Scattering by an Infinite Cylinder

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1. A Perfect Conductor $(E = (0, 0, u))$

$$\Delta_2 u + k^2 u = 0 \quad \text{in } R^2 \setminus \bar{D}$$

$$\begin{aligned} u &= u^i + u^s \\ u &= 0 \quad \text{on } \partial D \end{aligned}$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0$$

2. An Inhomogeneous Medium

$$\Delta_2 u + k^2 n(x) u = 0 \quad \text{in } R^2$$

$$u = u^i + u^s$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

Assume $u^i(x) = e^{ikx\cdot d}$, $n(x) = 1$ in $R^2 \setminus \bar{D}$, $Im\, n(x) \geq 0$ for $x \in D$, $Re\, n(x) > 0$, $k > 0$, $r = |x|$.

Scattering by an Infinite Cylinder

Let
$$\Phi(x,y) = \frac{i}{4} H_0^{(1)}(k|x-y|) \quad , x \neq y.$$
Then for $x \in R^2 \setminus \bar{D}$, Green's theorem implies that
$$u^s(x) = \int\limits_{\partial D} \left(u^s(y) \frac{\partial}{\partial \nu(y)} \Phi(x,y) - \frac{\partial u^s}{\partial \nu}(y) \Phi(x,y) \right) ds(y)$$
This is known as Green's representation formula.

Theorem: In $R^2 \setminus \bar{D}$, $u^s(x)$ is a real-analytic function of its independent variables.

Rellich's Lemma: Let $u \in C^2(R^2 \setminus \bar{D})$ be a solution of the Helmholtz equation satisfying
$$\lim_{R \to \infty} \int\limits_{|y|=R} |u(y)|^2 ds(y) = 0.$$
Then $u = 0$ in $R^2 \setminus \bar{D}$.

Definition: The condition
$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0$$
is called the Sommerfeld radiation condition.

The Perfect Conductor: Uniqueness and Existence

Uniqueness Theorem: Let $u^s \in C^2(R^2 \setminus \bar{D}) \cap C(R^2 \setminus D)$ be a solution of the Helmholtz equation in $R^2 \setminus \bar{D}$ satisfying the Sommerfeld radiation condition and $u^s = 0$ on ∂D. Then $u^s = 0$ in $R^2 \setminus D$.

Proof: Let B be a disk centered at the origin such that $B \supset D$. Then by Green's theorem

$$\int_{\partial B} \left(\bar{u}^s \frac{\partial u^s}{\partial r} - u^s \frac{\partial \bar{u}^s}{\partial r} \right) ds = 0. \tag{1}$$

The Perfect Conductor: Uniqueness and Existence

But for $x \in R^2 \setminus B$,

$$\begin{aligned} u^s(r,\theta) &= \sum_{-\infty}^{\infty} a_n(r) e^{in\theta} \qquad (2) \\ a_n(r) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} u^s(r,\theta) e^{-in\theta} d\theta \end{aligned}$$

$\Rightarrow$

$$a_n(r) = a_n H_n^{(1)}(kr).$$

Using the Wronskian relation for Hankel functions, 1), 2)$\Rightarrow$

$$\sum_{-\infty}^{\infty} |a_n|^2 = 0$$

$\Rightarrow u^s(x) = 0$ for $x \in R^2 \setminus B$ and, by analyticity, $u^s(x) = 0$ for $x \in R^2 \setminus \bar{D}$.

The Perfect Conductor: Uniqueness and Existence

We now try to construct a solution to the direct scattering problem for a perfect conductor. We first look for a solution in the form of a double layer potential

$$u^s(x) = \int\limits_{\partial D} \varphi(y) \frac{\partial}{\partial \nu(y)} \Phi(x,y) ds(y).$$

However, this approach fails if k^2 is an eigenvalue of the interior Dirchlet problem for the Laplacian in D! Hence, we look for a solution int he form of a modified double layer potential

$$u^s(x) = \int\limits_{\partial D} \varphi(y) \left\{ \frac{\partial \Phi(x,y)}{\partial \nu(y)} - i\eta \Phi(x,y) \right\} ds(y)$$

where $\varphi \in C(\partial D)$ and $\eta \neq 0$. u^s will be a solution of the scattering problem if

$$\varphi + 2 \int\limits_{\partial D} \varphi(y) \left\{ \frac{\partial \Phi(x,y)}{\partial \nu(y)} - i\eta \Phi(x,y) \right\} ds(y) = -2e^{ikx \cdot d}$$

It can be show using the Fredholm alternative that there exists a unique solution to this integral equation. Hence existence of a solution to the scattering problem for a perfect conductor has been established.

The Inhomogeneous Medium: Uniqueness and Existence

In this case uniqueness (and hence, by the Fredholm alternative, existence) is based on the following theorem:

Unique Continuation Principle: Let G be a domain in R^2 and suppose $u \in C^2(G)$ is a solution of

$$\Delta_2 u + k^2 n(x) u = 0$$

in G such that $n \in C(\bar{G})$ and u vanishes in a neighborhood of some $x_0 \in G$. Then u is identically zero in G.

