A Variational Approach to Non-Rigid Morphological Image Registration

Inverse Problems Workshop Series I
Emerging Applications of Inverse Problems Techniques to Imaging Science

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Outline

- Unimodal Registration
- Gradient flow perspective
- Relations to Tikhonov-Regularization

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- Multimodal Registration
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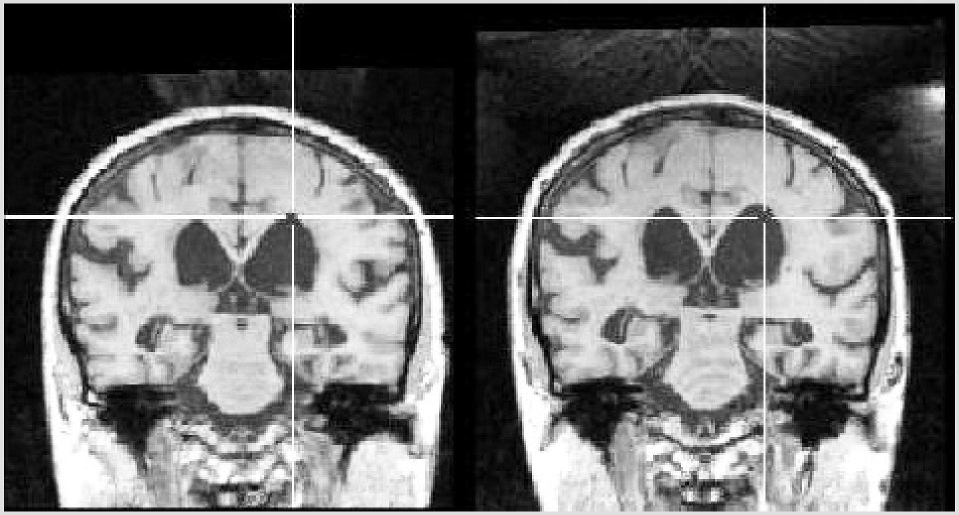
- Practical issues for solving the minimization problem

Registration of a pair of unimodal images

Given two C^1-images $R, T : \Omega \to \mathbb{R}^d$, find a deformation $\phi = \mathbb{1} + u : \Omega \to \Omega$, such that

$$T \circ \phi \approx R \text{ in the sense of image intensities.}$$

- Analysis of medical "time-series" of a single patient (intraindividual)



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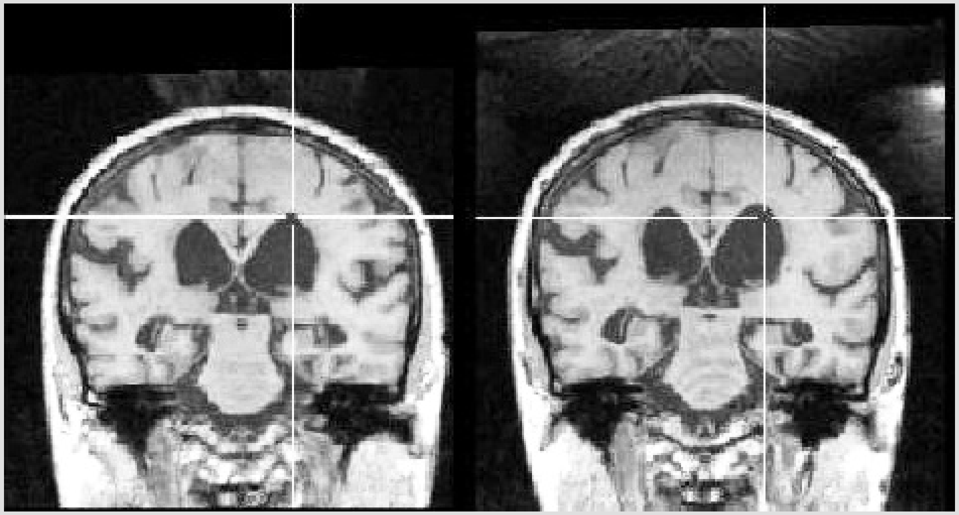

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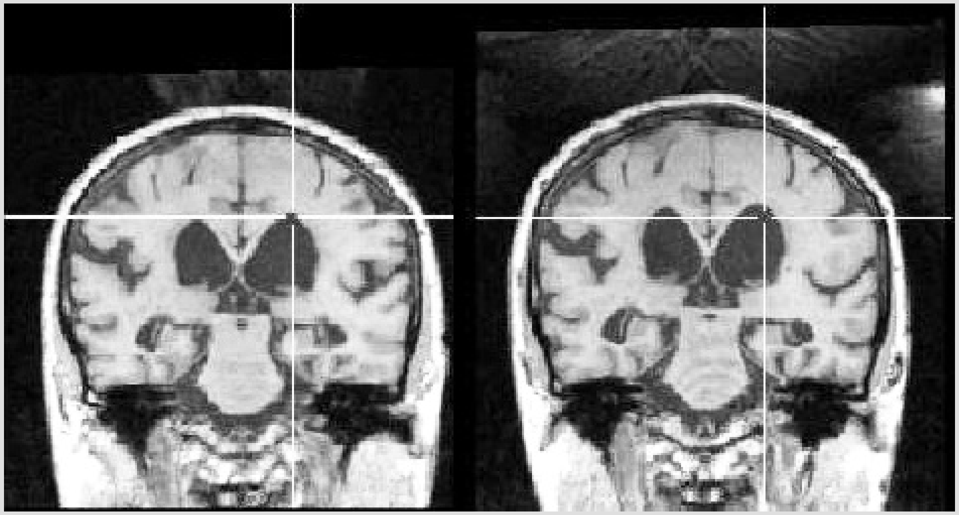
- Analysis of medical "time-series" of a single patient (intraindividual)
- Registration into a digital database (interindividual)
- Subtraction of angiographic images



Requirements of the deformation

Deformation maps on the background of the data:

- Bijectivity & topology-preservation: $\longrightarrow$ Homeomorphisms!!
- Rich space of deformations, allowing local dilations and contraction to resolve very fine anatomical details.
- Desirable: preservation of geometric features $\longrightarrow$ Diffeomorphisms.

Relation to *optical flow models*

Here, consider a time-dependent sequence of images: $I : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}$.
The *brightness constancy assumption* $I(x(t), t) =$ const for moving points described by $x(t)$ leads to the optical flow equation:

$$(\nabla I(x,t), \vec{v}(x,t)) + I_t(x,t) = 0,$$

where $\vec{v}$ describes the optical flow of the image.

This approach is *differential* and aims at determining of the movements in images, which are very close together.

Due to underdetermination of the equation, various variational approaches are considered (see *Hinterberger, Scherzer, Schnörr, Weickert '01*):

$$E[\vec{v}] := \int_\Omega \phi((\nabla I(x,t), \vec{v}(x,t)) + I_t(x,t)) + \psi(x, \vec{v}, \nabla \vec{v}) dx \to \min!$$

The Unimodal Registrationenergy

Images $T, R : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, $n = 2, 3$. Find $\phi \in \mathcal{V}$, $\mathcal{V} \subset \{\psi : \mathbb{R}^d \to \mathbb{R}^d\}$ such that the energy

$$E_m[\phi] := \frac{1}{2} \int\limits_{\Omega} |T \circ \phi - R|^2 dx$$

is minimal and $\phi(\Omega) = \Omega$. Gradient is given by

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$$\mathrm{grad} E_m[\phi] = (T \circ \phi - R) \nabla T \circ \phi$$

Highly nonlinear problem since the image T is nonlinear.

Assymmetric definition of the energy.

Ill-posedness

Define $\mathcal{M}_c^T := \{x \in \Omega \,|\, T(x) = c\}$
Consider deformation Λ, s. d. $\Lambda(\mathcal{M}_c) = \mathcal{M}_c$, $\forall c \in \mathbb{R}$.

Furthermore: Let T constant on $\tilde{\Omega} \subset \Omega$. $\tilde{x} \in T^{-1}(\tilde{\Omega})$.

$$\Lambda = 1\!\!\mathrm{I}\chi_{\Omega\setminus\{\tilde{x}\}} + z\chi_{\{\tilde{x}\}} \text{ for } z \in \tilde{\Omega} \text{ arbitrary.}$$

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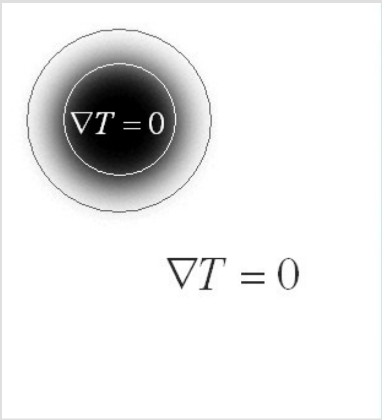
In both cases the following holds:

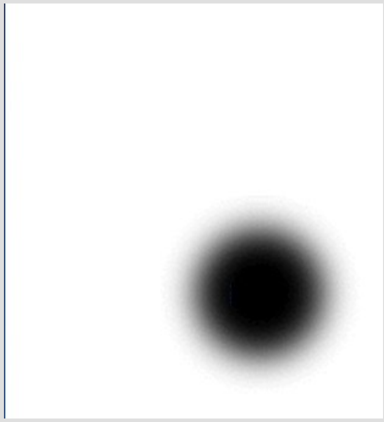
$$E_m[\Lambda \circ \phi] = E_m[\phi].$$

Set of minimizers is very irregular, depending on the variability of the images.

Non-overlapping shapes

Consider even very *simple* input images

T 

 R

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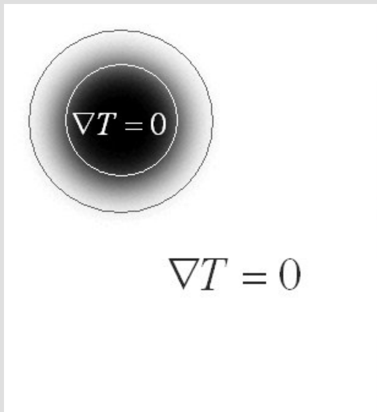
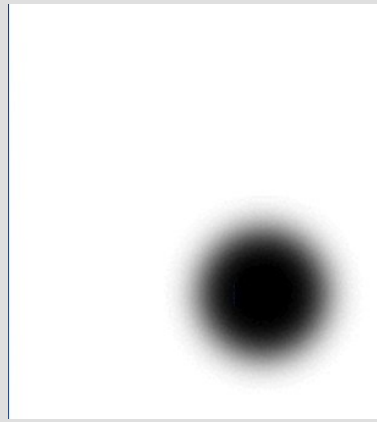
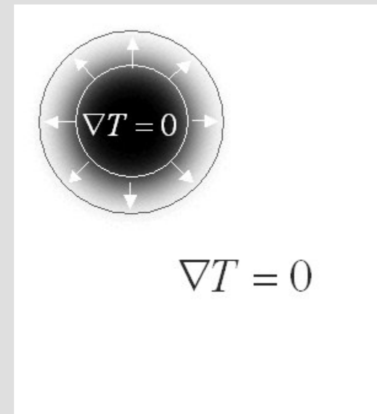
T 

 R $\Rightarrow$



$\nabla T = 0$ or $T \circ \phi = R \Rightarrow$ grad$E = 0$.

Gradient drives the deformation only at the transitions of flat regions, i. e. where $\nabla T \neq 0$. Here, this leads to a **concentration of the level sets, i. e. without regularization the gradient flow converges to a degenerate solution.**

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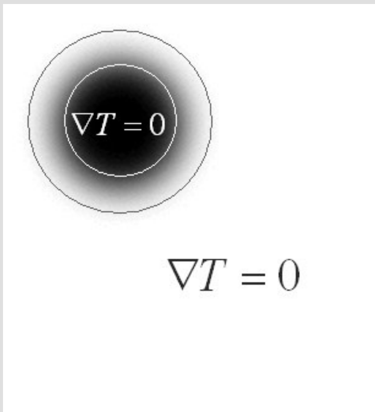
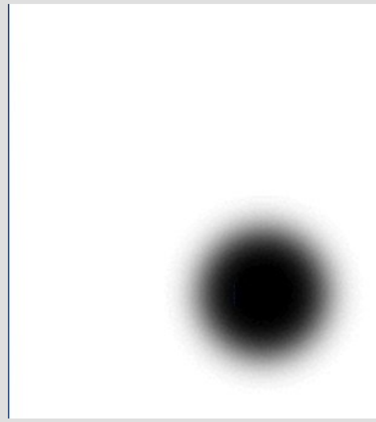
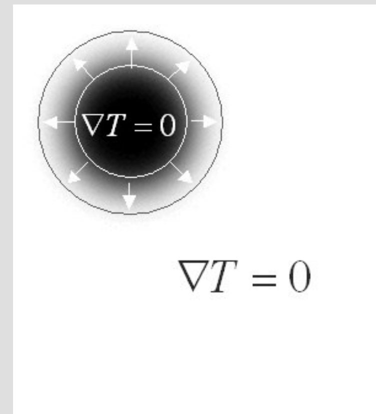
T 

 R $\Rightarrow$



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Matching of hypersurfaces: *Liao, Khuu, Bergschneider, Vese, Huang, Osher* use level sets combined with distance maps for converging globally.

