Interior Elastodynamics Inverse Problems II: *Algorithms*

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Interior Elastodynamics Inverse Problems

Data: Propagating Elastic Wave

Characteristics:

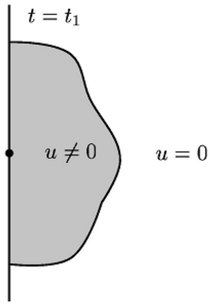
- Initially the medium is at rest
- Time and space dependent *interior* displacement measurements
- Wave has propagating fronts
- Wave amplitude is low $\rightarrow$ use linear model
- Medium is isotropic

Assumptions:

- Compression wavespeed, $\sqrt{(\lambda + 2\mu)/\rho}$, and shear wavespeed, $\sqrt{\mu/\rho}$ significantly different
- Compression wave is significantly lower amplitude contribution when shear wave arrives

Two Possible Arrival Time Equations

Two Models :
$$\begin{cases} \nabla \cdot (\mu(x) \nabla u) = \rho u_{tt} \\ \nabla(\lambda \nabla \cdot \vec{u}) + \nabla \cdot \left(\mu(\nabla \vec{u} + (\nabla \vec{u})^T)\right) = \rho \vec{u}_{tt} \end{cases}$$



Define arrival time:

$$\hat{T}(x) = \inf\{t \in (0, T) : u(x,t) \neq 0\}.$$

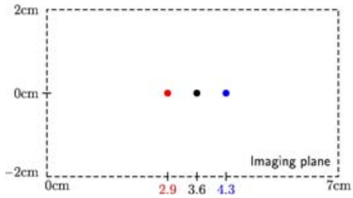
Eikonal equations
For each model
$$\begin{cases} |\nabla \hat{T}| \sqrt{\mu/\rho} = 1 \\ |\nabla \hat{T}| \sqrt{(\lambda + 2\mu)/\rho} = 1 \end{cases}$$

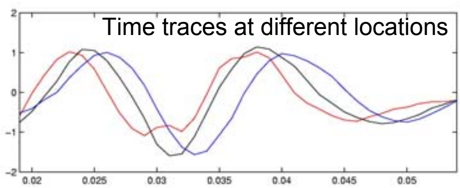
Practical Matters

How to find arrival time in the presence of noise?

- Reject simple thresholding
- Underlying idea: Track distinctive features in the data that propagate at the unknown wavespeed
- Extreme values or peaks are distinctive features
- Track distinctive features collectively using correlation

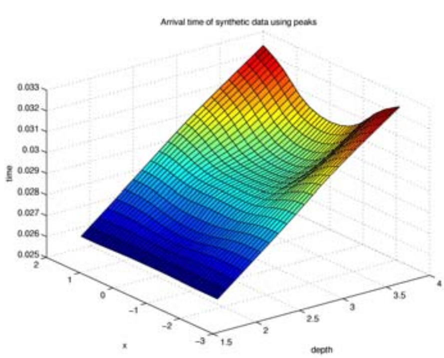
Time Trace + Peak & Correlation Description



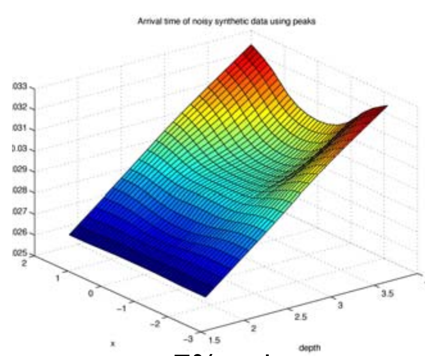


- Determine Arrival Time Surface by Peak Location:
  1. Zero of u_t = peak of u.
  2. Apply mollifier to u and approximate peak location by applying thresholding to u.
  3. Calculate u_t and use MATLAB zero finding routine.
- Determine Arrival Time Surface by Correlation:
  1. Apply mollifier to u;
  2. $\hat{T}(x_1) := \hat{T}(x_0) + \Delta\hat{t}$ where $x_1 = x_0 + \Delta x$ and
  $$\Delta\hat{t} = \arg\max_{\Delta t} \int u(x_0, t)u(x_1, t - \Delta t)\, dt.$$