Uniqueness Theorem: Let $u \in C^2(R^2)$ satisfy

$$\Delta_2 u + k^2 n(x) u = 0 \quad \text{in } R^2$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u}{\partial r} - iku \right) = 0$$

Then $u = 0$ in R^2.

Proof: Green's theorem implies that for $D \subset \{x : |x| < a\}$

$$\int\limits_{|x|=a} u \frac{\partial \bar{u}}{\partial \nu} ds = \int\limits_{|x| \leq a} \left\{ |\text{grad } u|^2 - k^2 \bar{n} |u|^2 \right\} dx$$

and hence

$$Im \int\limits_{|x|=a} u \frac{\partial \bar{u}}{\partial r} ds = k^2 \int\limits_{|x|<a} Im\ n |u|^2 dx \geq 0. \qquad (1)$$

The Inhomogeneous Medium: Uniqueness and Existence

But

$$Im \int_{|x|=a} u\frac{\partial\bar{u}}{\partial r}ds = \frac{1}{2i}\int\limits_{|x|=a}\left(u\frac{\partial\bar{u}}{\partial r} - \bar{u}\frac{\partial u}{\partial r}\right)ds$$

and the Wronskian relation for Hankel functions implies that

$$Im \int\limits_{|x|=a} u\frac{\partial\bar{u}}{\partial r}ds < 0$$

unless $u = 0$ for $|x| \geq a$. Hence, from 1) $u = 0$ for $|x| \geq a$ and the theorem follows by the unique continuation principle.

The direct scattering problem for an inhomogeneous medium is easily seen to be equivalent to the problem of solving the Lippmann Schwinger equation

$$u(x) = u^i(x) - k^2 \int\limits_{R^2} \Phi(x,y)m(y)u(y)dy, \; x \in R^2$$

where $m := 1 - n$. The above uniqueness theorem and the Fredholm alternative now imply the existence of a unique solution to the direct scattering problem for an inhomogeneous medium.

Far Field Patterns

Recall that for both a perfect conductor and an inhomogeneous medium,

$$u^s(x) = \int\limits_{\partial D} \left(u^s(y) \frac{\partial}{\partial \nu(y)} \Phi(x,y) - \frac{\partial u^s}{\partial \nu}(y) \Phi(x,y) \right) ds(y).$$

Letting $r = |x| \to \infty$ implies that

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + 0(r^{-3/2})$$

where $\hat{x} = x/|x|$ and

$$u_\infty(\hat{x}, d) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int\limits_{\partial D} (u^s \frac{\partial}{\partial \nu} e^{-ik\hat{x}\cdot y} - \frac{\partial u^s}{\partial \nu} e^{-ik\hat{x}\cdot y}) ds(y).$$

Definition: u_∞ is called the far field pattern corresponding to the specific scattering problem under consideration.

Theorem: Suppose $u_\infty = 0$. Then $u^s = 0$ in $R^2 \setminus D$.

Proof: $\int\limits_{|y|=R} |u^s(y)|^2 ds = \int\limits_{|\hat{x}|=1} |u_\infty(\hat{x}, d)|^2 ds(\hat{x}) + 0(\frac{1}{r})$ as $r \to \infty$. If $u_\infty = 0$ then by Rellich's lemma $u^s = 0$.

Far Field Patterns

Reciprocity Principle: $u_\infty(\hat{x}, d) = u_\infty(-d, -\hat{x})$.

It follows from the reciprocity principle that $u_\infty(\hat{x}, d)$ is infinitely differentiable with respect to its independent variables.

Example: Consider the direct scattering problem for a perfect conductor when D is a disk of radius a. Then using the Jacobi-Anger expansion

$$e^{ikr\cos\theta} = \sum_{-\infty}^{\infty} i^n J_n(kr) e^{in\theta}$$

we have that, for $d = (\cos\phi, \sin\phi)$,

$$u^s(r,\theta) = -\sum_{-\infty}^{\infty} i^n \frac{J_n(ka)}{H_n^{(1)}(ka)} H_n^{(1)}(kr) e^{in(\theta-\phi)}$$

and since

$$H_n^{(1)}(kr) = \sqrt{\frac{2}{\pi r}} \exp\left[i(kr - \frac{n\pi}{2} - \frac{\pi}{4})\right] + 0\left(r^{-3/2}\right)$$

we have that

$$u_\infty(\hat{x}, d) = -e^{-i\pi/4}\sqrt{\frac{2}{\pi}} \sum_{-\infty}^{\infty} \frac{J_n(ka)}{H_n(ka)} e^{in(\theta-\phi)}.$$

Far Field Operator for a Perfect Conductor

Let $\Omega := \{x : |x| = 1\}$. The far field operator $F : L^2(\Omega) \to L^2(\Omega)$ is defined by

$$(Fg)(\hat{x}) := \int_{\Omega} u_\infty(\hat{x}, d) g(d) ds(d).$$

From the smoothness of u_∞ we see that F is a compact operator. Note that $(Fg)(\hat{x})$ is the far field pattern corresponding to the incident field u^i being a Herglotz wave function $v_g(x)$ defined by

$$v_g(x) := \int_{\Omega} e^{ikx\cdot d} g(d) ds(d).$$

Theorem: If F is the far field operator corresponding to a perfect conductor then F is a normal operator.

