Interior Elastodynamics Inverse Problems II: 

**Algorithms**

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Interior Elastodynamics Inverse Problems

**Data:** Propagating Elastic Wave

**Characteristics:**
- Initially the medium is at rest
- Time and space dependent *interior* displacement measurements
- Wave has propagating fronts
- Wave amplitude is low → use linear model
- Medium is isotropic

**Assumptions:**
- Compression wavespeed, $\sqrt{(\lambda + 2\mu)/\rho}$, and shear wavespeed, $\sqrt{\mu/\rho}$ significantly different
- Compression wave is significantly lower amplitude contribution when shear wave arrives
Two Possible Arrival Time Equations

Two Models:
\[
\begin{align*}
\nabla \cdot (\mu(x) \nabla u) &= \rho u_{tt} \\
\nabla (\lambda \nabla \cdot \bar{u}) + \nabla \cdot (\mu (\nabla \bar{u} + (\nabla \bar{u})^T)) &= \rho \bar{u}_{tt}
\end{align*}
\]

Define arrival time:
\[
\hat{T}(x) = \inf \{ t \in (0, T) : u(x, t) \neq 0 \}.
\]

Eikonal equations
For each model
\[
\begin{align*}
|\nabla \hat{T}| \sqrt{\frac{\mu}{\rho}} &= 1 \\
|\nabla \hat{T}| \sqrt{\frac{(\lambda + 2\mu)}{\rho}} &= 1
\end{align*}
\]

Practical Matters

How to find arrival time in the presence of noise?

- Reject simple thresholding
- Underlying idea: Track distinctive features in the data that propagate at the unknown wavespeed
- Extreme values or peaks are distinctive features
- Track distinctive features collectively using correlation
Time Trace + Peak & Correlation Description

- Determine Arrival Time Surface by Peak Location:
  1. Zero of $u_t = \text{peak of } u$.
  2. Apply mollifier to $u$ and approximate peak location by applying thresholding to $u$.
  3. Calculate $u_t$ and use MATLAB zero finding routine.

- Determine Arrival Time Surface by Correlation:
  1. Apply mollifier to $u$;
  2. $\hat{T}(x_1) = \hat{T}(x_0) + \Delta \hat{t}$ where $x_1 = x_0 + \Delta x$ and
     \[ \Delta \hat{t} = \arg \max_{\Delta t} \int u(x_0, t)u(x_1, t - \Delta t) \, dt. \]

Model \[ \nabla \cdot (\mu(x) \nabla u) = \rho u_{tt} \]

Arrival Time Surface with Simulated Data
Using Peaks and Interpolation

- no noise
- 7% noise
Model $\nabla \cdot (\mu(x) \nabla u) = \rho u_{tt}$

Arrival Time Surface with Simulated Data Using Correlation (No Interpolation)

Simulation Model $\nabla \cdot (\mu(x) \nabla u) = \rho u_{tt}$

- $\rho \equiv 1$, $\mu(x_1, x_2) = \left(3 + 9 \exp \left\{ \frac{-(x_1 - 3)^2}{0.25^2} - \frac{x_2^2}{0.5^2} \right\} \right)^2$

  which makes

  \[
  \begin{align*}
  \text{Background wave speed} &= 3\text{m/sec} \\
  \text{Maximum wave speed} &= 12\text{m/sec}
  \end{align*}
  \]

- Boundary impulse central frequency $= 250$ Hz

- Discretization:

  - 250 time steps
  - $100 \times 100$ spatial grid

Normalized Target wave speed
Phantom for Laboratory Data
(Boundary impulse central frequency=50Hz)

Cross-section of the phantom

- Shear wavespeed is doubled in the included cylinder

Arrival Time Surface with Laboratory Data
(Boundary impulse central frequency=50Hz)

Arrival time surface with peak location (interpolation)

Arrival time surface with correlation method

Time traces along the center line
Level Curves of Arrival Time Surface
(Laboratory Data)