Model $\nabla \cdot (\mu(x)\nabla u) = \rho u_{tt}$



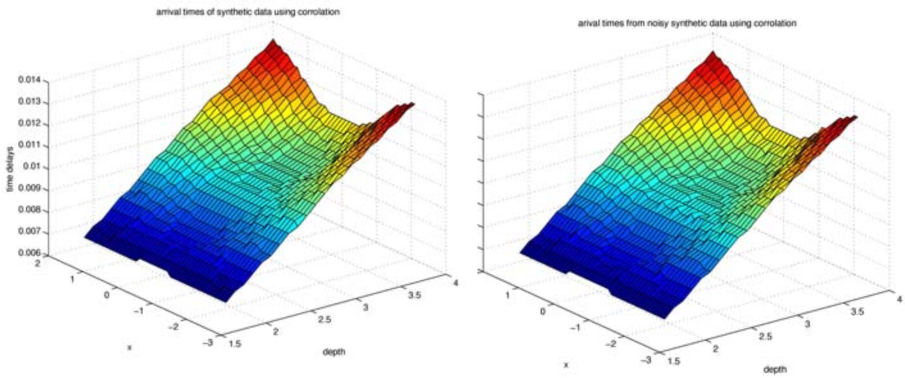
no noise



7% noise

Arrival Time Surface with Simulated Data
Using Peaks and Interpolation

Model $\nabla \cdot (\mu(x)\nabla u) = \rho u_{tt}$



no noise

7% noise

Arrival Time Surface with Simulated Data
Using Correlation (No Interpolation)

Simulation Model $\nabla \cdot (\mu(x)\nabla u) = \rho u_{tt}$

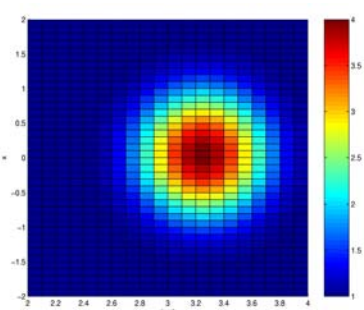
- $\rho \equiv 1, \quad \mu(x_1, x_2) = \left(3 + 9\exp\left\{-\frac{(x_1-3)^2}{0.25^2} - \frac{x_2^2}{0.5^2}\right\}\right)^2$

  which makes $\begin{cases} \text{Background wave speed} = 3m/sec \\ \text{Maximum wave speed} = 12m/sec \end{cases}$

- Boundary impulse central frequency $= 250$ Hz

- Discretization:

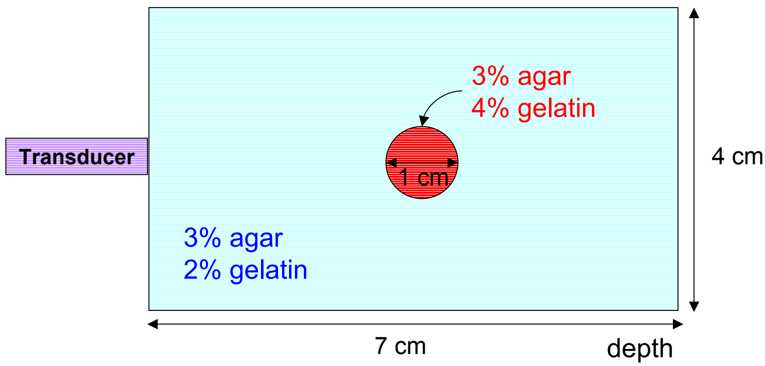
  250 time steps
  100×100 spatial grid



Normalized Target wave speed

Phantom for Laboratory Data

(Boundary impulse central frequency=50Hz)

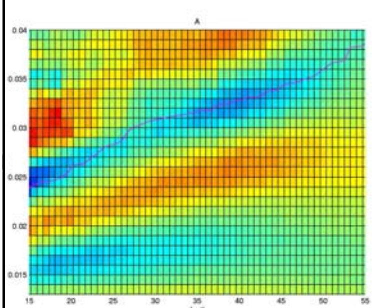


Cross-section of the phantom

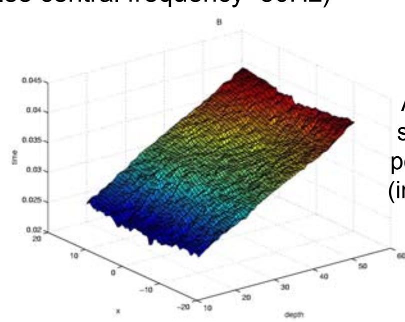
- Shear wavespeed is doubled in the included cylinder

Arrival Time Surface with Laboratory Data

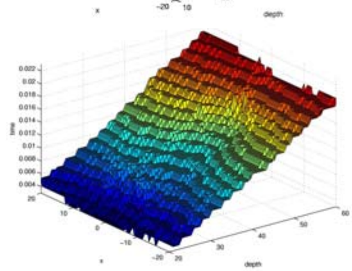
(Boundary impulse central frequency=50Hz)