Far Field Operator for a Perfect Conductor

Theorem: The far field operator corresponding to a perfect conductor is injective with dense range if and only if there does not exist a Dirichlet eigenfunction for D which is a Herglotz wave function.

Outline of Proof: The reciprocity principle implies that the adjoint operator F^* satisfies

$$(F^*h)(d) = \overline{(Fg)(-d)}$$

where $g(\hat{x}) = \overline{h(-\hat{x})}$. Hence F is injective if and only if F^* is injective. But $N(F^*)^\perp = \overline{F(L^2(\Omega))}$ and hence we only need to prove that F is injective.

$Fg = 0$ implies that the scattering problem with $u^i = v_g$ has vanishing far field pattern and hence using Rellich's lemma $v_g(x) = 0$ for $x \in \partial D$. Thus v_g is a Dirichlet eigenfunction unless $g = 0$.

Far Field Operator for an Inhomogeneous Medium

Recall that the far field operator $F : L^2(\Omega) \to L^2(\Omega)$ is defined by

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) ds(d).$$

Theorem: If F is the far field operator corresponding to an inhomogeneous medium, and $Im\, n(x) = 0$ for $x \in D$, then F is a normal operator.

Theorem: The far field operator corresponding to an inhomogeneous medium is injective with dense range if and only if there does not exist $w \in C^2(D) \cap C^1(\overline{D})$ and a Herglotz wave function v such that v, w is a solution of the interior transmission problem

$$\left.\begin{aligned} \Delta_2 v + k^2 v &= 0 \\ \Delta_2 w + k^2 n(x) w &= 0 \end{aligned}\right\} \quad \text{in } D$$

$$\left.\begin{aligned} v &= w \\ \frac{\partial v}{\partial \nu} &= \frac{\partial w}{\partial \nu}. \end{aligned}\right\} \quad \text{on } \partial D$$

Far Field Operator for an Inhomogeneous Medium

Definition: Values of k such that the interior transmission problem has a nontrivial solution are called transmission eigenvalues.

Theorem: If $Im\ n(x_0) \neq 0$ for some $x_0 \in D$ then k is not a transmission eigenvalue, i.e. the far field operator F is injective with dense range.

Proof: If there exists a nontrivial solution to the interior transmission problem then

$$\begin{aligned} 0 &= \int\limits_{\partial D} \left(v \frac{\partial \overline{v}}{\partial \nu} - \overline{v} \frac{\partial v}{\partial \nu} \right) ds = \int\limits_{\partial D} \left(w \frac{\partial \overline{w}}{\partial \nu} - \overline{w} \frac{\partial w}{\partial \nu} \right) ds \\ &= \int\limits_{D} \left(w \Delta \overline{w} - \overline{w} \Delta w \right) dx = 2ik^2 \int\limits_{D} Im\ n |w|^2 dx. \end{aligned}$$

If $Im\ n(x_0) \neq 0$ then $w(x) = 0$ in a neighborhood of x_0 and by the unique continuation principle $w(x) = 0$ for $x \in D$. Then v has vanishing Cauchy data and hence $v(x) = 0$ for $x \in D \Rightarrow\Leftarrow$.

Partially Coated Perfect Conductors

If a portion of a perfectly conducting cylinder is partially coated by a dielectric, we are led to the mixed boundary value problem

$$\Delta_2 u + k^2 u = 0 \quad \text{in} \;\; R^2 \setminus \overline{D}$$

$$u = u^i + u^s$$

$$u = 0 \quad \text{on } \Gamma_D$$

$$\frac{\partial u}{\partial \nu} + ik\lambda u = 0 \text{ on } \Gamma_I$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0$$

where λ is a positive constant and $\partial D = \Gamma_D \cup \Gamma_I$.

Theorem: The mixed boundary value problem has at most one solution.

Proof: Green's theorem and Rellich's lemma.

It is no longer appropriate to use integral equations of the second kind to obtain existence; instead integral equations of the first kind must be used.

Partially Coated Perfect Conductors

From Green's representation formula we have

$$u = S\frac{\partial u}{\partial \nu} - Du$$

where S and D are single layer and double layer potentials respectively. Applying the boundary conditions and letting ψ_I and ψ_D be the unknown boundary data for u on Γ_I and $\frac{\partial u}{\partial \nu} + ik\lambda u$ on Γ_D respectively leads to a system of integral equations of the first kind for the determination of ψ_I and ψ_D:

$$A\begin{pmatrix}\psi_D \\ \psi_I\end{pmatrix} = g$$

Theorem: In an appropriate function space, A is a Fredholm operator with index zero and A has a trivial kernel.

Corollary: A solution exists to the mixed boundary value problem.