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Regularization approaches

1. **Additional Regularization Energy:**

$$E[\phi] := E_m[\phi] + E_{\mathsf{reg}}[\phi] \to min!$$

$\rightsquigarrow$ *Tikhonov-Regularization*, (*Grötsch, Scherzer, Weickert*)

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2. **Gradient flows w.r.t. regularizing metric:**

$$\partial_t \phi = -\mathsf{grad}_g E[\phi]$$

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2. **Gradient flows w.r.t. regularizing metric:**

$$\partial_t \phi = -\mathsf{grad}_g E[\phi]$$

3. **Iterative smoothing of data:**

$$E_m^{\sigma_k}[\phi] := \int_\Omega |T^{\sigma_k} \circ \phi - R^{\sigma_k}| \, \mathrm{d}\mu \quad k = 1, 2, \ldots \quad I^\sigma = \text{smoothed version of} I$$

Regularization of gradient flows

Introduce a **metric** g on the space of deformations and consider the gradient of E with respect to this metric, i. e.:

$$g(\text{grad}_g E, \psi) = \langle E'[\phi], \psi \rangle \qquad \forall \psi \in \mathcal{V}$$

The general gradient flows becomes:

$$\partial_t \phi(t) = -\text{grad}_g E_m[\phi(t)]$$

which means $g(\partial_t \phi(t), \psi) = -\langle E'_m[\phi(t)], \psi \rangle \qquad \forall \psi \in \mathcal{V}$

Every metric g inhibits a linear representation $A : \mathcal{V} \to \mathcal{V}'$, $g(u, v) = \langle Au, v \rangle_{\mathcal{V}' \times \mathcal{V}}$, hence, we can also write

$$\partial_t \phi(t) = -A^{-1} \text{grad}_{L^2} E_m[\phi(t)].$$

Metrics

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  $\Rightarrow g(u,v) = \langle Au, v\rangle$ mit $A = \mathbb{1} + \frac{\sigma^2}{2}\Delta$

  A implicit time step of the linear diffusion equation.

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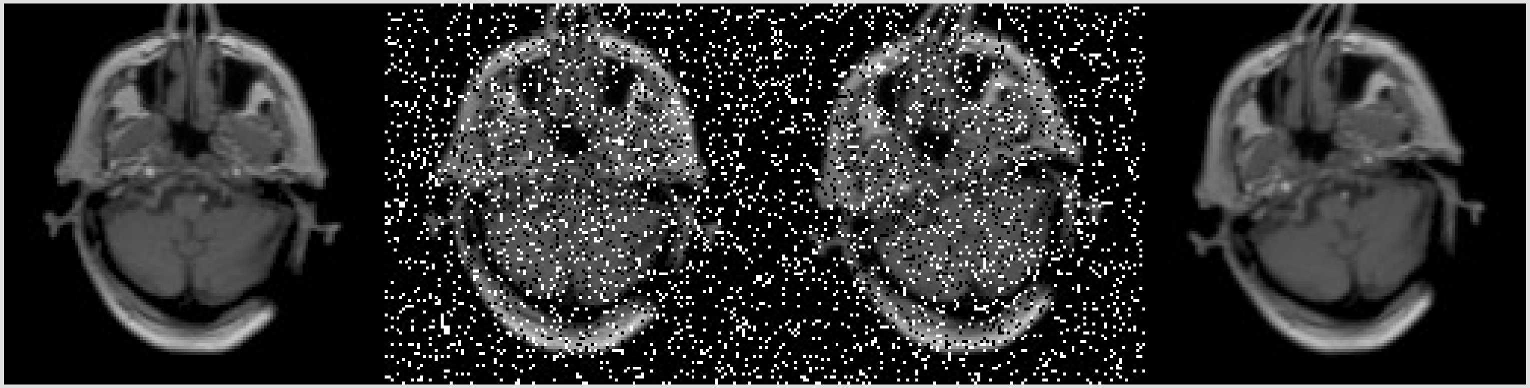
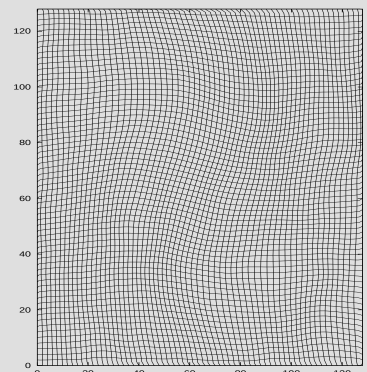
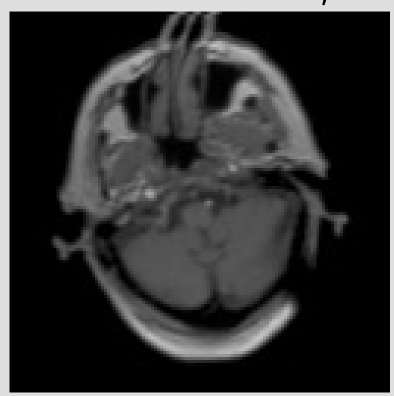
Numerical Results



T original, T with noise, R with noise, R original





deformation (ϕ), result of registration $T \circ \phi$

noise: 20% *Salt and Pepper noise*

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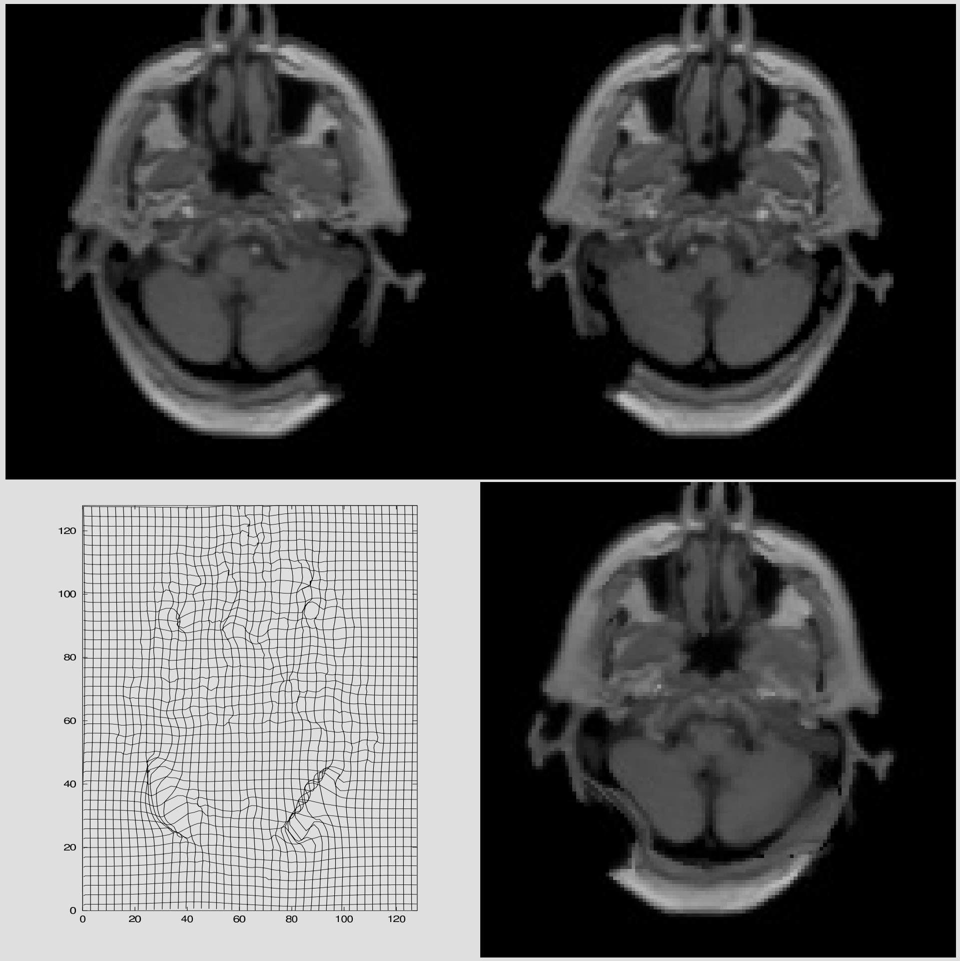

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Tikhonov Regularization: Linearization

Given $\phi^{(k)}$, find $\phi^{(k+1)}$, such that

$$E_m[\phi^{(k+1)}] < E_m[\phi^{(k)}]$$

Linearization: $$E_m[\phi^{(k+1)}] \approx E_m[\phi^{(k)}] + \langle E'_m[\phi^{(k)}], \phi^{(k+1)} - \phi^{(k)} \rangle$$

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Additional bilinearform $g(\phi^{(k+1)} - \phi^{(k)}, \phi^{(k+1)} - \phi^{(k)})$ for regularization ($Henn$, $Witsch$): find

$$\arg \min_{\phi^{(k+1)} \in \mathcal{V}} \left\{ \langle E'_m[\phi^{(k)}], \phi^{(k+1)} - \phi^{(k)} \rangle + \frac{\alpha}{2} g(\phi^{(k+1)} - \phi^{(k)}, \phi^{(k+1)} - \phi^{(k)}) \right\}$$

$$\alpha g(\phi^{(k+1)} - \phi^{(k)}, \psi) = -\langle E'_m[\phi^{(k)}], \psi \rangle$$

Tikhonov Regularization: The non-linear case

Consider the following non-linear minimization problem

$$\min_{\phi} \left\{ E[\phi] + \frac{\alpha}{2} Q[\phi] \right\}$$

residual error $(E) \leftrightarrow$ variability of the solution ϕ (Q).

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Inspect the behaviour for decreasing α:

$$\phi^{(k+1)}=\arg\min_{\phi}\{E[\phi]+\frac{\alpha_k}{2}g(\phi-\phi^{(k)},\phi-\phi^{(k)})\}$$

$\rightsquigarrow$ iterative Tikhonov Regularization ($Henn,\ Witsch$).

Important Regularization Methods for Registration

- *Horn-Schunk*-model (1981):

$$Q_{\mathsf{diff}}[\phi] := \frac{1}{2}\sum_{i=1}^{n}\int_{\Omega}|\nabla\phi_i|^2 dx$$

- Curvature-model *Modersitzki, Fischer '03*

$$Q_{\mathsf{curv}}[\phi] := \frac{1}{2}\sum_{i=1}^{n}\int_{\Omega}(\Delta\phi_i)^2 dx \qquad Q_{\mathsf{curv}}[Cx+B] = 0 \quad \forall C \in \mathbb{R}^{n,n},\ b \in \mathbb{R}^n$$

- *Nagel-Enkelmann*-model: (1987)

$$Q_{NE}[\phi] := g_{NE}(\phi,\phi) \text{ where}$$

$$g(\phi,\psi) := \int \mathrm{tr}\left\{\frac{1}{|\nabla R|^2 + 2\lambda}\left((\nabla R)^\perp \otimes (\nabla R)^\perp + \lambda^2 \mathbb{1}\right) \nabla\phi \cdot \nabla\psi\right\} dx$$

image based weight: preserve corners and edges, related to anisotropic diffusion (*Weickert*)

- linear elastic models

$$Q_{\mathsf{elast}}[\phi] := \int_\Omega 2\mu\epsilon(\phi) : \epsilon(\phi) + \frac{\lambda}{2}(\mathrm{div}\phi)^2 dx$$

- nonlinear elastic models

$$Q_{NLE}[\phi] := \int_\Omega W^*(D\phi, \mathbf{Cof}\, D\phi, \mathbf{det}\, D\phi)$$

Relation between reg. gradient flows $\leftrightarrow$ iterative Tikhonov

1. **linearized:**

$$\begin{aligned} \alpha g(\phi^{(k+1)} - \phi^{(k)}, \psi) &= -\langle E'_m[\phi^k], \psi \rangle \quad \forall \psi \in \mathcal{V} \\ \Rightarrow \phi^{(k+1)} &= \phi^{(k)} - \frac{1}{\alpha} \mathsf{grad}_g E_m[\phi^{(k)}] \quad g \text{ Metrik} \end{aligned}$$

Interpretation: **explicit** time step of the regularized gradient flow.

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   Interpretation: **explicit** time step of the regularized gradient flow.

2. **non-linear:** Sequence of minimization problems for $\alpha \to 0$.

$$E_m[\phi] + \frac{\alpha_k}{2} g(\phi - \phi^{(k)} \quad , \quad \phi - \phi^{(k)})$$

$$\begin{aligned} \text{Euler-Lagrange} \qquad \alpha_k g(\phi - \phi^k, \psi) &= -\langle E_m'[\phi], \psi\rangle \\ \phi^{(k+1)} &= \phi^{(k)} - \frac{1}{\alpha_k}\mathrm{grad}_g E_m[\phi^{(k+1)}] \end{aligned}$$

   Interpretation: **implicit** time step of length α_k^{-1}.

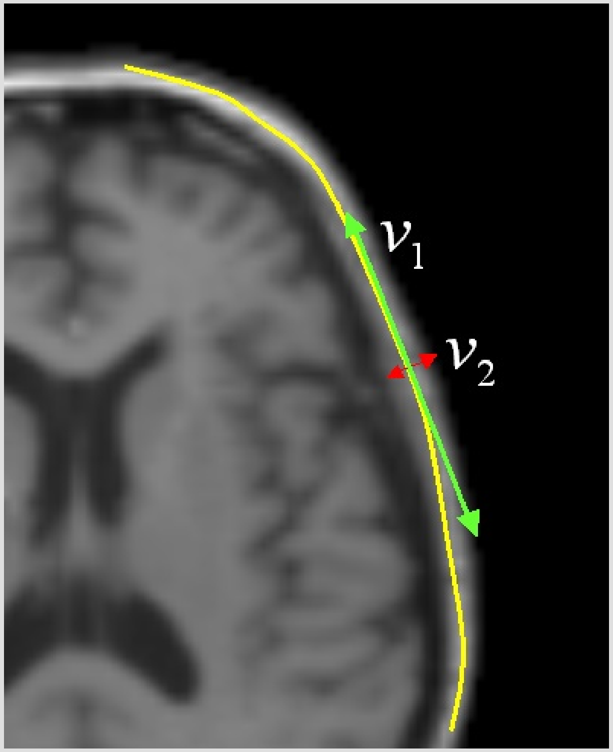
Application of non-homogenous metric

Nagel-Enkelmann-model

Metric $g_{NE}(\phi, \psi) = \mathrm{tr}(D(\nabla R)\nabla\phi \cdot \nabla\psi)$.

