Electrostatic Imaging via Conformal Mapping

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joint work with
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Or: A new solution method for inverse boundary value problems for the Laplace equation

Determine shape $\Gamma_0$ of a
- perfectly conducting or
- nonconducting inclusion or
- inclusion with different conductivity
from overdetermined Cauchy data on $\Gamma_1$

Applications in the field of nondestructive testing via electrostatic imaging or thermal imaging, e.g., impedance tomography
Here: Perfectly conducting inclusion, i.e., inverse Dirichlet problem
Extensions to other boundary conditions are in preparation

1. Brief survey on other methods
   (see 2. ed. of Linear Integral Equations)

2. Description of new method

3. Some numerical examples
The inverse problem

\[ \Delta u = 0 \quad \text{in } D \]

\[ u = 0 \quad \text{on } \Gamma_0 \]

\[ u = f \quad \text{on } \Gamma_1 \]

Inverse Problem:

Given \( g = \frac{\partial u}{\partial \nu} \) on \( \Gamma_1 \) (and \( f \)), **find** boundary \( \Gamma_0 \)

**Uniqueness!!!**
Uniqueness

\[ u = f, \quad \frac{\partial u}{\partial \nu} = g \]

\[ \Gamma_0 \leftrightarrow u \]
\[ \tilde{\Gamma}_0 \leftrightarrow \tilde{u} \]

In shaded domain: \( \Delta u = 0 \)

On boundary: \( u = 0 \)

**Schiffer** \( \approx 1960 \)
Existence???

For inverse boundary value problems, in general, **wrong question** to ask. Would need to characterize Cauchy data on $\Gamma_1$ for which the corresponding solution vanishes on a closed surface $\Gamma_0$ (or curve) within $\Gamma_1$.

**Main Task**: Assuming **correct data** or **perturbed** correct data, design method for approximate and stable solution.
Decomposition methods

1. Determine $u$ from Cauchy data on $\Gamma_1$, for example via

$$u(x) = u_0(x) + \int_{\Gamma} G(x, y) \varphi(y) \, ds(y)$$

and integral equation of the first kind

$$K\varphi = g - \frac{\partial u_0}{\partial \nu}$$

2. Find $\Gamma_0$ as location of the zeros of $u$ (in a least squares sense)


Pros:
- Conceptionally simple
- No need for forward solver

Contras:
- No high accuracy reconstructions
- Gap between theory and numerics
Newton type iterations

1. Interpret inverse problem as operator equation $F(\Gamma_0) = g$ where
   \[ F : \Gamma_0 \mapsto \frac{\partial u}{\partial \nu}|_{\Gamma_1} \]

2. Solve by regularized Newton iterations

**Pros:**
- Conceptionally simple
- Accurate reconstructions

**Contras:**
- Need forward solver
- Need good a priori information
- Convergence analysis difficult

Hohage (1999), Potthast (2001)
Hybrid of decomposition and Newton methods

1. Determine \( u \) via
\[
u(x) = u_0(x) + \int_{\Gamma} G(x, y) \varphi(y) \, ds(y)
\]
and integral equation \( K\varphi = \tilde{g} \)

2. Update
\( \Gamma \rightarrow \Gamma_h = \{ x + h(x) : x \in \Gamma \} \)
via Newton step
\[
u + \text{grad} \, u \cdot h = 0
\]

Does not need a forward solver!

Inverse obstacle scattering:
Kirsch’s factorization method

Characterize unknown domain via spectral data of the Dirichlet-to-Neumann operator

\[ A : u \mapsto \frac{\partial u}{\partial \nu} \]

on \( \Gamma_1 \)

Brühl, Hanke (2000)

Pros:
- Elegant mathematics
- Simple implementation
- No a priori information needed

Contras:
- Need a lot of data
- No sharp boundaries (\( \infty = ? \))
- Very sensitive to noise
Our method

1. Solve nonlocal nonlinear ordinary differential equation for boundary values of a conformal mapping (by successive iterations)