Using peak location
Using correlation method

How to Use Formula $\sqrt{\frac{\mu}{\rho}} = \frac{1}{|\nabla T|}$
Simplistic Method

- Try taking partial derivatives with noisy data

Target wave speed
Recovered wave speed
Distance Method: Use Level Curve

\[ \hat{T}(\hat{x}) = \hat{T}(x) + \Delta t \]

\[ \hat{T} = \hat{T}(x) \]

\[ \hat{x} \]

\[ \sqrt{\frac{\mu}{\rho}}(x) = \frac{1}{\Delta t} \inf \{ \| x - \hat{x} \| : \hat{T}(\hat{x}) = \hat{T}(x) + \Delta t \} \]  
1\textsuperscript{st} Order Method

\[ \sqrt{\frac{\mu}{\rho}}(x) = \frac{1}{2\Delta t} \left[ \inf \{ \| x - \hat{x} \| : \hat{T}(\hat{x}) = \hat{T}(x) + \Delta t \} + \inf \{ \| x - \hat{x} \| : \hat{T}(\hat{x}) = \hat{T}(x) - \Delta t \} \right] \]  
2\textsuperscript{nd} Order Method

Simulated Data (1\textsuperscript{st} Order Distance Method)

Peak Location

Target wave speed

Correlation Method

no noise 7% noise
Simulated Data (2nd Order Distance Method)

- Peak Location
- Target wave speed
- Correlation Method

Reconstruction using Real Data and Peak Location (2nd Order Method)

- Time trace
- Level curves
- Arrivals time
- Recovered wave speed
Final Step with Real Data (Total Variation Denoising)

\[
\min_{\mu, \rho \in \text{BV}} \left[ \gamma \int_{\Omega} \left\{ \left( \frac{\mu}{\rho} \right)_D - \left( \frac{\mu}{\rho} \right)_F \right\}^2 + \int_{\Omega} \left| \nabla \left( \frac{\mu}{\rho} \right)_F \right| \right]
\]

with regularized Euler-Lagrange equation

\[
\nabla \cdot \left[ \frac{\nabla \left( \frac{\mu}{\rho} \right)_F}{\left| \nabla \left( \frac{\mu}{\rho} \right)_F + \epsilon \right|} + 2\gamma \left[ \left( \frac{\mu}{\rho} \right)_D - \left( \frac{\mu}{\rho} \right)_F \right] = 0
\]

Introducing the evolving variable \( s \), solve

\[
\frac{\partial}{\partial s} \sqrt{\frac{\mu}{\rho}} = \nabla \cdot \left[ \frac{\nabla \left( \frac{\mu}{\rho} \right)_F}{\left| \nabla \left( \frac{\mu}{\rho} \right)_F + \epsilon \right|} + 2\gamma \left[ \left( \frac{\mu}{\rho} \right)_D - \left( \frac{\mu}{\rho} \right)_F \right]
\]

\[
\sqrt{\frac{\mu}{\rho}}_{s=0} = \left( \frac{\mu}{\rho} \right)_D
\]

We can speed this up considerably

1. Using Distance Method (finding distance to previous level set); 
   \( O(N^{3/2}) \), \( N \) is the number of grid points.

2. With level set algorithm; \( O(N \log N) \).
Key Ideas for Level Set Method for Inverse Problems

(1) Recall \(|\nabla \hat{T}|\sqrt{\mu/\rho} = 1\).

(2) \(t = \hat{T}(x)\) is the zero level set of a function \(\phi\);

\[\phi(x,\hat{T}(x)) = 0.\]

(3) For each \(t\), \(\phi(x,t) = \pm \inf_{\hat{T}(x)=t} |x - \hat{x}|.\)

(4) \(\hat{T}(x)\) Lipschitz continuous \(\Rightarrow \phi\) is Lipschitz continuous.