With eigenvalues $\lambda_1 > \lambda_2 > 0$.
Eigenvector v_1 in direction of $\nabla R^{\perp}$. λ_1 large.
Eigenvector v_2 in direction of ∇R. λ_2 small.

Corresponds to a stronger regularization in the set of deformations which are invariant under the registration energy.



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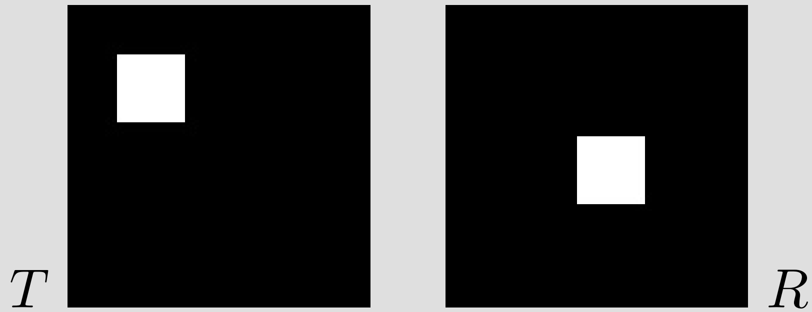
input images



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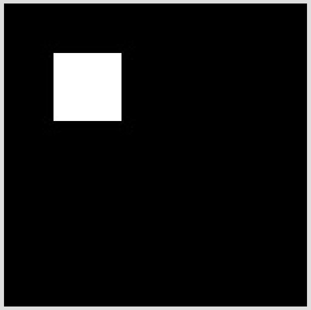
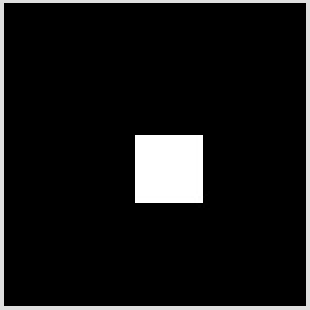
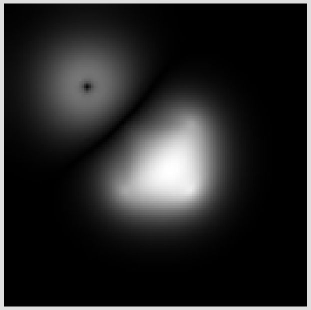
T   R



homogeneous metric

Application of non-homogenous metric

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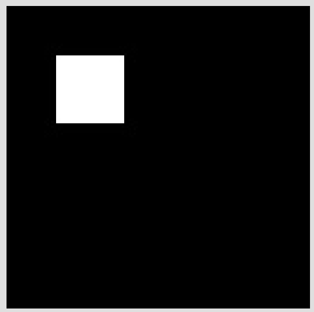
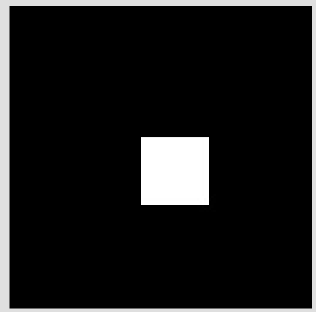
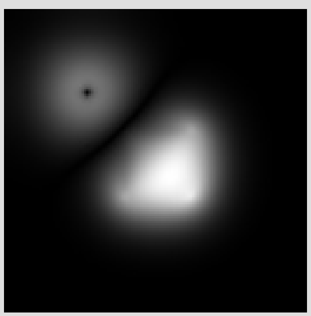
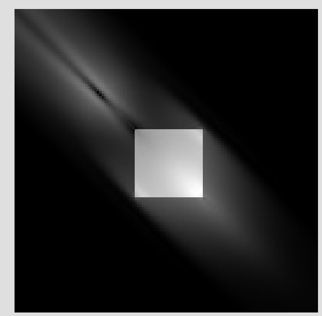
T   R

 homogeneous metric

heterogenous metric 

intensity corresponds to $|\phi - \mathbb{1}| = |u|$

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- variations in illuminations for the study of image sequences (cf. *optical flow*)

Overview of Multimodal similarity measures

- Global Mutual information (*Viola, Wells '96*, *Collignon, et al. '95*)

$$MI(X,Y) = \underbrace{H(X) + H(Y)}_{\text{entropy of } X \text{ resp. } Y} + \underbrace{H(X,Y)}_{\text{joint entropy of } X \text{ and } Y}$$

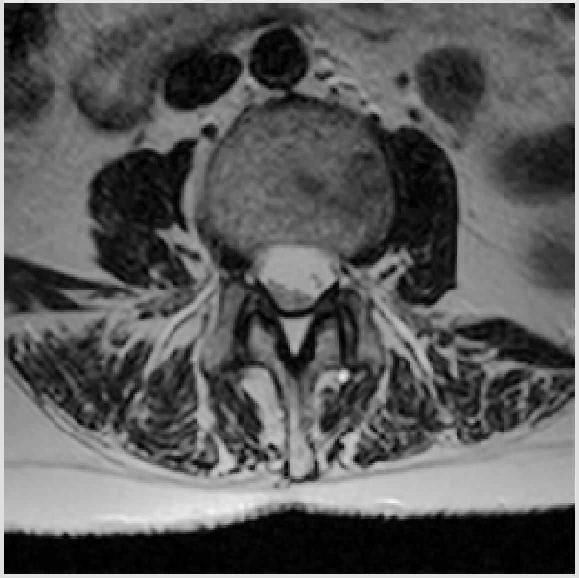
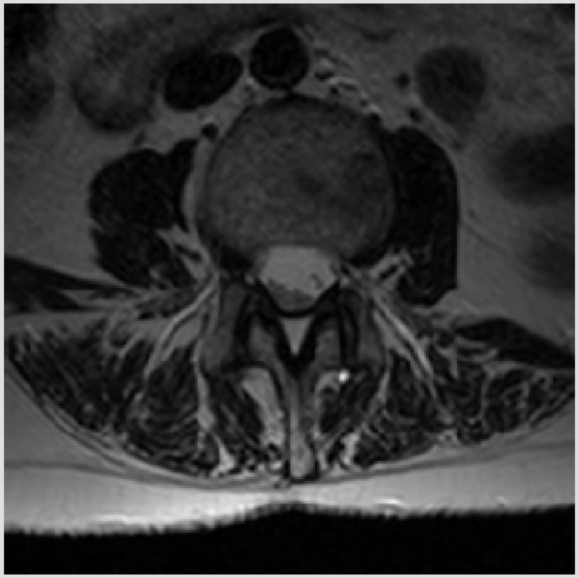
$$H(x) := -\int p(x) \ln p(x) dx \qquad H(x,y) := -\int p(x,y) \ln p(x,y) dx\, dy$$

- Local Cross-Covariance

$$CC(X,Y) := -\int_\Omega \frac{V_{X,Y}^2}{V_X \cdot V_Y} dx$$

$V_{X,Y}$ local correlation of X and Y $\quad$ V_X local variance of X

Morphological Methods in Image Processing

Contrast invariant description of images:
Def. Morphology $M[I] := \{\mathcal{M}_c^I \,|\, c \in \mathbb{R}\}$
Upper topographic map $UTM[I] := \{[I \geq c] \,|\, c \in \mathbb{R}\}$ (*Caselles, Coll, Morel*).

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Morphological filter $\mathcal{F}$:

$$\mathcal{CC} \circ \mathcal{F} = \mathcal{F} \circ \mathcal{CC}$$

where $\mathcal{CC}$ is a contrast-change of the image, i. e. $\mathcal{CC}(I) = g(I)$ for a nondecreasing function g.

$\rightsquigarrow$ morphological filters do only depend on $M[I]$ resp. $UTM[I]$.

Motivation:

- Nonlinear response of sensors
- Different display devices in general have different contrast.

Example: **Level-Set-Methods** with evolution speed only depending on the shape of the level sets of a function $\phi : \Omega \rightarrow \mathbb{R}$.

$$\partial_t + F(S)\|\nabla\phi\| = 0 \quad \text{in } (0,T)\times\Omega \qquad \phi(0,\cdot) = I$$

$$\text{Shape operator} S := DN = \frac{1}{\|\nabla\phi\|}(\mathbb{1} - N\otimes N)D^2\phi \quad N = \frac{\nabla\phi}{\|\nabla\phi\|}$$

corresponding to the *Weingarten-Map* $DX^{-1}\circ DN$ on the tangent space $T_x M_c \phi$.

- **Mean-Curvature-Flow** $F_{\text{MCM}}(S) = -\text{tr}S = -\text{div}\left(\frac{\nabla\phi}{\|\nabla\phi\|}\right)$
- **Affine morphological scale space** (*Alvarez, et al.*) $F_{\text{AMSS}}(S) = -(\text{tr}S)^{\frac{1}{3}}$.
- **Anisotropic curvature flow** (*Preusser, Rumpf*) $F_{\text{aniso}}(S) = -\text{div}\left(a^\sigma \frac{\nabla\phi}{\|\nabla\phi\|}\right)$

Multimodal Registration based on Image Morphology

Aim: Construct a geometric similarity measure as registration energy.

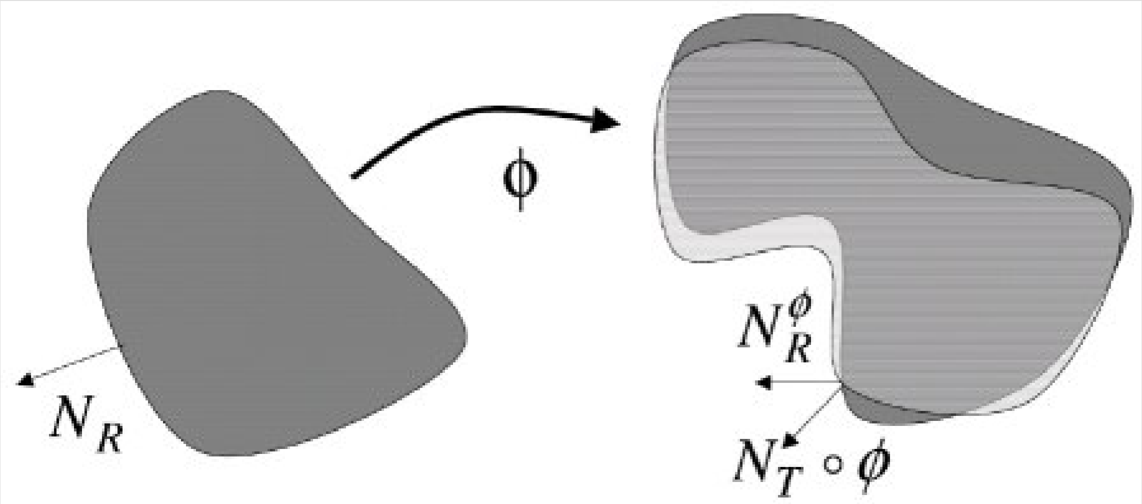
$$\text{Identification } M[I] \leftrightarrow T\mathcal{M}_c^I \quad c \in \mathbb{R}$$

Find $\phi : \Omega \to \Omega$ such that

$$M[T \circ \phi] = M[R]$$

Consider **Gauss map**: $N_I : \Omega \to S^{d-1}$, $x \mapsto \frac{\nabla I(x)}{\|\nabla I(x)\|}$.

Alignment of tangent spaces $\rightsquigarrow$ matching of Gauss maps



Find: $\phi : \Omega \to \Omega$ such that $\boxed{N_T \circ \phi \,||\, N_R^\phi}$

where N_R^ϕ is the transformed normal from R onto $T_{\phi(x)}\phi(\mathcal{M}^R_{R(x)})$ which is given by

$$D\phi u \times D\phi v = \mathbf{Cof}\, D\phi(u \times v) \Rightarrow N_R^\phi = \frac{\mathbf{Cof}\, D\phi N_R}{|\mathbf{Cof}\, D\phi N_R|}$$

where $\mathbf{Cof}\, A = \mathbf{det}\, A \cdot A^{-T}$ for invertible A.

General Framework for a Multimodal Registrationenergy

Define function $g : S^{d-1} \times S^{d-1} \times R^{d,d}$ measuring the deviation of the normals.

$$
\begin{aligned}
E_m[\phi] &:= \int_\Omega g_0(\nabla T \circ \phi, \nabla R, \mathbf{Cof}\, \nabla \phi)\, \mathrm{d}\mu \\
g_0(v,w,A) &= \begin{cases} 0 & ; \quad v = 0 \text{ or } w = 0 \\ g(\frac{v}{|v|}, \frac{w}{|w|}, A) & ; \quad \text{else} \end{cases} \\
\text{e. g.} \qquad g(v,w,A) &= \hat{g}\left((\mathbb{1} - v \otimes v) \frac{Aw}{|Aw|} \right) .
\end{aligned}
$$

where $\hat{g}$ convex.

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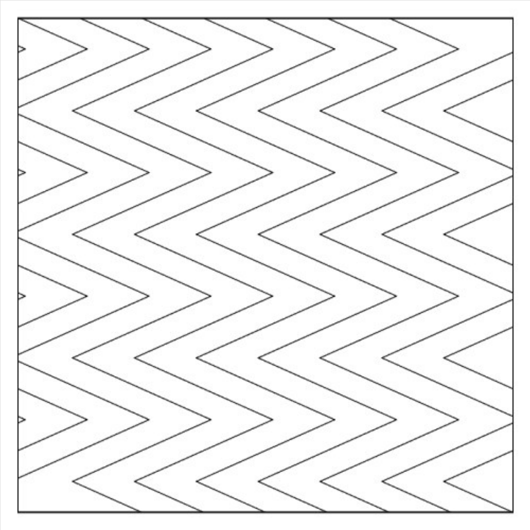
\end{aligned}
$$

where $\hat{g}$ convex. We need convexity in A.

