Scattering by an Infinite Cylinder

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1. A Perfect Conductor $(E = (0, 0, u))$

$$\Delta_2 u + k^2 u = 0 \quad \text{in } R^2 \setminus \bar{D}$$

$$\begin{aligned} u &= u^i + u^s \\ u &= 0 \quad \text{on } \partial D \end{aligned}$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0$$

Scattering by an Infinite Cylinder

2. An Inhomogeneous Medium

$$\Delta_2 u + k^2 n(x) u = 0 \quad \text{in } R^2$$

$$u = u^i + u^s$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0$$

Assume $u^i(x) = e^{ikx \cdot d}$, $n(x) = 1$ in $R^2 \setminus \bar{D}$, $Im\, n(x) \geq 0$ for $x \in D$, $Re\, n(x) > 0$, $k > 0$, $r = |x|$.

The Inverse Scattering Problem for a Perfect Conductor

The scattered field has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + 0(r^{-3/2})$$

where $\hat{x} = x/|x|$. Let $\Omega := \{x : |x| = 1\}$ and assume that k is fixed.

Inverse Scattering Problem: Give $u_\infty(\hat{x}, d)$ for $\hat{x} \in \Omega$ and either 1) $d = -\hat{x}$, 2) d fixed or 3) $d \in \Omega$, find D.

In practice it is often necessary to consider the limited aperature problem where $u_\infty(\hat{x}, d)$ is only known for $\hat{x}, -d \in \Omega_0 \subset \Omega$.

Uniqueness Theorems for the Inverse Problem

Theorem (Kirsch–Kress): Assume that D_1 and D_2 are two perfect conductors such that the far field patterns coincide for all $\hat{x},\ d \in \Omega$. Then $D_1 = D_2$.

Theorem (Colton-Sleeman): Let D_1 and D_2 be two perfect conductors which are contained in a disk of radius R such that $kR < \gamma_0$ where γ_0 is the first zero of the Bessel function $J_0(z)$. Then if the far field patterns coincide for one incident plane wave, $D_1 = D_2$.

Remark: The conclusion of the above theorem is an open problem for Neumann boundary conditions. The a priori assumption on D can be replaced by assuming D is a polyhedral domain (Alessandrini and Rondi, Cheng and Yamamoto, to appear).

Physical Optics Approximation

We now attempt to reconstruct D from a knowledge of $u_\infty(d, -d)$, $d \in \Omega$, under the assumption that D is convex and the wave number k is large.

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d

ν

∂D_-

D

∂D_+

The physical optics approximation is, for k large,

$$\frac{\partial u}{\partial \nu} = 2\frac{\partial u^i}{\partial \nu} \qquad \text{on } \partial D_-$$

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial D_+$$

∂D_- is called the illuminated region and ∂D_+ is called the shadow region.

Physical Optics Approximation

Using the identity

$$0 = \int_{\partial D} \left(u^i(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)} \;-\; \frac{\partial u^i}{\partial \nu}(y)\Phi(x,y)\right) ds(y)$$

for $x \in R^3 \setminus \overline{D}$ and

$$\Phi(x,y) = \frac{i}{4}H_0^{(1)}(k|x-y|), \quad x \neq y$$

we now have that

$$\begin{aligned} u_\infty(\hat{x}, d) &= -\frac{e^{i\pi/4}}{\sqrt{2\pi k}} \int_{\partial D_-} \frac{\partial}{\partial \nu} e^{iky\cdot d} e^{-ik\hat{x}\cdot y} ds(y) \\ &= -\frac{ike^{i\pi/4}}{\sqrt{2\pi k}} \int_{\partial D_-} \nu(y)\cdot d e^{ik(d-\hat{x})\cdot y} ds(y). \end{aligned}$$

We now set $\hat{x} = -d$, replace d by $-d$ and add the two results.

Physical Optics Approximation

The previous calculations now yield the Bojarski identity

$$u_\infty(-d,d) + \overline{u_\infty(d,-d)} =$$

$$= \frac{-ike^{i\pi/4}}{\sqrt{2\pi k}} \int\limits_{\partial D} \frac{\partial}{\partial \nu(y)} e^{2ikd\cdot y} ds(y)$$

$$= \frac{-ike^{i\pi/4}}{\sqrt{2\pi k}} \int\limits_{D} \Delta_2 e^{2ikd\cdot y} dy$$

$$= \frac{4ik^3 e^{i\pi/4}}{\sqrt{2\pi k}} \int\limits_{R^3} \mathcal{X}(y) e^{2ikd\cdot y} dy$$

where $\mathcal{X}$ is the characteristic function of D. Hence we have a linear integral equation to solve to determine $\mathcal{X}$ and hence D.