(5) Chain Rule applies to \(\phi(x,\hat{T}(x))\)

\[\Rightarrow \quad \phi_t \nabla \hat{T} + \nabla_x \phi = 0 \quad \text{a.e. } t = \hat{T}(x),\]

\[\Rightarrow \quad \phi_t = \sqrt{\mu/\rho} |\nabla_x \phi| \quad \text{a.e. } t = \hat{T}(x).\]

(6) Extend this equation to narrow band around \(t = \hat{T}(x)\):

\( (\phi_t, \nabla_x \phi) \) exists \hspace{1cm} \text{a.e.}\n
\( f_{ext} = \phi_t/|\nabla_x \phi| \) \hspace{1cm} \text{a.e.}\n
\( \phi_t = f_{ext} |\nabla_x \phi| \) \hspace{1cm} \text{a.e.}\n
\( f_{ext} = \sqrt{\mu/\rho} \) \hspace{1cm} \text{a.e. } t = \hat{T}(x).\)

(7) From (3), \(|\nabla_x \phi| = 1\) for fixed \(t\).

(8) Calculate

\[\sqrt{\frac{\mu}{\rho}}(x) = \frac{\phi(x,\hat{T}(x) + \Delta t) - \phi(x,\hat{T}(x))}{\Delta t}, \] \((1\text{st order})\)

or

\[\sqrt{\frac{\mu}{\rho}}(x) = \frac{1}{2\Delta t} \left[ \phi(x,\hat{T}(x) + \Delta t) + \phi(x,\hat{T}(x) - \Delta t) \right], \] \((2\text{nd order})\)
Recovery with Synthetic Data Using Level Set (first order) Method

Using Peak Locations

Target wave speed

Level Set Method

O($N^{3/2}$) Method

Sources of Error

(1) Both peak location and correlation experience some distortion due to backscattering

(2) Low amplitude compression wave treated as noise in the data

(3) Use of ultrasound to determine interior displacement yields errors

(4) Modeling errors – Viscoelastic effects, Anisotropic model
Conclusion for Arrival Time Algorithm

(1) Data is subset of time and space dependent interior displacement subset = arrival time surface
(2) Arrival time determined by correlating feature propagation
(3) Simplistic Approach yields poor results
(4) Distance Method yields $O(N^{3/2})$ algorithm, $N = \#$ of grid pts
(5) Level set method yields $O(N \log N)$ algorithm
(6) Final Total Variation step is needed to ‘clean up’ noisy reconstruction when using real data
(7) Excellent Recovery when wavespeed increases up to 4 times over the background wavespeed

Unsolved Problems

(1) Reduce smoothness assumptions in uniqueness theorem for equations of elasticity – almost complete
(2) Identify more coefficients
(3) Determine uniqueness theorems for anisotropic case - choosing models to fit applications
(4) Justify arrival time algorithm for the shear wave in an elastic system
(5) Extend arrival time algorithm to anisotropic case
(6) Investigate elasticity equations for an incompressible medium
(7) Model viscoelastic and nonlinear effects
Second algorithm:
Use Central Frequency Content

- Applies to Dynamic Sinusoidal Excitation
- Single Frequency

**Theorem** (Richter) Let $\Omega$ be bounded and $\mu_1, \mu_2 \in C^1(\bar{\Omega})$ with $\mu_1 = \mu_2$ on $\partial\Omega$, and $g \in C^0(\partial\Omega)$. Let $\hat{u} \in C^2(\Omega)$ be a common solution to $\nabla \cdot (\mu_j \nabla \hat{u}) + \kappa^2 \hat{u} = 0$ in $\Omega$ with $\mu_j \nabla \hat{u} \cdot \nu = g$ on $\partial\Omega$ for $j = 1, 2$.

Suppose further that $\hat{u}$ satisfies $\inf_{x \in \Omega} \{ \max \{|\nabla \hat{u}(x)|, |\Delta \hat{u}(x)|\} \} > 0$. Then $\mu_1 = \mu_2$ in $\Omega$.