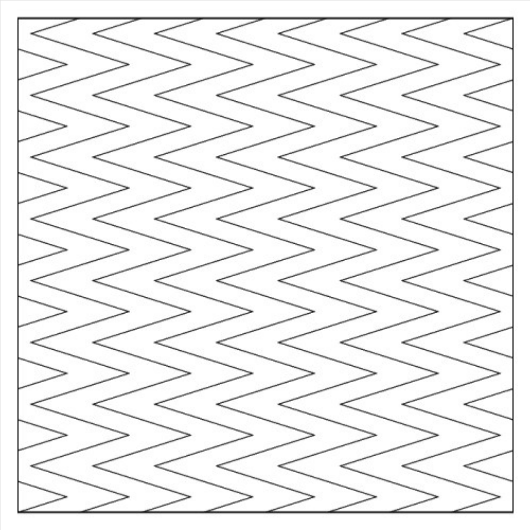
$$
\begin{aligned}
g(u,v,A) &:= \|(\mathbb{1} - v \otimes v) \cdot Aw\|^\gamma \quad 1 \leq \gamma \\
\rightarrow E_m[\phi] &= \int_\Omega \|(\mathbb{1} - (N_T \circ \phi) \otimes (N_T \circ \phi)) \cdot \mathbf{Cof}\, D\phi N_R\|^\gamma
\end{aligned}
$$



Microstructures

$$N_T \equiv e_1 \quad N_R \equiv e_2$$

$$\phi_\epsilon(x) = 1\!\!\mathrm{I} + \epsilon^\alpha \psi\left(\frac{x_2}{\epsilon}\right) e_1 \qquad 1 < \alpha < 2$$



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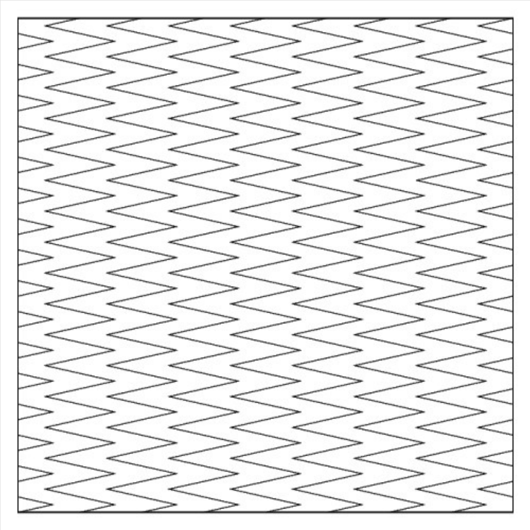

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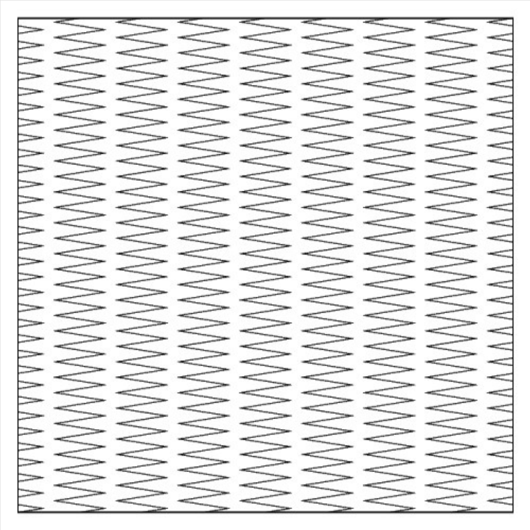

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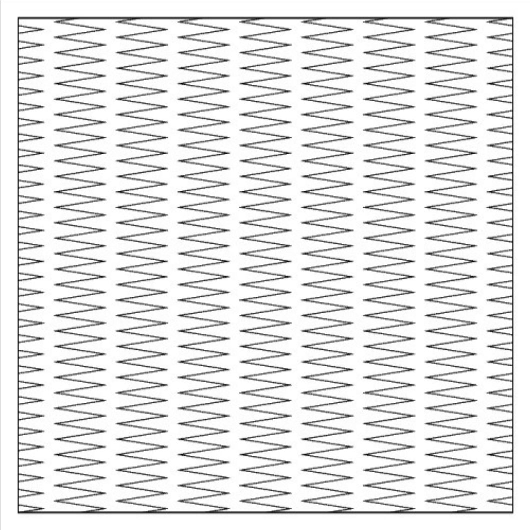
$$\phi_\epsilon \rightharpoonup 1\!\mathrm{I} \text{ in } H^{1,2}$$

but

$$E[1\!\mathrm{I}] = |\Omega| g_0(e_1, e_2, 1\!\mathrm{I}) > 0 = \liminf_{\epsilon \to 0+} E[\phi_\epsilon]$$

Thus, we don't have existence without regularization of the matching energy.



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and

The simpler the images, the higher the degree of ill–posedness.

Outline

- Unimodal Registration
- Gradient flow perspective
- Relations to Tikhonov-Regularization
- Multimodal Registration
- **Hyperelastic polyconvex Regularization**
- Practical issues for solving the minimization problem

Hyperelastic polyconvex Regularization

$$E[\phi] := E_m[\phi] + E_{reg}[\phi] \to \min!$$

Consider an elastic *polyconvex* regularizationenergy, i.e.,

$$E_{reg}[\phi] = \int \hat{W}(D\phi) = \int W^*(D\phi, \mathbf{Cof}\, D\phi, \mathbf{det}\, D\phi)$$

and W^* convex.

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and W^* convex.

E. g.: *Mooney-Rivlin-Energy* for elastic compressible materials.

$$E_{MR}[\phi] := \int_\Omega \alpha \|D\phi\|^2 + \beta \|\mathbf{Cof}\, D\phi\|^2 + \Gamma(\mathbf{det}\, D\phi) dx$$

with $\Gamma(s) \rightarrow \infty$ for $s \rightarrow 0, \infty$, **corresponding to the** *a priori* **information, that deformation must be a homeomorphism.**

Space of Images

Set of degenerate points $\mathcal{D}_I := \{x \in \Omega \,|\, \nabla I = 0\}$

We suppose that for the Lebesgue measure

$$\mu(B_\epsilon(\mathcal{D}_I)) \stackrel{\epsilon \longrightarrow 0}{\longrightarrow} 0$$

and the corresponding space of images

$$\mathcal{I}(\Omega) \;\; := \;\; \Big\{ I : \Omega \to \mathbb{R} \,\Big|\, I \in C^1(\bar{\Omega}), \exists\, \mathcal{D}_I \subset \Omega \text{ s. t. } \nabla I \neq 0 \text{ on } \Omega \setminus \mathcal{D}_I, \\ \mu(B_\epsilon(\mathcal{D}_I)) \stackrel{\epsilon \longrightarrow 0}{\longrightarrow} 0 \Big\} \; .$$

Theorem 1. [Existence in three dimensions] *, $T, R \in \mathcal{I}(\Omega)$, admissible deformations*

$$\mathcal{A} \;:=\; \{\phi : \Omega \to \Omega \,\big|\, \phi \in H^{1,p}(\Omega), \mathbf{Cof}\, D\phi \in L^q(\Omega),$$
$$\mathbf{det}\, D\phi \in L^r(\Omega), \mathbf{det}\, D\phi > 0 \text{ a.e. in } \Omega, \phi = 1\!\!\mathrm{I} \text{ on } \partial\Omega\}$$

where p, $q > 3$ and $r > 1$. W be polyconvex, $\exists!\ \beta, s \in \mathbb{R}$, $\beta > 0$, and $s > \frac{2q}{q-3}$ such that

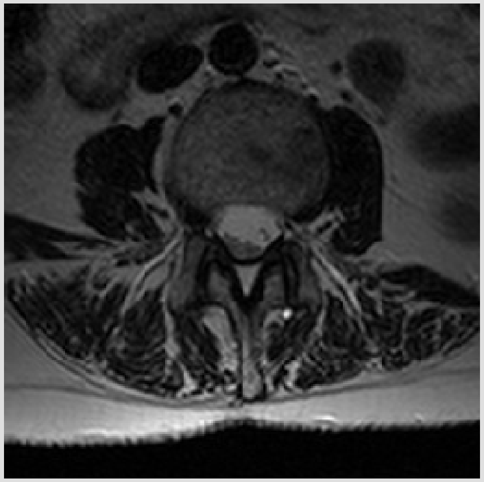
$$W(A, C, D) \geq \beta \left(\|A\|_2^p + \|C\|_2^q + D^r + D^{-s}\right) \quad \forall A, C \in \mathbb{R}^{3,3},\ D \in \mathbb{R}^+ \tag{1}$$

$g_0(v, w, A) = g(\frac{v}{|v|}, \frac{w}{|w|}, A)$ be continuous in $\frac{v}{|v|}$, $\frac{w}{|w|}$, convex in A and for $m < q$

$$g(v, w, A) - g(u, w, A) \leq C_g \,\|v - u\|\, (1 + \|A\|_2^m).$$

holds. $\boxed{\Rightarrow \exists \text{ homeomorphism } \phi \in \mathcal{A}.}$

Boundary conditions



Dirichlet cond. $\phi = \mathbb{1}$ on $\partial\Omega$ is not always approriate.
We have to allow $\phi(\Omega) \not\subset \Omega$, but integrand is only defined on $\phi^{-1}(\mathrm{Im}(\phi) \cap \Omega)$.

$\Rightarrow$ define energy on $\Omega^\phi := \{x \in \Omega \,|\, \phi(x) \in \Omega\}$ and consider the energy $\tilde{E}_m[\phi] := \int_{\Omega^\phi} g_0(\nabla T \circ \phi, \nabla R, \mathbf{Cof}\, D\phi)\,\mathrm{d}\mu$.

But then

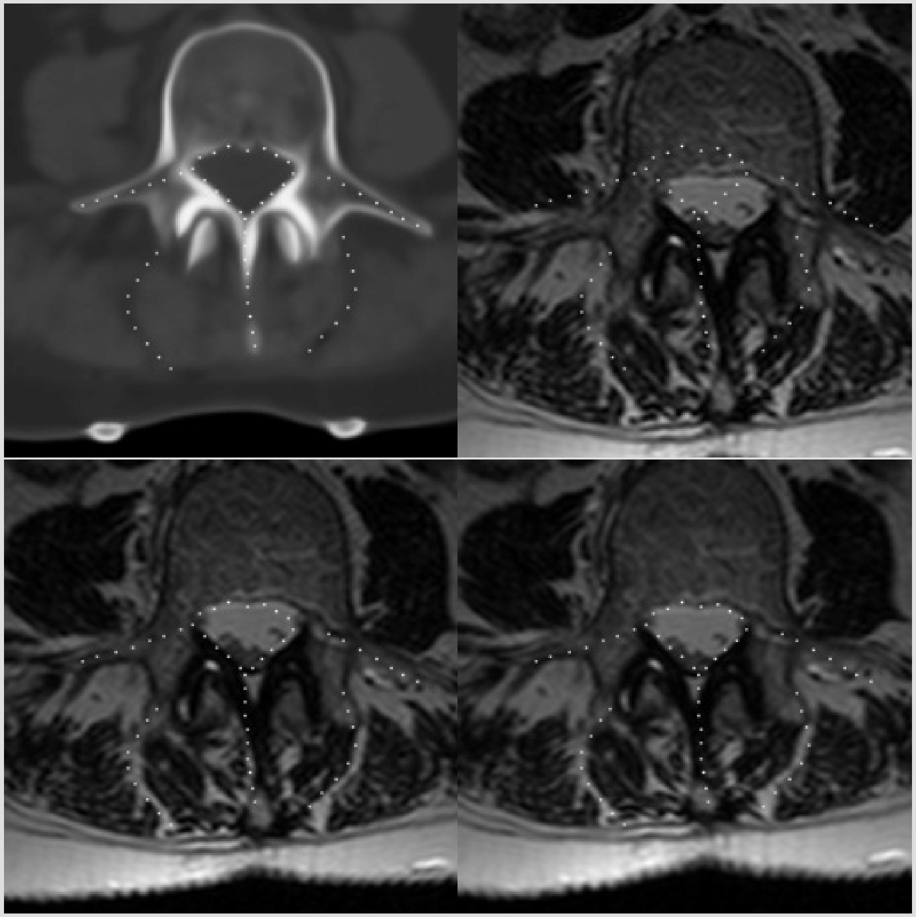
$$\tilde{E}_m[\phi] = 0 \text{ for all } \phi \text{ s.t. } \phi(\Omega) \cap \Omega = \emptyset.$$

To avoid the problem, minimize

$$\tilde{E}_m + E_{\mathrm{reg}} + \int_\Omega |d(\phi(\cdot), \mathcal{F}_T) - d(\cdot, \mathcal{F}_R)|^2\,\mathrm{d}\mu,$$