2. Solve Cauchy problem for a holomorphic function in an annulus

Pros:
- Conceptionally simple
- Satisfactory reconstructions
- Domain for Cauchy problem is known

Contras:
- Restricted to two dimensions
- and to Laplace equation
\[ u = f \]
\[ \frac{\partial u}{\partial \nu} = g \]

\[ \Gamma_1 = \{ \gamma(s) : s \in [0, L] \} \]
\[ C_1 = \{ e^{it} : t \in [0, 2\pi] \} \]

\[ \varphi : \text{arc length on } C_1 \quad \mapsto \quad \text{arc length on } \Gamma_1 \]

\[ \Psi(e^{it}) = \gamma(\varphi(t)) \]

knowing \( \varphi \) equivalent to knowing \( \Psi|_{C_1} \)
\[ u = f \]
\[ \frac{\partial u}{\partial \nu} = g \]

\[ \Gamma_0 \quad \Gamma_1 \]

\[ u = 0 \]

\[ \rightarrow \Psi, \varphi \]

\[ D \]

\[ B \]

\[ C_0 \quad C_1 \]

\( u, \tilde{u} \) and \( v = u \circ \Psi, \tilde{v} = \tilde{u} \circ \Psi \) conjugate harmonics

\[ \tilde{v}(t) = \tilde{u}(\varphi(t)) \quad \Rightarrow \quad \frac{\partial \tilde{v}}{\partial t} = \frac{\partial \tilde{u}}{\partial s} \frac{d\varphi}{dt} \]

Cauchy–Riemann equations \( \Rightarrow \quad \frac{\partial v}{\partial \nu} = \frac{\partial u}{\partial \nu} \frac{d\varphi}{dt} \)

\[ \frac{d\varphi}{dt} = \frac{A(f \circ \varphi)}{g \circ \varphi} \quad A = \text{Dirichlet-to-Neumann map for } B \]
\[ u = f, \quad \frac{\partial u}{\partial \nu} = g \quad \text{on} \quad \Gamma_1 \]

\[ \int_{C_0} \frac{\partial v}{\partial \nu} \, ds = \int_{C_1} \frac{\partial v}{\partial \nu} \, ds = \int_{\Gamma_1} g \, ds \]

\[ \int_{C_0} \left\{ \ln |x| \frac{\partial v}{\partial \nu} - v \frac{1}{|x|} \right\} \, ds = \int_{C_1} \left\{ \ln |x| \frac{\partial v}{\partial \nu} - v \frac{1}{|x|} \right\} \, ds \]

\[ \ln \rho \int_{C_0} \frac{\partial v}{\partial \nu} \, ds = - \int_{C_1} v \, ds \]

\[ \rho = \exp \left( \frac{- \int_0^{2\pi} f \circ \varphi \, dt}{\int_{\Gamma_1} g \, ds} \right) \]
\[ \frac{d\varphi}{dt} = \frac{A(f \circ \varphi)}{g \circ \varphi} \quad A = \text{Dirichlet-to-Neumann map for } B \]

\[ L = \int_{0}^{2\pi} A(f \circ \varphi) \frac{1}{g \circ \varphi} \, dt \]

\[ \frac{d\varphi}{dt} = \frac{A(f \circ \varphi)}{g \circ \varphi} + \frac{L}{2\pi} - \frac{1}{2\pi} \int_{0}^{2\pi} A(f \circ \varphi) \frac{1}{g \circ \varphi} \, dt \]

Makes sure that

\[ \varphi(2\pi) = L \]

throughout the iteration
\[ \rho = \exp \left( -\frac{\int_0^{2\pi} f \circ \varphi \, dt}{\int_{\Gamma_1} g \, ds} \right) \]

\[ \frac{d\varphi}{dt} = \frac{A(f \circ \varphi)}{g \circ \varphi} + \frac{L}{2\pi} - \frac{1}{2\pi} \int_0^{2\pi} \frac{A(f \circ \varphi)}{g \circ \varphi} \, dt \]
\[ \rho_n = \exp \left( -\frac{\int_0^{2\pi} f \circ \varphi_n \, dt}{\int_{\Gamma_1} g \, ds} \right) \]

\[ \frac{d\varphi_{n+1}}{dt} = \frac{A_n(f \circ \varphi_n)}{g \circ \varphi_n} + \frac{L}{2\pi} - \frac{1}{2\pi} \int_0^{2\pi} \frac{A_n(f \circ \varphi_n)}{g \circ \varphi_n} \, dt \]

\[ \varphi_0(t) = \frac{L}{2\pi} \, t, \quad \text{correct if } D \text{ annulus} \]