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**Geometric Optics Algorithm (1)**

- 2-D wave equation:
  \[
  \nabla \cdot (c_0^2 \mu \nabla w) - w_{tt} = 0, \quad x_1 > 0
  \]
  \[
  \left. \frac{\partial w}{\partial x_1} \right|_{x_1=0} = g(t, x_2)
  \]

- The corresponding Helmholtz equation:
  \[
  \nabla \cdot (\mu \nabla u) + \left( \frac{\tau}{c_0} \right)^2 u = 0, \quad x_1 > 0
  \]
  \[
  \left. \frac{\partial u}{\partial x_1} \right|_{x_1=0} = f(\tau, x_2)
  \]

**Inverse Problem**

$w(t, x)$ \(\xrightarrow{?}\) $\mu$

Fourier Transform $|u(\tau, x)|$ \(\xrightarrow{\text{inverse algorithm}}\) $\mu$
Geometric Optics Algorithm (2)

- Single frequency content
- The key:
  \[ \mu \quad \text{algebraic equation} \quad \frac{\mu}{(\text{approximate relation})} \quad |u(\tau, x)| \]
- Asymptotic expansion of geometrical optics:
  \[ u(\tau, x) = a(\kappa, x)e^{i\kappa \phi(x)} \]
  where
  \[ a \sim a_0(x) + \frac{a_1(x)}{i\kappa} + \frac{a_2(x)}{(i\kappa)^2} + \cdots \]
  as \( \kappa \to \infty \)
  \[ |u(\tau, x)| \sim a_0(x) \]

Geometric Optics Algorithm (3)

- Eikonal equation for the phase: \( |\nabla \phi|^2 - \frac{1}{\mu} = 0 \)
- Transport equation for the amplitude:
  \[ 2\mu \nabla a_0 \cdot \nabla \phi + \mu a_0 \Delta_x \phi + a_0 \nabla \mu \cdot \nabla \phi = 0 \]
- Characteristic ODE for the Eikonal equation:
  \[ \frac{dx_2}{dx_1} = \frac{p_2}{p_1}, \quad p_1 = \sqrt{\frac{1}{\mu} - p_2^2} > \alpha_1 > 0, \quad \vec{p} := \nabla \phi, \]
  \[ \frac{dp_2}{dx_1} = \frac{1}{2p_1} \frac{\partial}{\partial x_2} \frac{1}{\mu}. \]
**Geometric Optics Algorithm (4)**

ODE-algebraic system \( \dot{\gamma} = d/dx_1 \)

\[
\begin{align*}
\dot{x}_2 &= \frac{p_2}{p_1}, \\
p_1 &= \sqrt{\frac{1}{\mu} - (p_2)^2} \geq \alpha_0 > 0, \\
p_2 &= \frac{1}{2p_1} \frac{\partial}{\partial x_2} \left( \frac{1}{\mu} \right), \\
J &= F \left( J, A, p_2, \frac{1}{\mu}, \frac{\partial^2 1}{\partial x_2} \mu \right), \\
A &= \frac{\mu(x_1, x_2) a_0^2(x_1, x_2) p_1(x_1, x_2) J(x_1)}{\mu(0, y_1) a_0^2(0, y') p_1(0, y')} = 1 \\
\alpha_0 &\sim |u|
\end{align*}
\]

**Numerical Experiment (Mild Stiffness)**

![Numerical Experiment Diagram](image)
Recovery From Exact Data

The recovered $\mu$ from the exact $a_0$ (left) and the data, $u(\tau, x)$ (right).

Recovery From Noisy Data

The recovered $\mu$ (right) from the data, $u(\tau, x)$, with random noise (left).

The random noise is created as $\tilde{w}(t, x) = w(t, x) + 0.05w_{\text{max}}\tau(t, x)$. 
Numerical Experiment (Strong Stiffness)

Conclusion for Single Frequency Algorithm

(a) **Successful** when wavespeed increases up to \(~1.7\) times the background wavespeed

(b) **Robust** to noise

(c) Large backscattering limits application of this algorithm (so far)

(d) Algorithm needs to be advanced to include *caustics*
Unsolved Problems

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