Newton's Method

We now attempt to reconstruct the support of a perfect conductor without the restrictive assumptions of the physical optics approximation. We assume $u_\infty(\hat{x}, d)$ is known for $\hat{x} \in \Omega$, d fixed, and, for the sake of simplicity, that ∂D can be represented as

$$x = r(\hat{x})\hat{x} \qquad , \hat{x} \in \Omega.$$

Consider the mapping

$$F : \ r \to u_\infty$$

for d fixed.

Theorem (Kirsch, Potthast): $F : \ C^2(\Omega) \to L^2(\Omega)$ has a Fréchet derivative F'_r. The linear operator F'_r is compact and injective with dense range.

Newton's Method

To determine ∂D, $F(r) = u_\infty$ is replaced by the linearized equation

$$F(r) + F'_r q = u_\infty$$

which from an initial guess $r = r_0$ yields the new approximation $r_1 = r_0 + q$. Newton's method consists in iterating this procedure. Partial results on the convergence of Newton's method in this case can be found in

R. Potthast, Point Sources and Multipoles in Inverse Scattering Theory, Chapman and Hall/CRC, 2001.

Note that, for Newton's method, a priori information is needed on the number of scatterers as well as the boundary condition (which is also true for the physical optics approximation).

The Linear Sampling Method

The far field operator $F : L^2(\Omega) \to L^2(\Omega)$ is defined by

$$(Fg)(\hat{x}) := \int_{\Omega} u_\infty(\hat{x}, d) g(d) ds(d).$$

If $\Phi_\infty(\hat{x}, z)$ is the far field pattern of the fundamental solution

$$\Phi(x, z) = \frac{i}{4} H_0^{(1)}(k|x - z|)$$

then the far field equation is given by

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z)$$

Unfortunately, in general no solution exists to the far field equation! However, defining the Herglotz wave function by

$$v_g(x) := \int_{\Omega} e^{ikx \cdot d} g(d) ds(d)$$

we have the following theorem:

The Linear Sampling Method

Theorem: Assume k^2 is not a Dirchlet eigenvalue for $-\Delta$ in D. Then

1. If $z \in D$ then for every $\epsilon > 0$ there exists a solution $g(\cdot; z) \in L^2(\Omega)$ of the inequality

$$\|Fg - \Phi_\infty(\cdot, z)\|_{L^2(\Omega)} < \epsilon$$

   such that

$$\lim_{z \to \partial D} \|v_g(\cdot; z)\|_{H^{1/2}(\partial D)} = \infty$$

   and

$$\lim_{z \to \partial D} \|g(\cdot; z)\|_{L^2(\Omega)} = \infty.$$

2. If $z \in R^3 \setminus \overline{D}$ then for every $\epsilon > 0$ and $\delta > 0$ there exists a solution $g(\cdot; z) \in L^2(\Omega)$ of the inequality

$$\|Fg - \Phi_\infty(\cdot, z)\|_{L^2(\Omega)} < \epsilon + \delta$$

   such that

$$\begin{aligned} \lim_{\delta \to 0} \|v_g(\cdot; z)\|_{H^{1/2}(\partial D)} &= \infty \\ \lim_{\delta \to 0} \|g(\cdot; z)\|_{L^2(\Omega)} &= \infty. \end{aligned}$$

The function g in the above theorem is now sought for using Tikhonov regularization and the Morozov discrepancy principle to solve the far field equation $Fg = \Phi_\infty$.

The Inverse Scattering Problem for an Inhomogeneous Medium

We now consider the case of scattering by an inhomogeneous medium. We again assume that k is fixed.

Inverse Scattering Problem: Given $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$ find either 1) D or 2) $n(x)$.

In R^2, it is not known if $u_\infty(\hat{x}, d)$ uniquely determines $n(x)$ for fixed k. However, the following is known:

Theorem (Sun and Uhlmann, Potthast): The far field pattern $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$ uniquely determines D.

Theorem (Eskin): Except for possibly a countable set of value of k, $u_\infty(\hat{x}, d)$ for $\hat{x}, d \in \Omega$ uniquely determines $n(x)$.