**Theorem** Under appropriate assumptions on \( D \) and \( f \) the successive approximations converge in \( H^1[0, 2\pi] \).
\[
\frac{d\varphi_{n+1}}{dt} = \frac{A_n(f \circ \varphi_n)}{g \circ \varphi_n} + \frac{L}{2\pi} - \frac{1}{2\pi} \int_0^{2\pi} \frac{A_n(f \circ \varphi_n)}{g \circ \varphi_n} \, dt
\]

Numerical implementation:

\[
\varphi_n(t) \approx \frac{L}{2\pi} t + \alpha_{n,0} + \sum_{k=1}^{N} [\alpha_{n,k} \cos kt + \beta_{n,k} \sin kt]
\]

\[
L_n := \frac{1}{2\pi} \int_0^{2\pi} \frac{A_n(f \circ \varphi_n)}{g \circ \varphi_n} \, dt - \frac{L}{2\pi}
\]

\[
(g \circ \varphi_n)(t_j) \{\varphi_{n+1}'(t_j) + L_n\} - A_n(f \circ \varphi_n)(t_j) = 0, \quad j = 1, \ldots, J
\]

Solve by least squares to update coefficients
Use trigonometric interpolation for \( f \circ \varphi \) and \( g \circ \varphi \)
Now, we know the radius $\rho$ and $\Psi = \gamma \circ \varphi$ on the outer circle $C_1$, i.e., Fourier series

$$\gamma(\varphi(t)) = \sum_{k=-\infty}^{\infty} b_ke^{ikt}, \ 0 \leq t \leq 2\pi$$

Solve Cauchy problem for $\Psi$ in $B$ via Laurent expansion

$$\Psi(z) = \sum_{k=-\infty}^{\infty} b_kz^k, \ \rho \leq |z| \leq 1$$

and obtain unknown boundary by

$$\Gamma_0 = \Psi(C_0) \approx \left\{ \sum_{k=-M}^{M} \rho^k b_k e^{ikt}, \ 0 \leq t \leq 2\pi \right\}$$

Need to truncate because of exponential ill-posedness.
Our method

1. Solve nonlocal nonlinear ordinary differential equation for boundary values of a conformal mapping (by successive iterations)

2. Solve Cauchy problem for a holomorphic function in an annulus
For numerical examples $\Gamma_1$ unit circle

$$f(t) = 6 + \exp(\cos t) + \exp(\sin t)$$

degree of trigonometric polynomials: $N = 6 \ldots 8$

cutoff in Laurent expansion: $M = 6 \ldots 8$

number of collocation points: $J = 32$

Between 6 to 10 iterations
Remarks:

- **Idemen, Akduman** (1988)

\[ f = f_0 = \text{const} \quad \Rightarrow \quad \frac{d\varphi}{dt} = -\frac{1}{\ln \rho} \frac{f_0}{g \circ \varphi} \]

- Nonconstant \( f \) required for extensions to other boundary conditions. However, no flux \( \int_{\Gamma_0} g \, ds = 0 \) has to be observed.

- **Inverse Problems 18, 1659–1672 (2002)**
Cracks!!!
\[ u = f \]
\[ \frac{\partial u}{\partial \nu} = g \]

Try \( \Gamma_0 \approx \Psi(\tilde{C}_0) \) with radius \( \rho(1 + \lambda) \) for \( \tilde{C}_0 \)
Open problems:

- Other boundary conditions
- Incomplete data
- Satisfactory analysis for cracks
- Other regularizations in second step