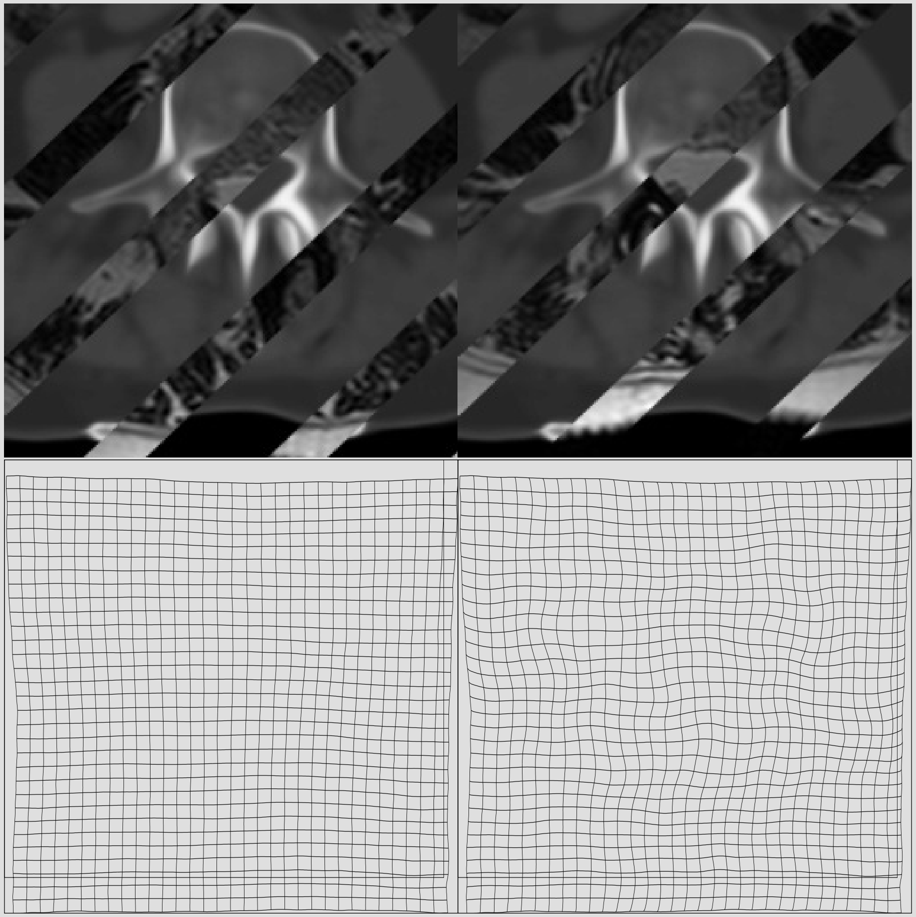
where $\mathcal{F}_T$, $\mathcal{F}_R$ are corresponding selected features of T and R.
$d(x, A) := \hat{d} \circ \mathrm{dist}(x, A)$.

Spine Registration CT-MR



Top Left: reference R, CT, Top Right: template T, MR. Bottom Left: after feature based registration $T \circ \phi_f$. Bottom Right: final deformed template $T \circ \phi$.

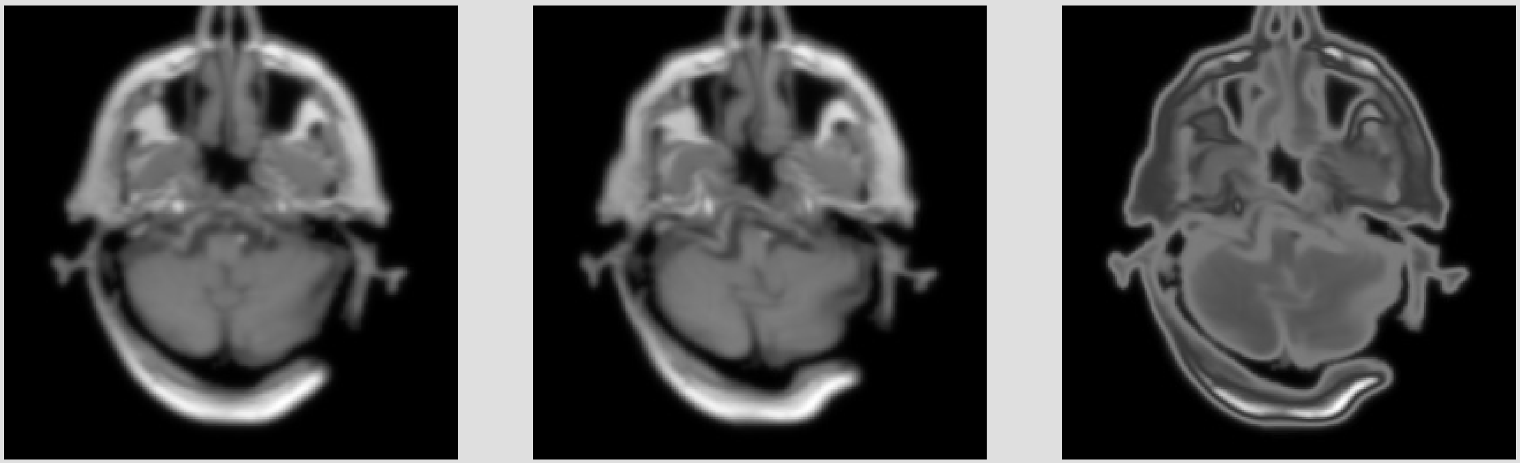
Spine Registration CT-MR



Comparison of superimposed template and reference before (left) and after (right) registration and deformation after the preregistration (feature-based) against final deformation.

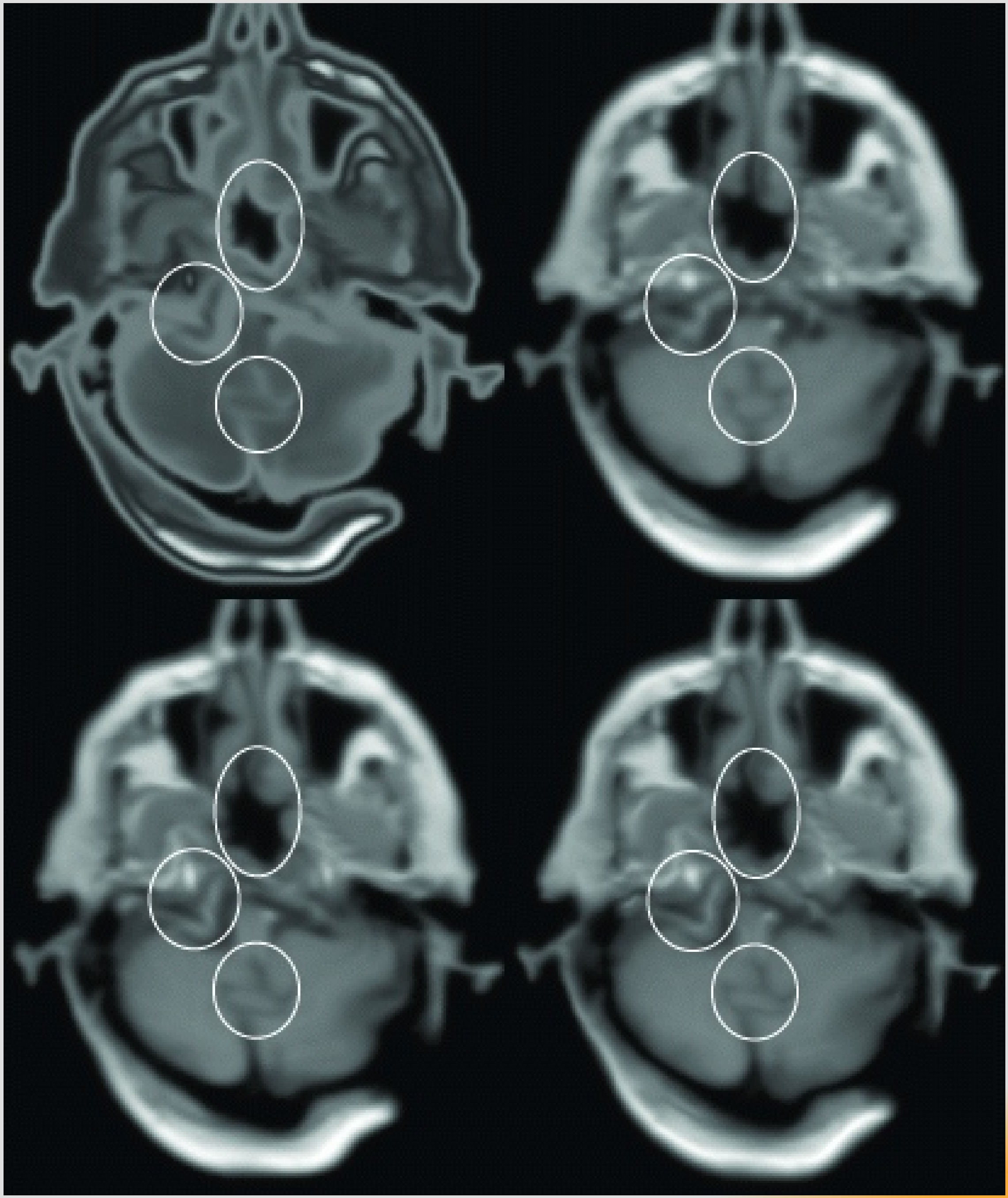
Validation

Generation of pair of test images.

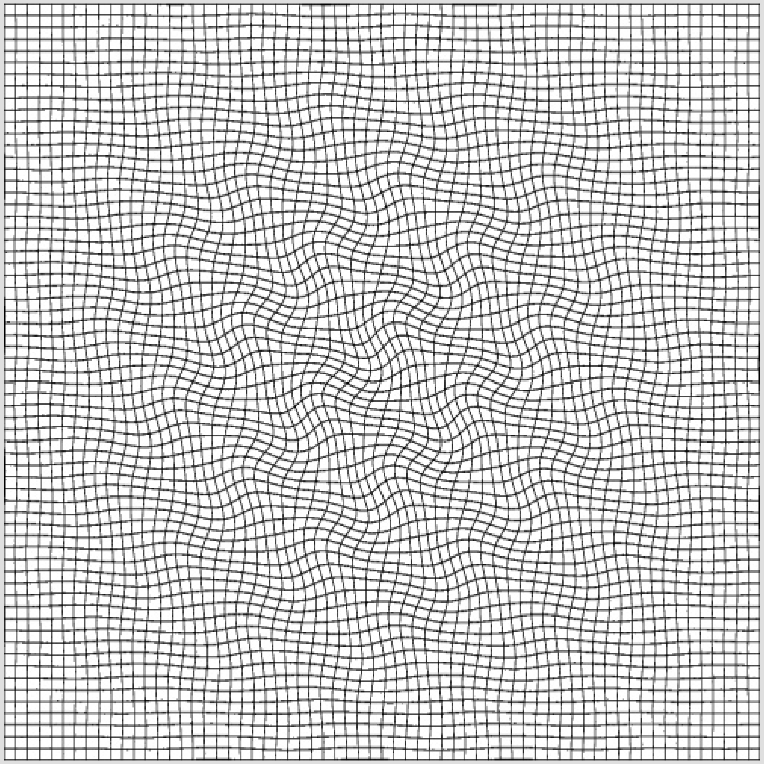


original T deformed $T \circ \phi^*$ contrast change $g \circ T \circ \phi^*$

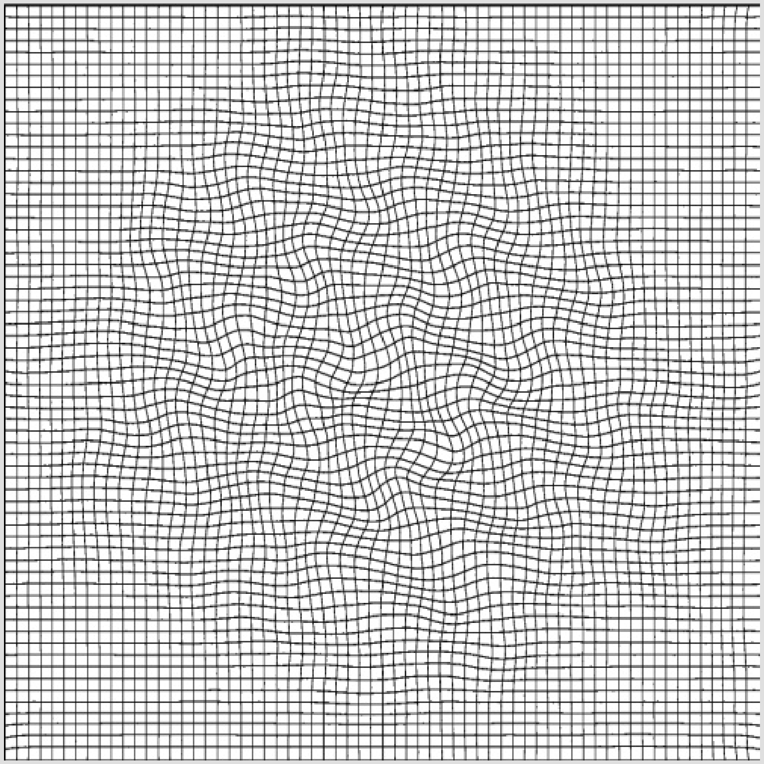
Use $R := g \circ T \circ \phi^*$ for computation and compare solution ϕ to ϕ^*