Time traces along the center line

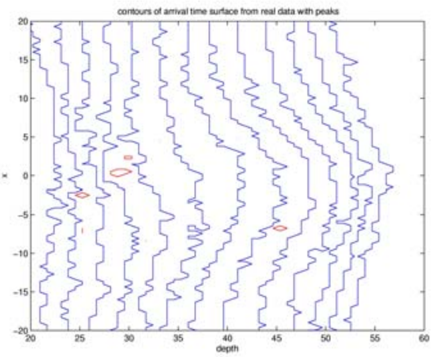


Arrival time surface with peak location (interpolation)

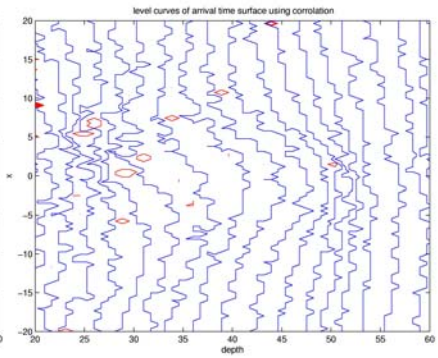


Arrival time surface with correlation method

Level Curves of Arrival Time Surface (Laboratory Data)



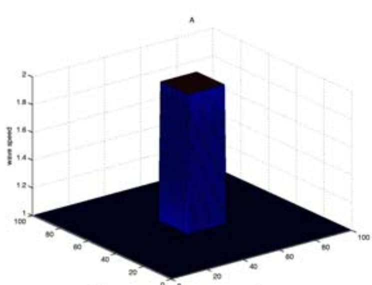
Using peak location



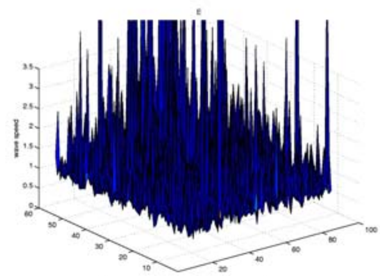
Using correlation method

How to Use Formula $\sqrt{\mu/\rho} = 1/|\nabla \hat{T}|$ Simplistic Method

- Try taking partial derivatives with noisy data

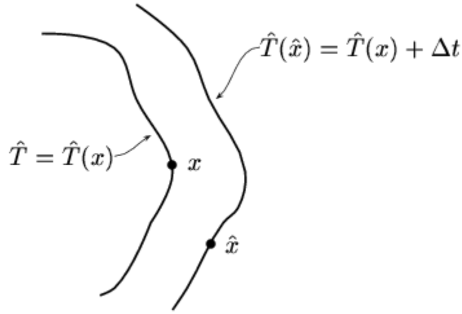


Target wave speed



Recovered wave speed

Distance Method: Use Level Curve



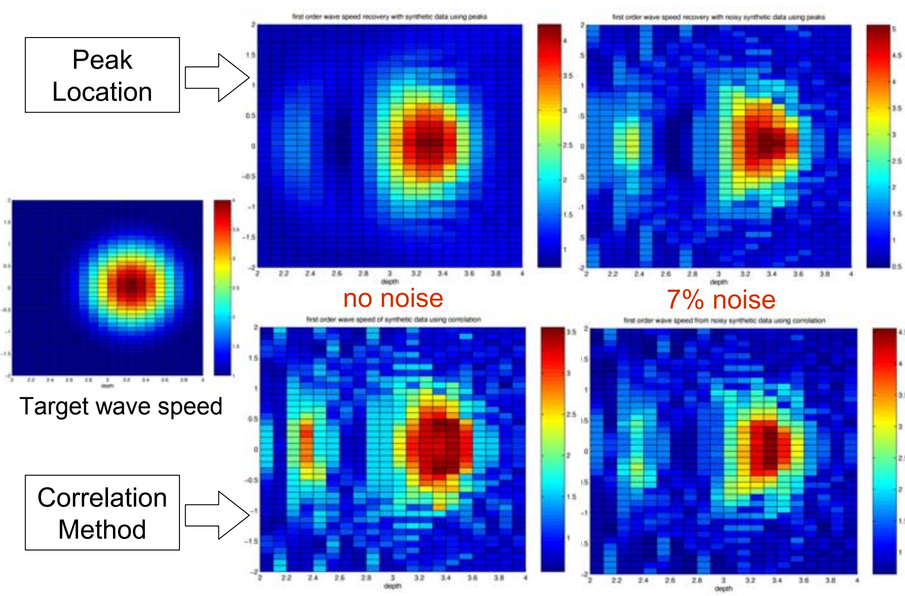
$$\sqrt{\frac{\mu}{\rho}(x)} = \frac{1}{\Delta t}\inf\{|x-\hat{x}| : \hat{T}(\hat{x}) = \hat{T}(x) + \Delta t\}$$

