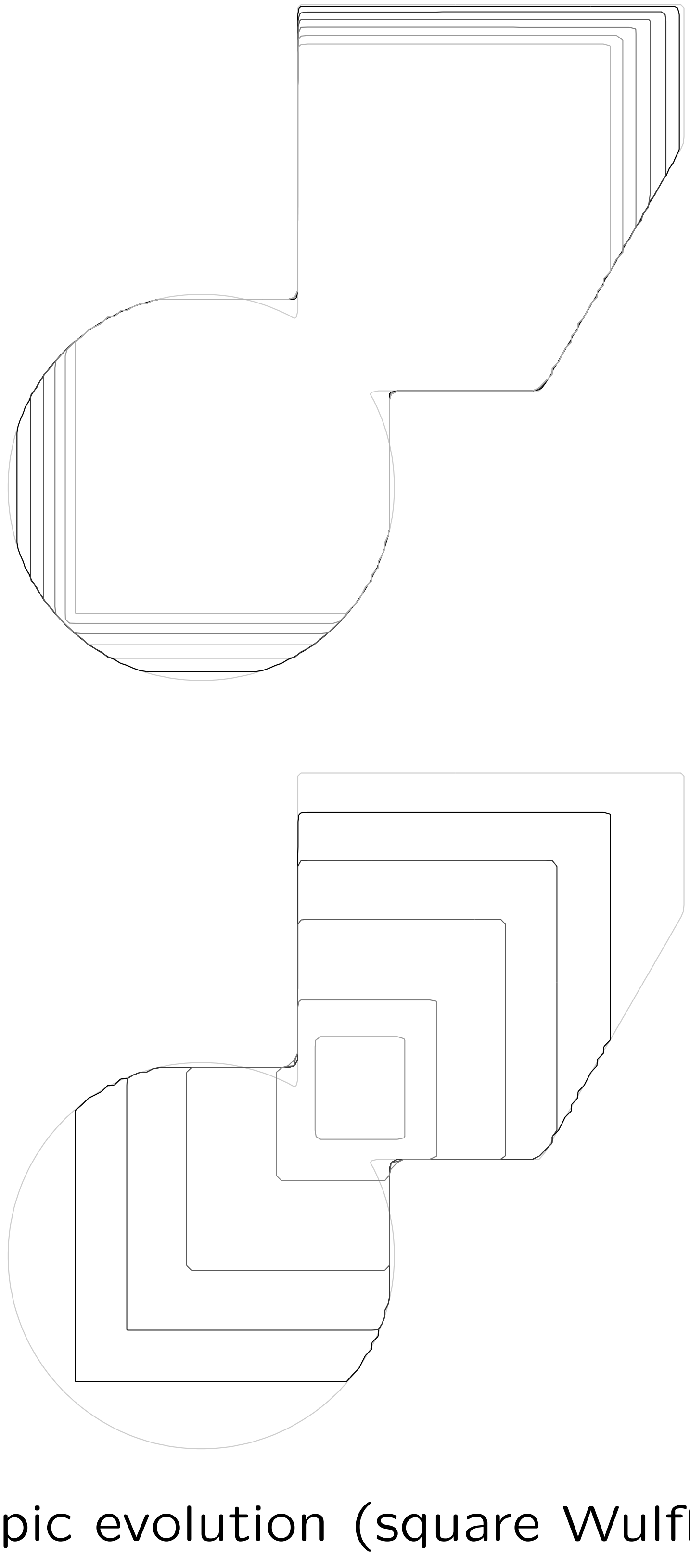
An advantage of (7) is not only that it yields the monotonicity of (ATW), other properties are also easier to study. Example:

Theorem [AC+Vicent Caselles]: Assume E is convex: then $T_h E$ is also convex (any anisotropy). Hence $E_h(t)$ converges to a convex evolution $E(t)$. In the crystalline case, we deduce the existence of an evolution for convex sets (in a quite weak sense, but any dimension), preserving convexity.

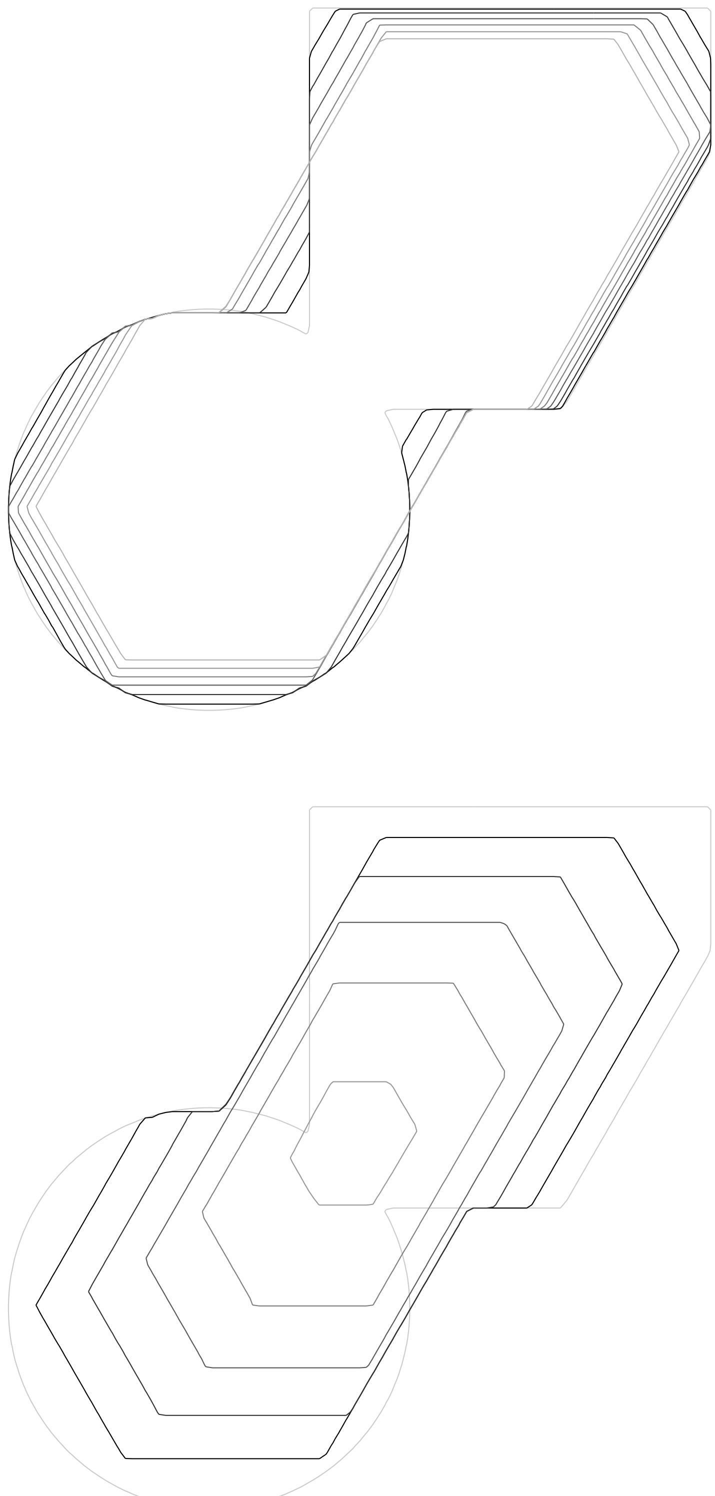
Examples of evolutions



An isotropic evolution at different times



Anisotropic evolution (square Wulff shape)



Anisotropic evolution (hexagonal Wulff shape)