Comparison of deformation

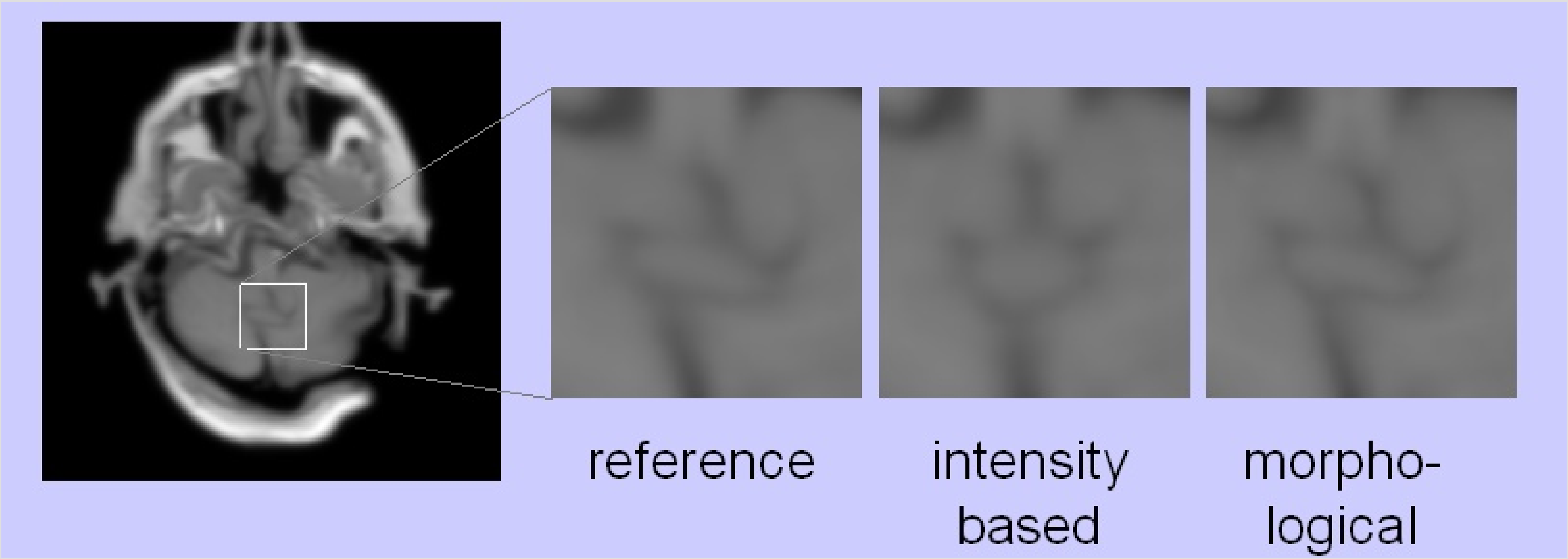


exact solution ϕ^*



computed solution ϕ

Comparison to unimodal Registration



In regions of *low contrast* but *high geometric variability*, the morphological registration can be superior to the unimodal registration, even if only one modality is considered.

Outline

- Unimodal Registration
- Gradient flow perspective
- Relations to Tikhonov-Regularization
- Multimodal Registration
- Hyperelastic polyconvex Regularization
- **Practical issues for solving the minimization problem**

Multiscale minimization approach

Energy E_m in general possesses many local minima for given images T and R.

Strategy: Consider a continuous scale of images T^σ, R^σ, generated by a **scale-space-operator** $\mathcal{S}(\sigma)$, where $\sigma \geq 0$ denotes the **scale-parameter**.

$$I^\sigma = S(\sigma)I \quad \text{e. g. } \mathcal{S}(\sigma) = HESG(\frac{\sigma^2}{2}), \mathcal{S}(\sigma) = MCM(\sigma)$$

$$E_m^\sigma[\phi] \; := \; \frac{1}{2}\int |T^\sigma \circ \phi - R^\sigma|^2 \,\mathrm{d}\mu \text{ resp. } \int g_0(\nabla T^\sigma, \nabla R^\sigma, \mathbf{Cof}\, D\phi)\,\mathrm{d}\mu$$

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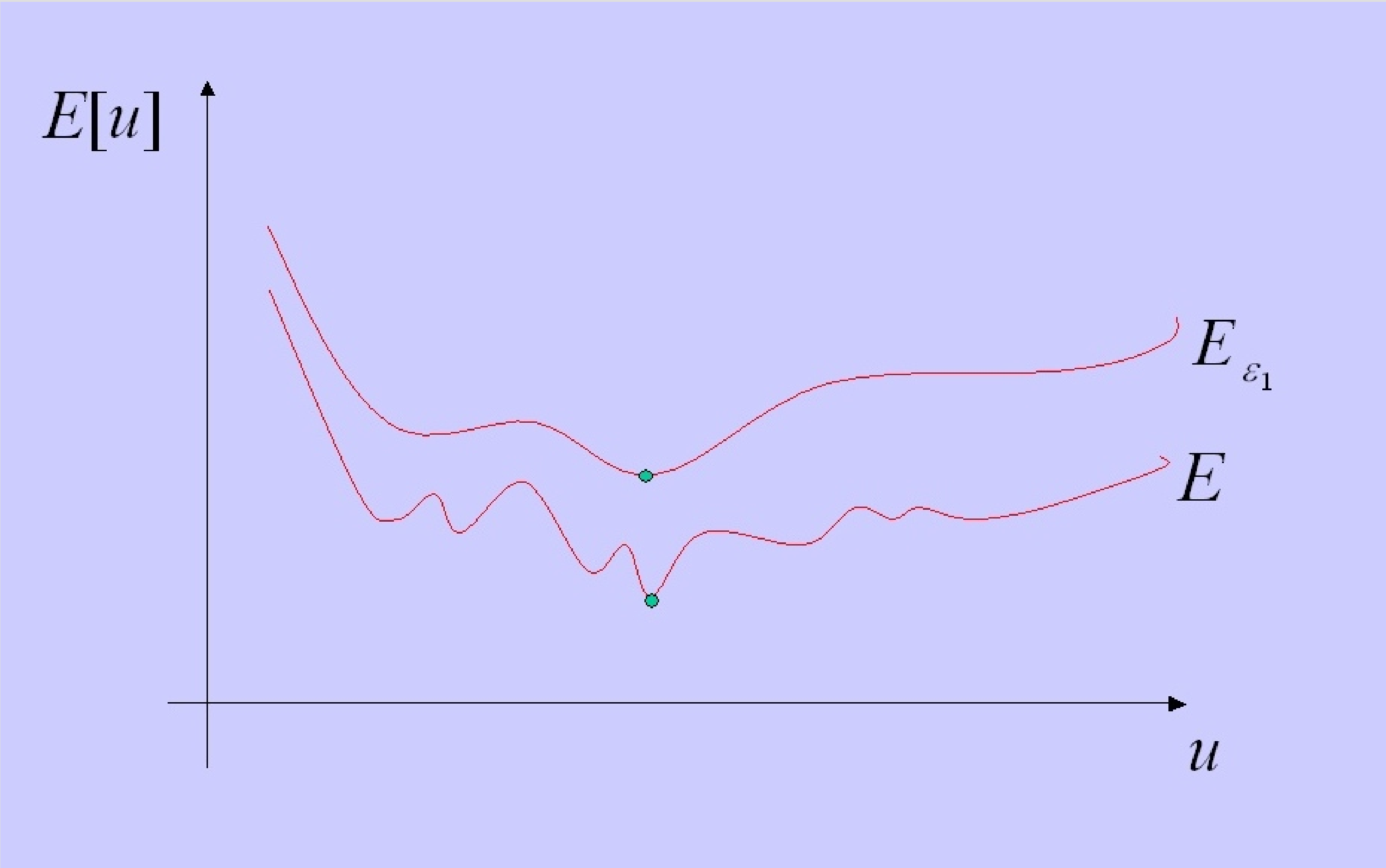
g(\partial_t \phi_\sigma, \psi) &= -\langle {E_m^\sigma}'[\phi_\sigma], \psi\rangle \\
\phi_\sigma(0) &= \phi_{0,\sigma} \\
\sigma_k &= \beta_1 2^{-\beta_2 k} \qquad \beta_1, \beta_2 > 0 \\
\phi_{\sigma_k}(0) &= \phi_{\sigma_{k-1}}(T_k)
\end{aligned}$$

Energylandscapes

Energylandscapes



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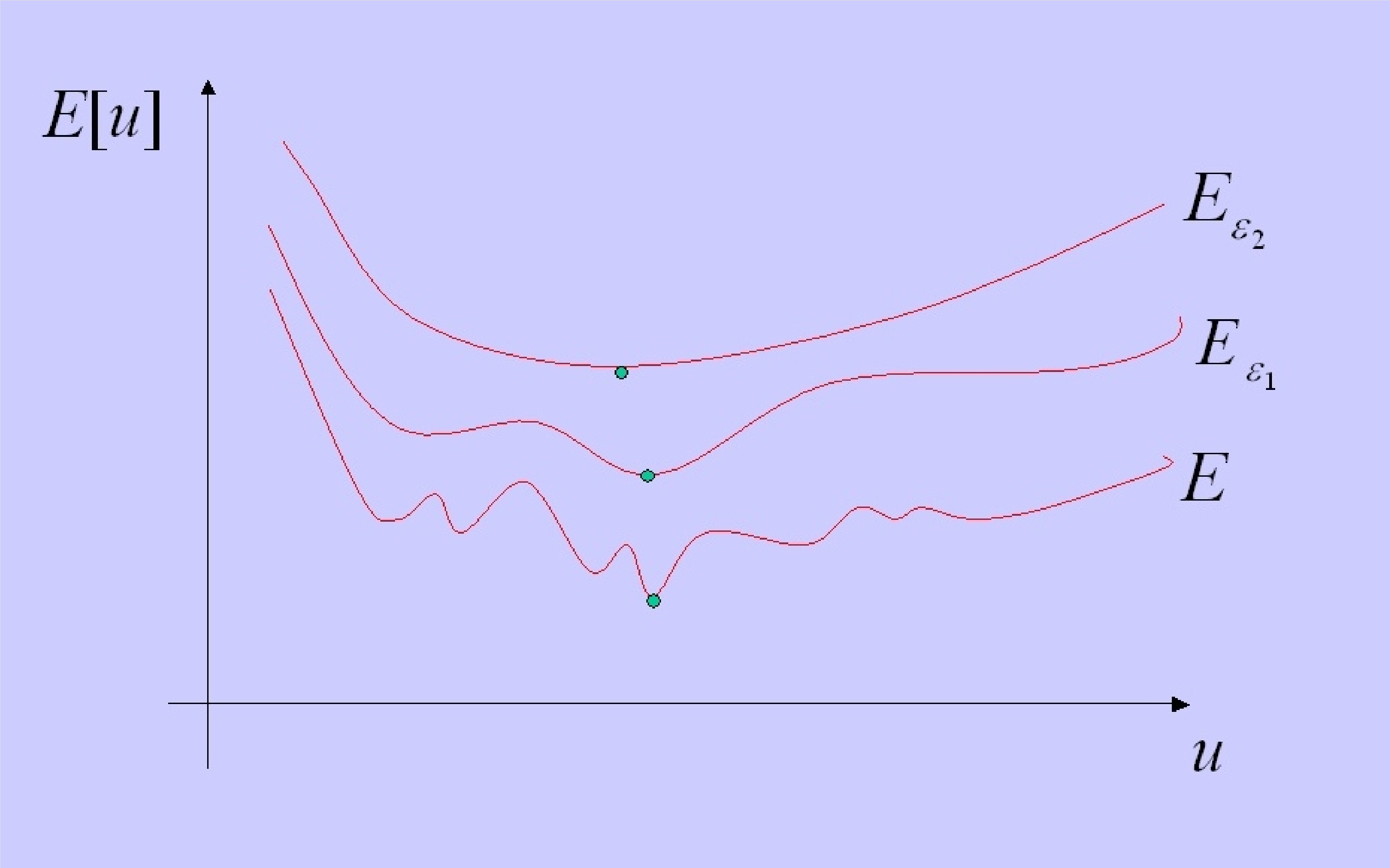

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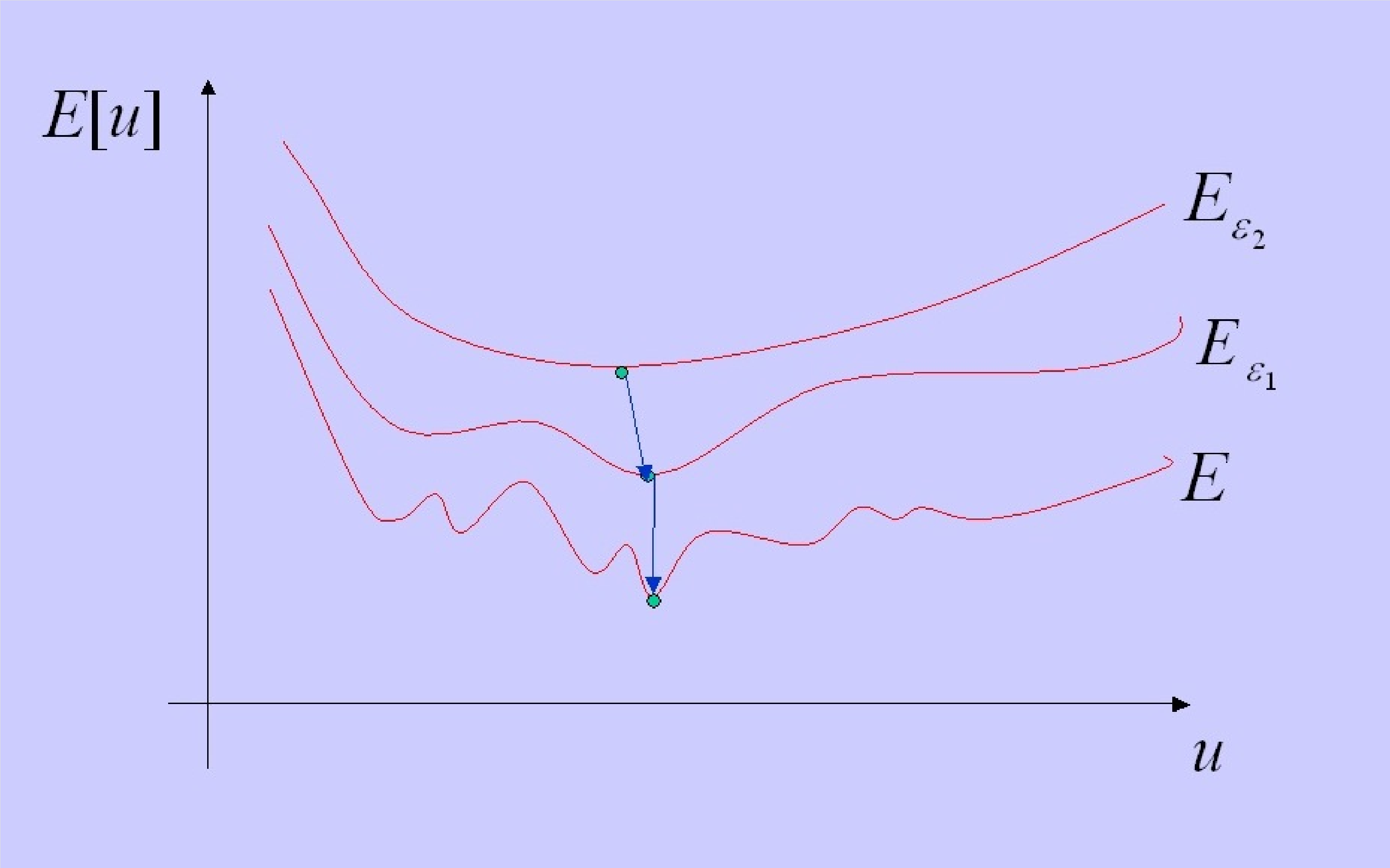