1st Order Method

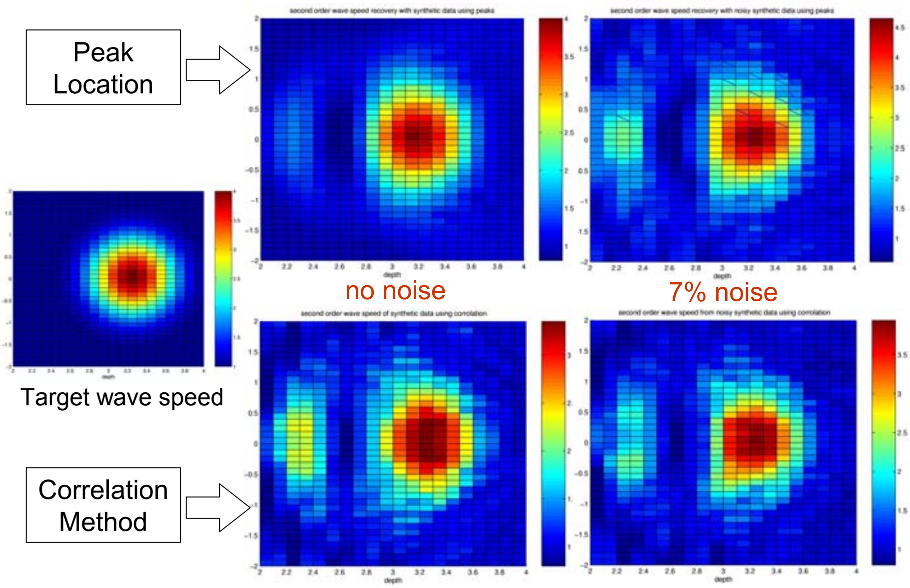
$$\sqrt{\frac{\mu}{\rho}(x)} = \frac{1}{2\Delta t}\Big[\inf\{|x-\hat{x}| : \hat{T}(\hat{x}) = \hat{T}(x) + \Delta t\} + \inf\{|x-\hat{\hat{x}}| : \hat{T}(\hat{\hat{x}}) = \hat{T}(x) - \Delta t\}\Big]$$

2nd Order Method

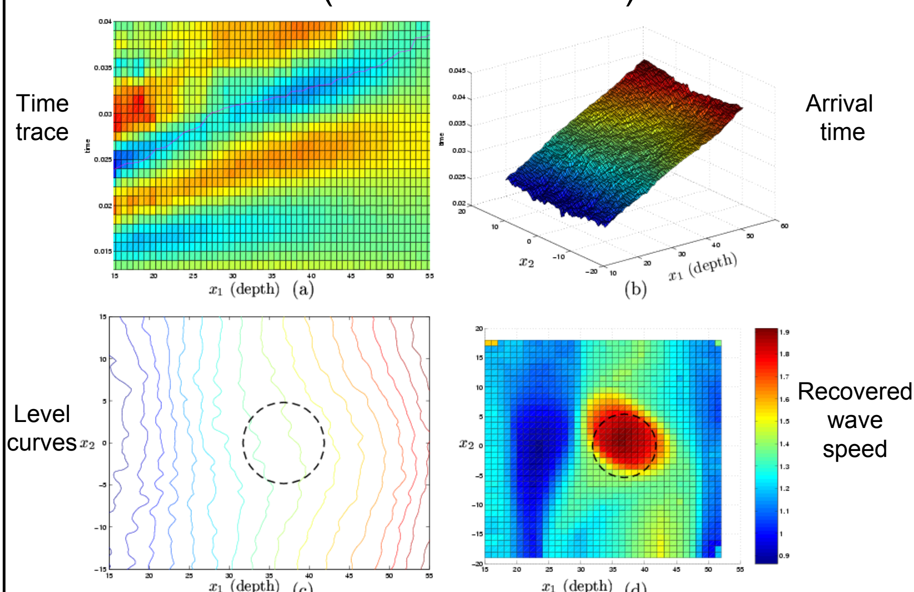
Simulated Data (1st Order Distance Method)