Reconstruction of $n(x)$

1. The Born approximation

Recall that for an inhomogeneous medium u is the unique solution of the Lippmann–Schwinger equation

$$u(x) = u^i(x) - k^2 \int\limits_{R^2} \Phi(x,y)m(y)u(y)dy \ , \ x \in R^2$$

where $m := 1 - n$. If k is small this can be solved by successive approximations and replacing u by the first term in this iterative process and letting $r = |x| \to \infty$ we obtain the Born approximation

$$u_\infty(\hat{x}, d) = -\frac{k^2 e^{i\pi/4}}{\sqrt{8\pi k}} \int\limits_{R^2} e^{-ik\hat{x}\cdot y} m(y) u^i(y) dy.$$

This is a linear integral equation for the determination of m.

Reconstruction of $n(x)$

2. Newton's Method

Newton's method can be used to determine $n(x)$ with no assumptions on k being small. A very nice paper on this topics is

T. Hohage, On the numerical solution of a three dimensional inverse medium scattering problem, *Inverse Problems* 17 (2001), 1743-1763.

3. The Linear Sampling Method

The linear sampling method can be used to determine the support D of an inhomogeneous medium by solving the far field equation where u_∞ is now the far field pattern corresponding to an inhomogeneous medium. See

D. Colton and A. Kirsch, A simple method for solving inverse scattering problems in the resonance region, *Inverse Problems* 12 (1996), 383-393.

D. Colton, M. Piana and R. Potthast, A simple method using Morozov's discrepancy principle for solving inverse scattering problems, *Inverse Problems* 13 (1997), 1477-1493.

Maxwell's Equations

The previous results on scattering by an infinite cylinder have their analogues for Maxwell's equations in R^3. We consider only the case of a perfect conductor:

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(E^i, H^i)
(E^s, H^s)
D

$$\begin{aligned} \Delta \times E - ikH &= 0 \\ & \qquad\qquad \text{in } R^3 \setminus D \\ \Delta \times H + ikE &= 0 \end{aligned}$$

$$\begin{aligned} E &= E^i + E^s \\ H &= H^i + H^s \end{aligned}$$

$$\nu \times E = 0 \quad \text{on } \partial D$$

$$\lim_{r \to \infty} (H^s \times x - rE^s) = 0$$

where

$$\begin{aligned} E^i(x) &= \tfrac{i}{k} \bigtriangledown \times \bigtriangledown \times p e^{ikx\cdot d} \\ H^i(x) &= \bigtriangledown \times p e^{ikx\cdot d} \end{aligned}$$

and $p \in R^3$ is the polarization.

Maxwell's Equations

It can be shown that

$$E^s(x) = \frac{e^{ikr}}{r} E_\infty(\hat{x}, d, p) + 0(\frac{1}{r^2})$$

and the inverse scattering problem is to determine D from $E_\infty(\hat{x}, d, p)$ for $\hat{x}, d \in \Omega : \{x : |x| = 1\}$ and three linearly independent polarizations.

Theorem: D is uniquely determined by $E_\infty(\hat{x}, d, p)$ for $\hat{x}, d \in \Omega$ and three linearly independent polarizations p_1, p_2, p_3.

Physical optics, Newton's method and the linear sampling method can all be generalized to the present case and used to determine D. We consider only the linear sampling method.

Maxwell's Equations

Define the electric dipole with source at $z \in R^3$ by

$$\begin{aligned} E_e(x,z,q) &= \frac{i}{k} \nabla_x \times \nabla_x \times q\Phi(x,z) \\ H_e(x,z,q) &= \nabla_x \times q\Phi(x,z) \end{aligned}$$

where

$$\Phi(x,z) := \frac{1}{4\pi} \frac{e^{ik|x-z|}}{|x-z|}$$

and $q \in R^3$. The far field operator $F : L_t^2(\Omega) \to L_t^2(\Omega)$ is defined by

$$(Fg)(\hat{x}) := \int_{\Omega} E_\infty(\hat{x}, d, g(d)) ds(d)$$

and the far field equation is

$$(Fg)(\hat{x}) = E_{e,\infty}(\hat{x}, z, q)$$

where $E_{e,\infty}$ is the far field pattern of E_e. D can again be characterized by those $z \in R^3$ for which the regularized solution of the far field equation satisfies $||g(\cdot, z)|| \le C$ for C an appropriately chosen constant.