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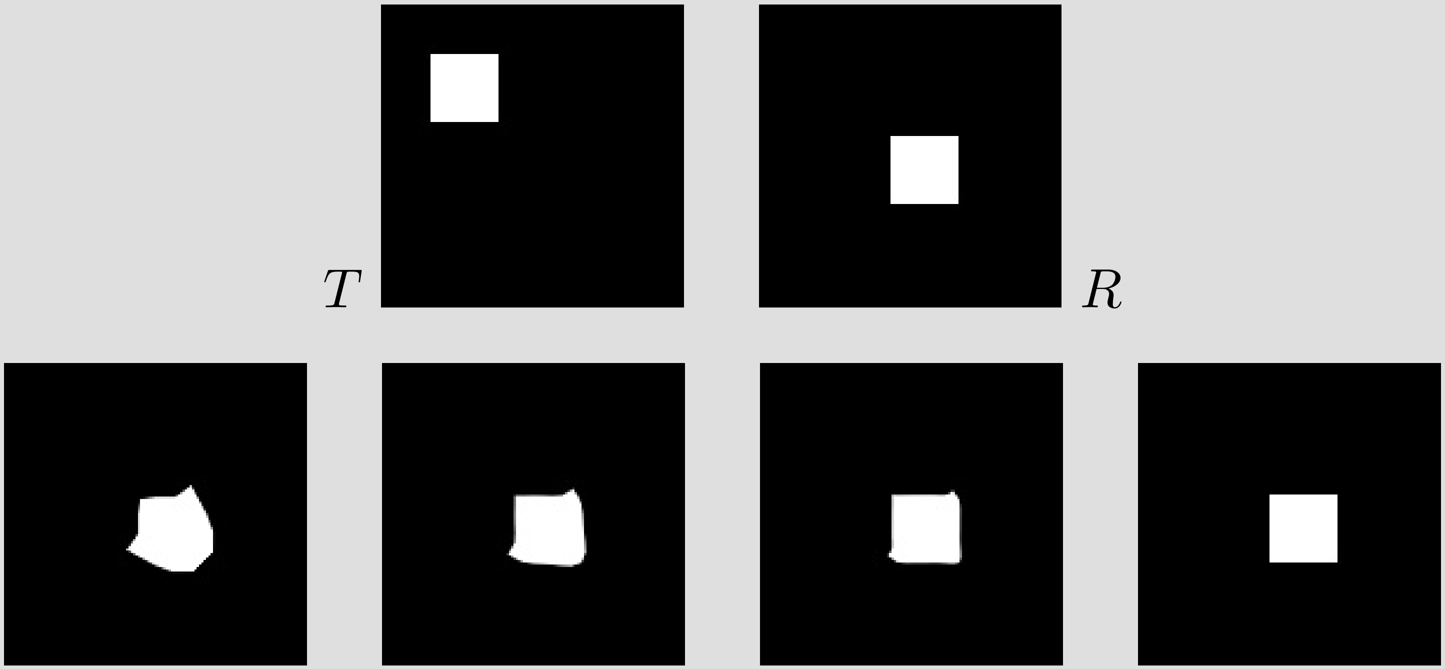
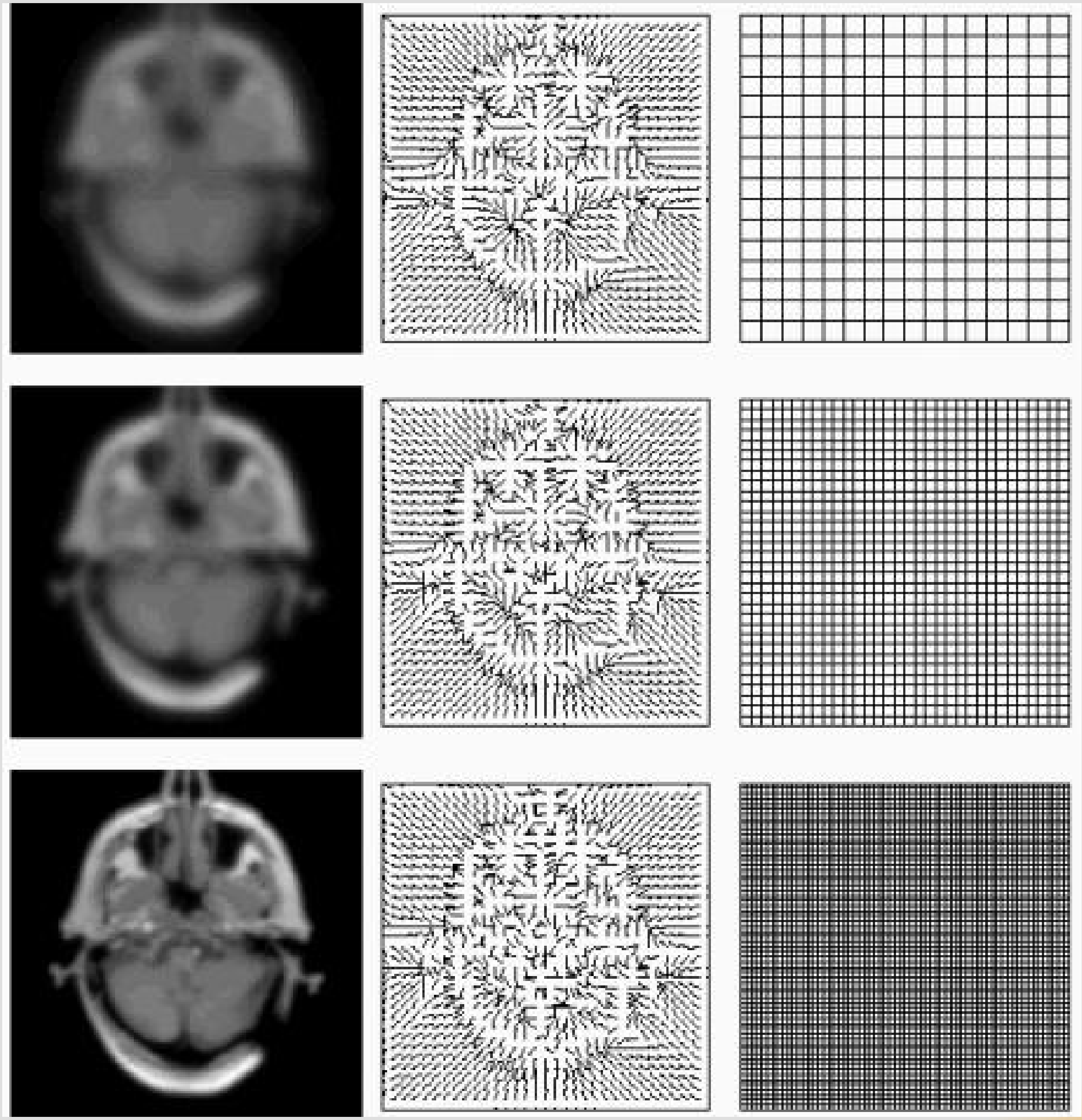
Refinement of the solution w.r.t. to scale



scale 1 scale 2 scale 3 scale 4



Coupling Scale and Resolution

Multiscale onf images: $T^{\sigma_k} = \mathcal{S}(\sigma_k)T$, $R^{\sigma_k} = \mathcal{S}(\sigma_k)R$ given by the scale-operator $\mathcal{S}(\sigma_k)$ and $\sigma_k \to 0$ for $k \to \infty$.

Solve the problem on a **multilevel hierarchy** $(\mathcal{M}_{h_l}, \mathcal{V}_{h_l})$ of grids, with $\mathcal{M}_{h_{l_{\max}}} \subset \ldots \subset \mathcal{M}_{h_{l+1}} \subset \mathcal{M}_{h_l} \subset \mathcal{M}_{h_{l-1}} \subset \ldots \mathcal{M}_{h_{l_0}}$ and corresponding discrete functionspaces $\mathcal{V}^l$.

During step k with assigned scaleparameter σ_k find minimal $l(k) \in \mathbb{N}$, such that

$$h_{l(k)} \leq \alpha\sigma_k$$

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During step k with assigned scaleparameter σ_k find minimal $l(k) \in \mathbb{N}$, such that

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change the scale $k \to k+1$, if

$$\|\phi_k^{n+1} - \phi_k^n\| \leq \gamma \sigma_k$$

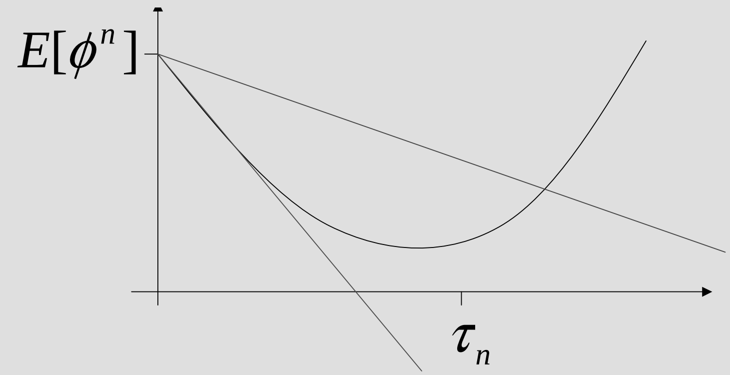
Multigrid for regularized metric and scales

Consider $HESG$ (heat equation semi-group)

$$\begin{aligned} g(u,v) &= (u,v)_{L^2} + \frac{\sigma^2}{2}(\nabla u, \nabla v)_{L^2} \\ \rightarrow \partial_t \phi &= -\left(\mathbb{1} - \frac{\sigma^2}{2}\Delta\right)^{-1} E'_m[\phi] \\ \text{Furthermore} \quad I^\sigma &= \left(\mathbb{1} - \frac{\sigma^2}{2}\Delta\right)^{-1} I \end{aligned}$$

Solve $(\mathbb{1} - \frac{\sigma^2}{2}\Delta)^{-1}$ by multigrd V-cycle. Smoothing properties $\longrightarrow$ robustness of gradient flow.

Step size control

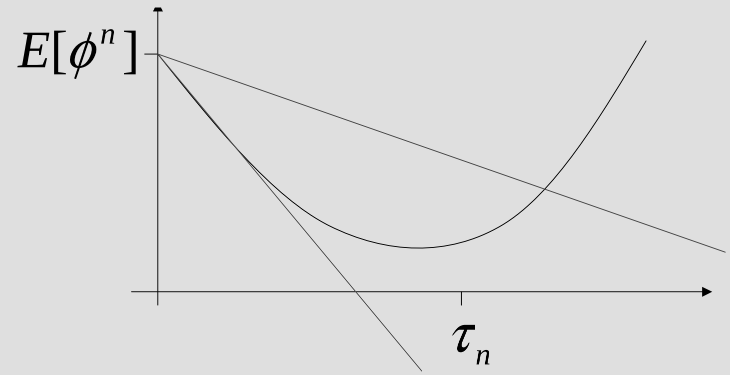


Gradient flow perspective with metric g
allows to apply well-known step-control criteria.