Simulated Data (2nd Order Distance Method)



Reconstruction using Real Data and Peak Location (2nd Order Method)



Final Step with Real Data (Total Variation Denoising)

$$\min_{\sqrt{\mu/\rho}\in \mathrm{BV}} \left[\gamma \int_\Omega \left\{\left(\sqrt{\frac{\mu}{\rho}}\right)_D - \left(\sqrt{\frac{\mu}{\rho}}\right)_F\right\}^2 + \int_\Omega \left|\nabla\left(\sqrt{\frac{\mu}{\rho}}\right)_F\right|\right]$$

with regularized Euler-Lagrange equation

$$\nabla \cdot \left[\frac{\nabla\left(\sqrt{\frac{\mu}{\rho}}\right)_F}{\left|\nabla\left(\sqrt{\frac{\mu}{\rho}}\right)_F\right| + \epsilon}\right] + 2\gamma\left[\left(\sqrt{\frac{\mu}{\rho}}\right)_D - \left(\sqrt{\frac{\mu}{\rho}}\right)_F\right] = 0$$

Introducing the evolving variable s, solve

$$\frac{\partial}{\partial s}\sqrt{\frac{\mu}{\rho}} = \nabla \cdot \left[\frac{\nabla\left(\sqrt{\frac{\mu}{\rho}}\right)_F}{\left|\nabla\left(\sqrt{\frac{\mu}{\rho}}\right)_F\right| + \epsilon}\right] + 2\gamma\left[\left(\sqrt{\frac{\mu}{\rho}}\right)_D - \left(\sqrt{\frac{\mu}{\rho}}\right)_F\right]$$

$$\left.\sqrt{\frac{\mu}{\rho}}\right|_{s=0} = \left(\sqrt{\frac{\mu}{\rho}}\right)_D$$

We can speed this up considerably

1. Using Distance Method (finding distance to previous level set);
   O($N^{3/2}$), N is the number of grid points.

2. With level set algorithm; O($N \log N$).

Key Ideas for Level Set Method for Inverse Problems

(1) Recall $|\nabla\hat{T}|\sqrt{\mu/\rho} = 1.$

(2) $t = \hat{T}(x)$ is the zero level set of a function ϕ;

$$\phi(x, \hat{T}(x)) = 0.$$

(3) For each t, $\phi(x,t) = \pm \inf_{\hat{T}(\hat{x})=t} |x - \hat{x}|.$

(4) $\hat{T}(x)$ Lipschitz continuous $\Rightarrow$ ϕ is Lipschitz continuous.

(5) Chain Rule applies to $\phi(x, \hat{T}(x))$

$$\Rightarrow \quad \phi_t \nabla\hat{T} + \nabla_x \phi = 0 \qquad \text{a.e. } t = \hat{T}(x),$$

$$\Rightarrow \quad \phi_t = \sqrt{\mu/\rho}\,|\nabla_x \phi| \qquad \text{a.e. } t = \hat{T}(x).$$

(6) Extend this equation to narrow band around $t = \hat{T}(x)$:

$$(\phi_t, \nabla_x \phi) \text{ exists} \qquad \text{a.e.}$$

$$f_{ext} = \phi_t / |\nabla_x \phi| \qquad \text{a.e.}$$

$$\phi_t = f_{ext} |\nabla_x \phi| \qquad \text{a.e.}$$

$$f_{ext} = \sqrt{\mu/\rho} \qquad \text{a.e. } t = \hat{T}(x).$$

(7) From (3), $|\nabla_x \phi| = 1$ for fixed t.

(8) Calculate

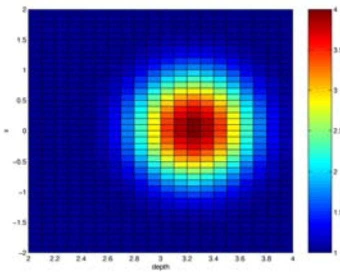
$$\sqrt{\frac{\mu}{\rho}}(x) = \frac{\phi(x, \hat{T}(x) + \Delta t) - \cancelto{0}{\phi(x, \hat{T}(x))}}{\Delta t}, \qquad \text{(1st order)}$$

or

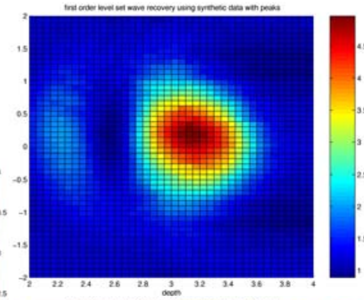
$$\sqrt{\frac{\mu}{\rho}}(x) = \frac{1}{2\Delta t}\left[\phi(x, \hat{T}(x) + \Delta t) + \phi(x, \hat{T}(x) - \Delta t)\right]. \qquad \text{(2nd order)}$$

Recovery with Synthetic Data Using Level Set (first order) Method

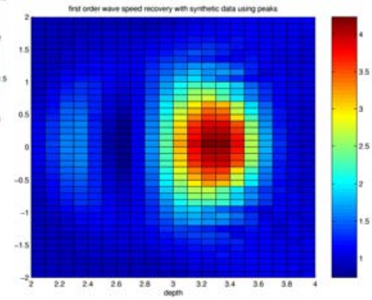
Using Peak Locations



Level Set Method



Target wave speed



O($N^{3/2}$) Method

Sources of Error

(1) Both peak location and correlation experience some distortion due to backscattering

(2) Low amplitude compression wave treated as noise in the data

(3) Use of ultrasound to determine interior displacement yields errors

(4) Modeling errors – Viscoelastic effects, Anisotropic model

Conclusion for Arrival Time Algorithm

(1) Data is subset of time and space dependent interior displacement subset = arrival time surface

(2) Arrival time determined by correlating feature propagation

(3) *Simplistic Approach* yields poor results

(4) *Distance Method* yields $O(N^{3/2})$ algorithm, N = # of grid pts

(5) *Level set method* yields $O(N \log N)$ algorithm

(6) Final *Total Variation* step is needed to *'clean up'* noisy reconstruction when using real data

(7) Excellent Recovery when wavespeed increases up to 4 times over the background wavespeed

Unsolved Problems

(1) Reduce smoothness assumptions in uniqueness theorem for equations of elasticity – almost complete

(2) Identify more coefficients

(3) Determine uniqueness theorems for *anisotropic* case - choosing models to fit applications

(4) Justify arrival time algorithm for the shear wave in an *elastic system*

(5) Extend arrival time algorithm to *anisotropic* case

(6) Investigate elasticity equations for an *incompressible* medium

(7) Model *viscoelastic* and *nonlinear* effects

Second algorithm: Use Central Frequency Content

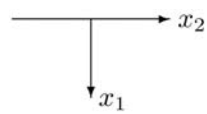
- Applies to Dynamic Sinusoidal Excitation
- Single Frequency

Theorem (Richter) *Let* Ω *be bounded and* $\mu_1, \mu_2 \in \mathcal{C}^1(\bar{\Omega})$ *with* $\mu_1 = \mu_2$ *on* $\partial\Omega$, *and* $g \in \mathcal{C}^0(\partial\Omega)$. *Let* $\hat{u} \in \mathcal{C}^2(\bar{\Omega})$ *be a common solution to* $\nabla \cdot (\mu_j \nabla \hat{u}) + \kappa^2 \hat{u} = 0$ *in* Ω *with* $\mu_j \nabla \hat{u} \cdot \nu = g$ *on* $\partial\Omega$ *for* $j = 1, 2$.

Suppose further that $\hat{u}$ *satisfies* $\inf_{x\in\Omega}\{\max\{|\nabla \hat{u}(x)|, |\Delta \hat{u}(x)|\}\} > 0$. *Then* $\mu_1 = \mu_2$ *in* Ω.

Geometric Optics Algorithm(1)

- 2-D wave equation:



$$\nabla \cdot (c_0^2 \mu \nabla w) - w_{tt} = 0, \quad x_1 > 0$$

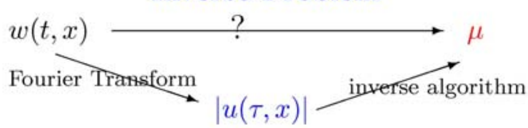
$$\left.\frac{\partial w}{\partial x_1}\right|_{x_1=0} = g(t, x_2)$$

- The corresponding Helmholtz equation:

$$\nabla \cdot (\mu \nabla u) + (\frac{\tau}{c_0})^2 u = 0, \quad x_1 > 0$$

$$\left.\frac{\partial u}{\partial x_1}\right|_{x_1=0} = f(\tau, x_2)$$

Inverse Problem



Geometric Optics Algorithm(2)

- Single frequency content
- The key:

$$\mu \xleftrightarrow[\text{(approximate relation)}]{\text{algebraic equation}} |u(\tau, x)|$$