E. g. *Armijo's Rule:* $\sigma \in (0,1)$

$$\begin{aligned} E[\phi^{(k+1)}] - E[\phi^{(k)}] &\leq -\sigma\tau\langle E'[\phi^{(k)}], A^{-1}E'[\phi^{(k)}]\rangle \\ &= -\sigma\tau\|A^{-1}E'[\phi^{(k)}]\|_g^2 \end{aligned}$$

Step size control



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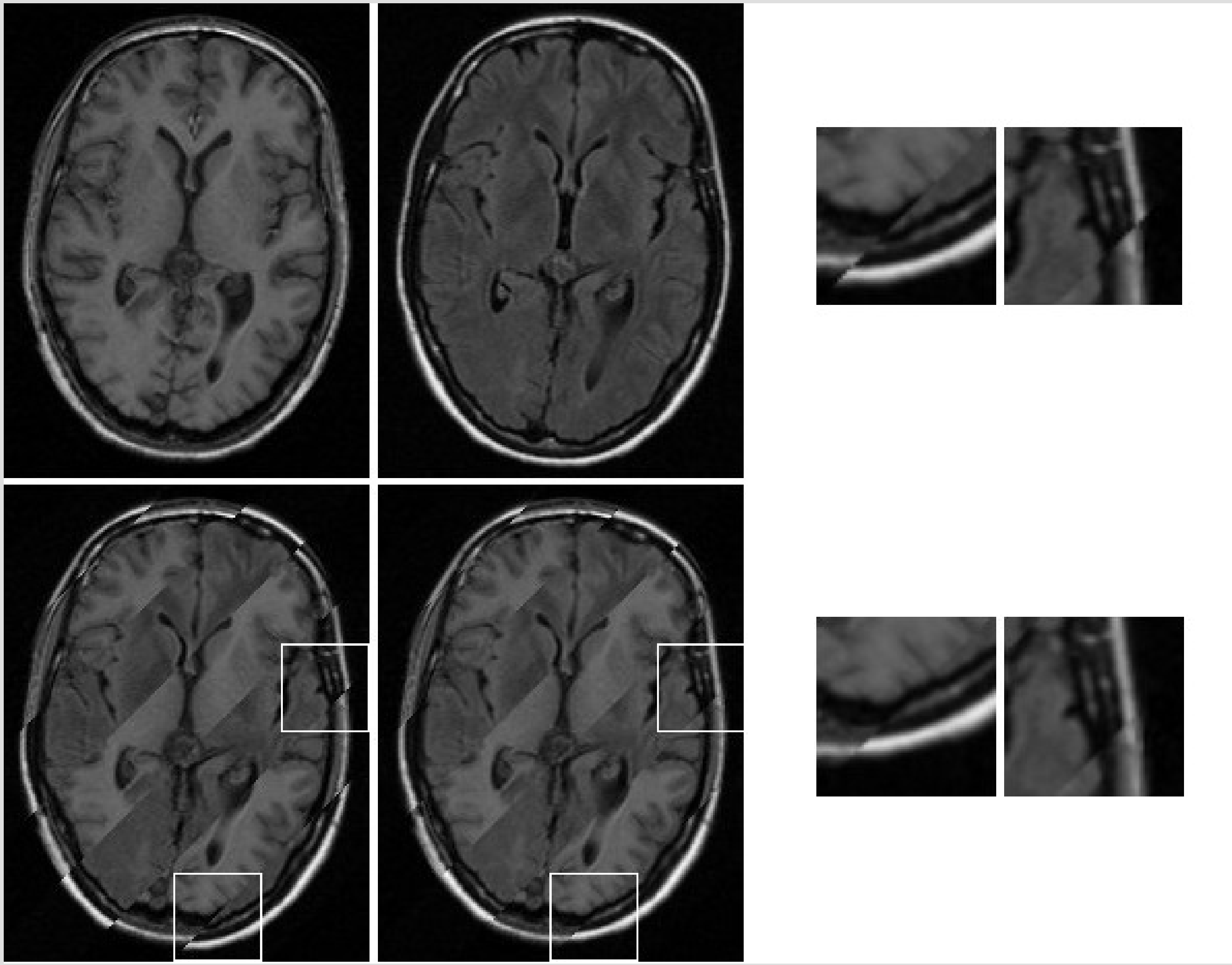
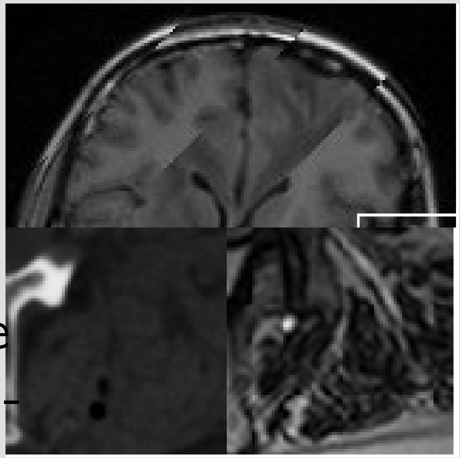
Given $\beta \in (0,1)$ (typically $\beta = \frac{1}{2}$), find minimal $k \in \mathbb{Z}$, such that

$$E[\phi^{(k)} - \beta^k A^{-1}E'[\phi^{(k)}]] - E[\phi^{(k)}] \leq \sigma\beta^k\|A^{-1}E'[\phi^{(k)}]\|_g^2$$

Other linesearch criteria easily incorporated in the regularized gradient flow framework.

Morphological Registration: MRT T1 $\leftrightarrow$ FLAIR





Issues & Outlook

- Registration may stop, when edges are only parallel (i. e. tangent spaces coincide) but *not correctly aligned*. Classical approaches using *gradient magnitudes* may not be used without leaving the morphological framework. Registration may aim at additionally align *discontinuitiy sets*.

$$u \in SBV(\Omega) \Rightarrow Du = D^{ac}u + D^{j}u = \nabla u \mu + (u^{+} - u^{-})\nu_u \mathcal{H}^{n-1} \llcorner S(u)$$

  Let integrand g_0 vanish where $Du = 0$ and use normals ν_I of essential boundary $\partial^\star\{I < t\}$ as N_I to generalize the theorem.

  Application of Γ-convergence for a PDE-based approach to compute the jump set $S(u)$.

- In practice, multimodal pairs of images are obviously *not morphologically* equivalent.

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Implementation in DX9 Graphics Hardware by *R. Strzodka, caesar, Germany*

- Discrete scheme of the gradient descent is well suited for parallel computing.
  - **Many identical computations** $\rightsquigarrow$ good for SIMD (Single Instruction Multiple Data).
  - **Requires only local data** $\rightsquigarrow$ good for distributed memory.

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- **Why use graphics hardware?**
  - Outstanding price-performance ratio
  - Readily available in any modern PC
  - Performance doubles in less than 9 months $\rightsquigarrow$ Moore's Law squared

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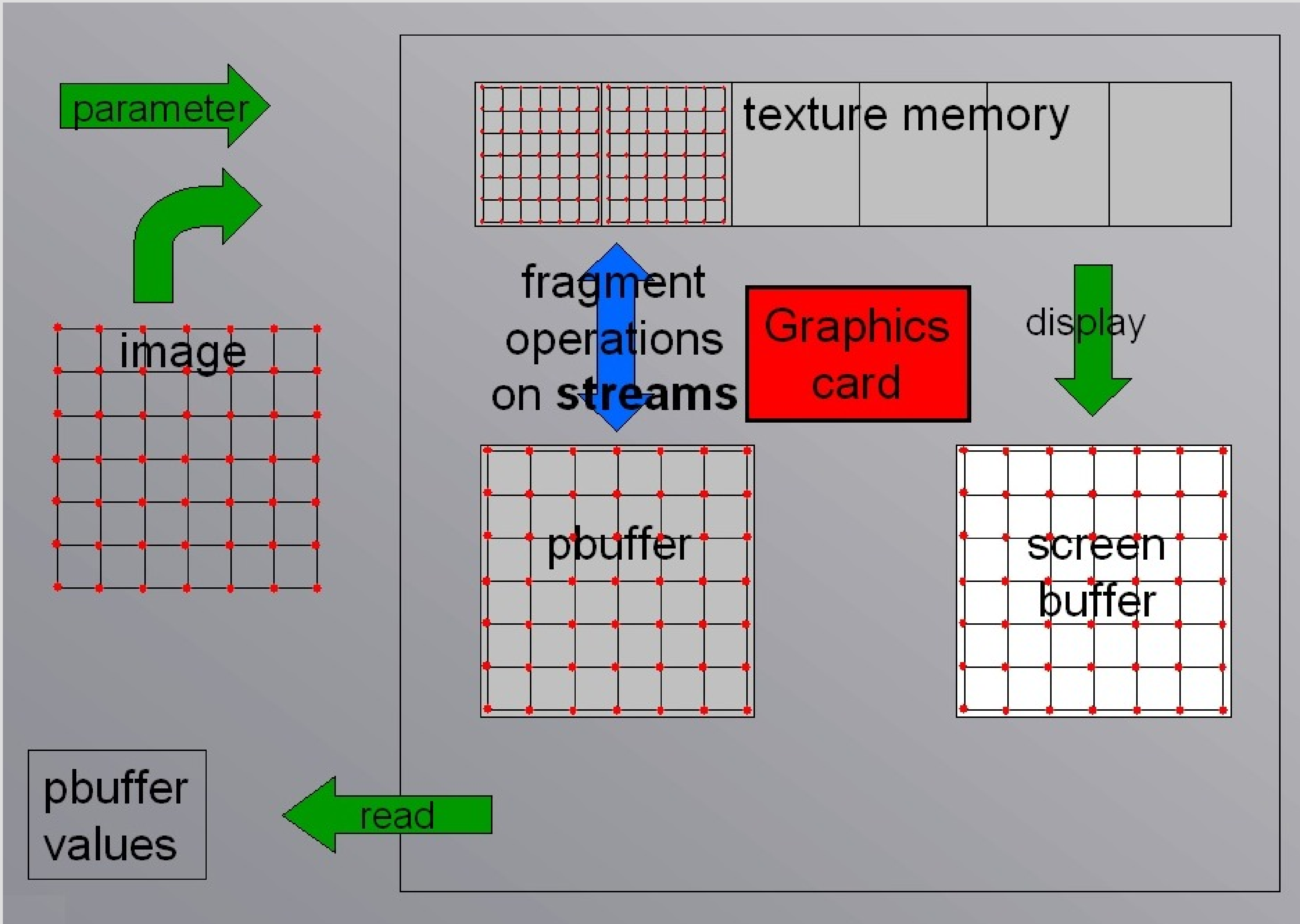
- **Constraints**
  - ⋆ Process data in large streams, i.e. avoid frequent changes in the graphics pipeline
  - ⋆ Minimize data transfer between main memory and graphics card



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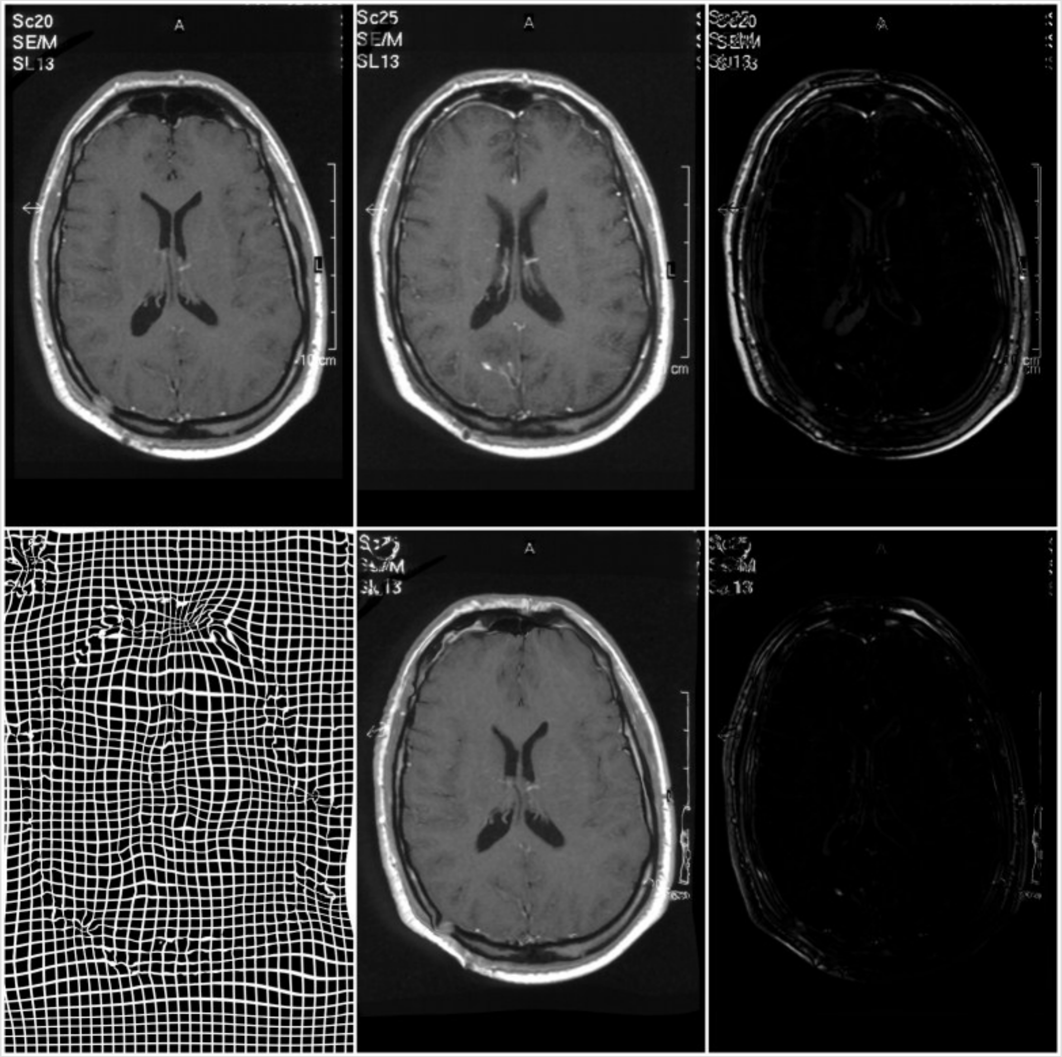

513×769 images registered in 5.9 seconds $\rightsquigarrow$ ca. saving of factor 10

End

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- M. Rumpf, *On the matching of images with edge discontinuities*, in prep.
- MD, R. Strzodka, M. Rumpf, *Image registration by a regularized gradient flow, A streaming Implementation in DX9 Graphics Hardware*, Computing, submitted.