- Asymptotic expansion of geometrical optics:

$$u(\tau, x) = a(\kappa, x)e^{i\kappa\phi(x)}, \quad \kappa = \frac{\tau}{c_0}$$

where

$$a \sim a_0(x) + \frac{a_1(x)}{i\kappa} + \frac{a_2(x)}{(i\kappa)^2} + \cdots \quad \text{as} \quad \kappa \to \infty$$

$$|u(\tau, x)| \sim a_0(x)$$

Geometric Optics Algorithm(3)

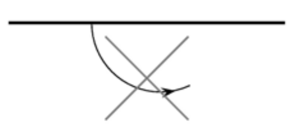
- Eikonal equation for the phase: $|\nabla\phi|^2 - \frac{1}{\mu} = 0$
- Transport equation for the amplitude:

$$2\mu\nabla a_0 \cdot \nabla\phi + \mu a_0 \Delta_x \phi + a_0 \nabla\mu \cdot \nabla\phi = 0$$

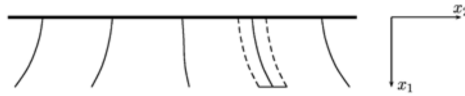
- Characteristic ODE for the Eikonal equation:

$$\frac{dx_2}{dx_1} = \frac{p_2}{p_1}, \quad p_1 = \sqrt{\frac{1}{\mu} - p_2^2} > \alpha_1 > 0, \quad \vec{p} := \nabla\phi,$$

$$\frac{dp_2}{dx_1} = \frac{1}{2p_1}\frac{\partial}{\partial x_2}\frac{1}{\mu}.$$



Geometric Optics Algorithm(4)



ODE-algebraic system ($\dot{}$ $= d/dx_1$)

$$\dot{x}_2 = \frac{p_2}{p_1}, \qquad p_1 = \sqrt{\frac{1}{\mu} - (p_2)^2} \geq \alpha_0 > 0,$$

$$\dot{p}_2 = \frac{1}{2p_1}\frac{\partial}{\partial x_2}\left(\frac{1}{\mu}\right),$$

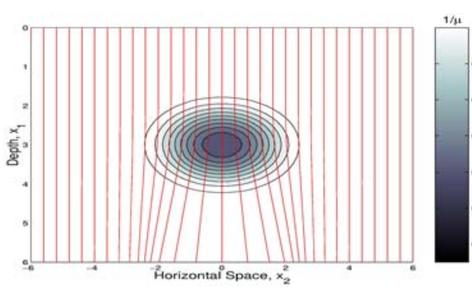
$$\dot{\begin{pmatrix} J \\ A \end{pmatrix}} = \vec{F}\left(J, A, p_2, \frac{1}{\mu}, \frac{\partial}{\partial x_2}\frac{1}{\mu}, \frac{\partial^2}{\partial x_2^2}\frac{1}{\mu}\right),$$

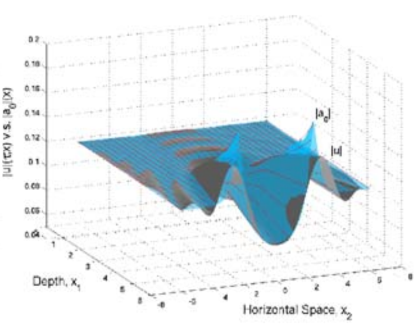
where $x_2(0) = y_2,\ p_2'(0) = 0,\ J(0) = 1,\ A(0) = 0.$

$$\frac{\mu(x_1, x_2)a_0^2(x_1, x_2)p_1(x_1, x_2)J(x_1)}{\mu(0, y_1)a_0^2(0, y')p_1(0, y')} = 1$$

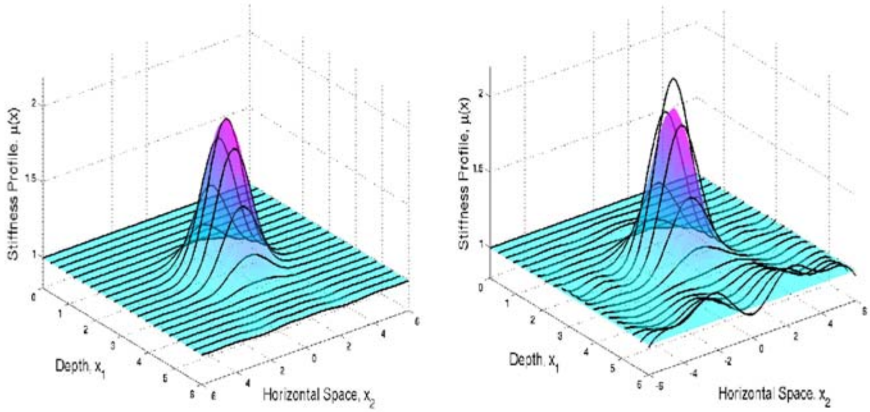
$$a_0 \sim |u|$$

Numerical Experiment (Mild Stiffness)



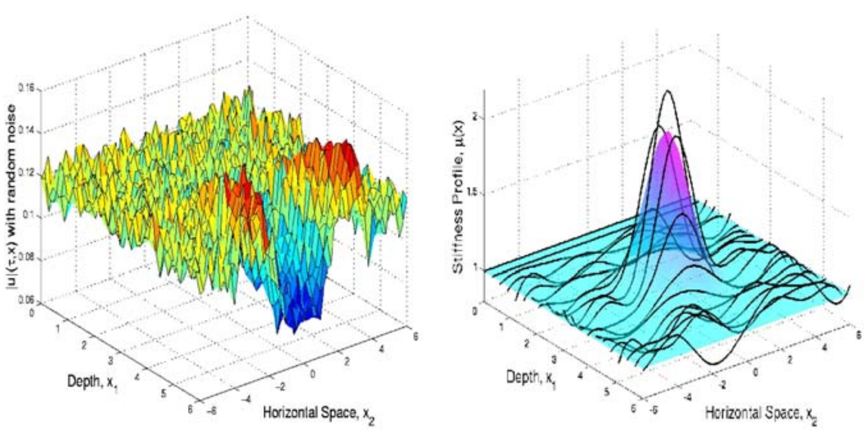


Recovery From Exact Data



The recovered μ from the exact a_0 (left) and the data, $u(\tau, x)$ (right).

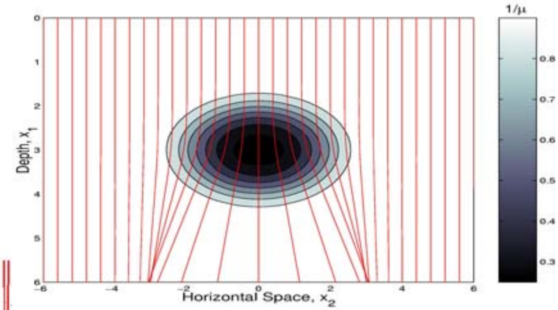
Recovery From Noisy Data

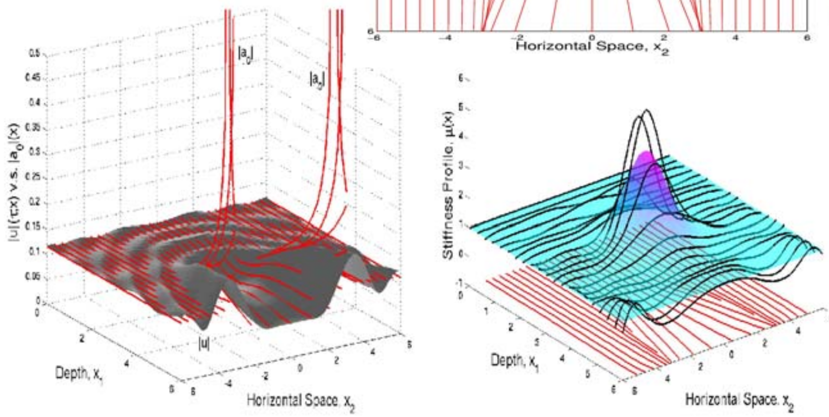


The recovered μ (right) from the data, $u(\tau, x)$, with random noise (left).

The random noise is created as $\tilde{w}(t, x) = w(t, x) + 0.05 w_{max} r(t, x)$.

Numerical Experiment
(Strong Stiffness)





Conclusion for Single Frequency Algorithm

(a) Successful when wavespeed increases up to ~1.7 times the background wavespeed

(b) Robust to noise

(c) Large backscattering limits application of this algorithm (so far)

(d) Algorithm needs to be advanced to include *caustics*

Unsolved Problems

(1) Reduce smoothness assumptions in uniqueness theorem for equations of elasticity – almost complete

(2) Identify more coefficients

(3) Determine uniqueness theorems for *anisotropic* case
- choosing models to fit applications

(4) Justify arrival time algorithm for the shear wave in an *elastic system*

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(6) Investigate elasticity equations for an *incompressible* medium

(7) Model *viscoelastic* and *nonlinear* effects