Time reversal and Imaging in clutter

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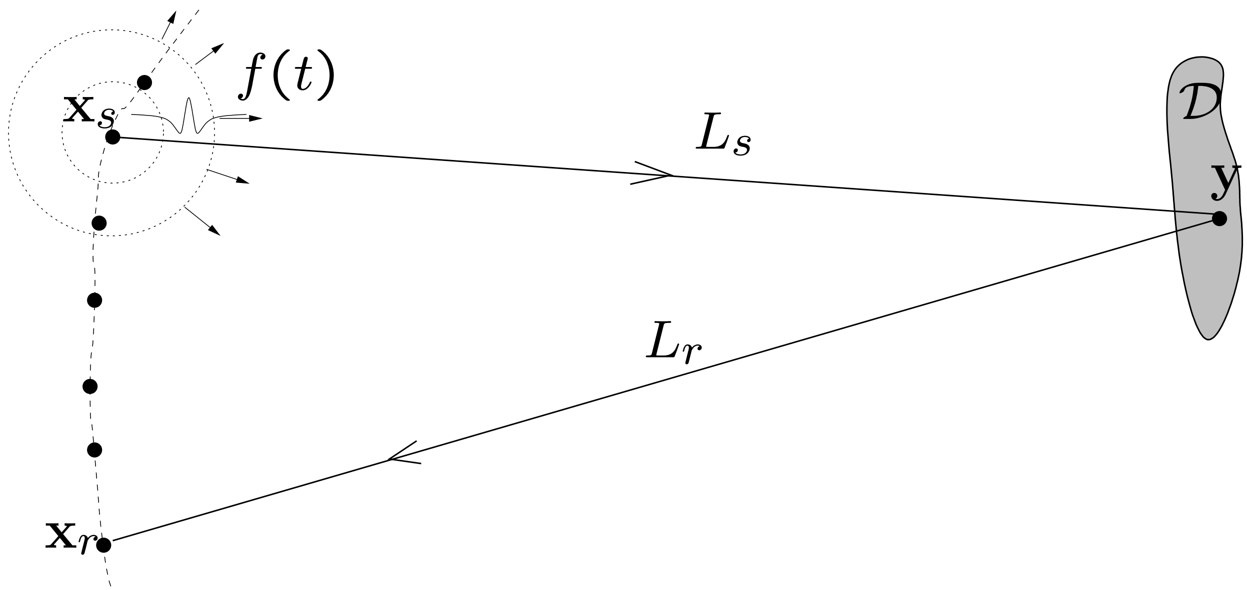


Array data: $P(\mathbf{x}_s, \mathbf{x}_r, t)$ for $(\mathbf{x}_s, \mathbf{x}_r, t)$ a set of source-receiver locations in $R^2 \times R^2$ and time in R_+. A five-dimensional parametrization of the data.

Different data acquisition geometries: Synthetic aperture imaging (zero-offset, large linear apertures, broadband), Ultrasonic imaging arrays (many sources and receivers, broadband signals), etc.

Look carefully at resolution and noise issues.

Born approximation (single scattering for the reflector only)

$$P(\mathbf{x}_s, \mathbf{x}_r, t) \approx \int d\omega \frac{\omega^2 \widehat{f}_B(\omega - \omega_o)}{2\pi c_0^2} e^{-i\omega t} \int_{\mathcal{D}} d\mathbf{y}\, \varrho(\mathbf{y}) \widehat{G}(\mathbf{x}_s, \mathbf{y}, \omega) \widehat{G}(\mathbf{x}_r, \mathbf{y}, \omega)$$

The reflectivity $\varrho(\mathbf{y})$, $\mathbf{y} \in \mathcal{D}$ is the contrast in the index of refraction that is to be determined.

The background, outgoing random Green's function satisfies

$$\Delta \widehat{G}(\mathbf{x}, \mathbf{y}, \omega) + k^2 n^2(\mathbf{x}) \widehat{G}(\mathbf{x}, \mathbf{y}, \omega) = -\delta(\mathbf{x} - \mathbf{y}) \quad \text{in } \mathbb{R}^3, \; k = \frac{\omega}{c_o}$$

with index of refraction:

$$n(\mathbf{x}) = \frac{c_0}{c(\mathbf{x})} \left(1 + \sigma \mu(\frac{\mathbf{x}}{l})\right)^{1/2}$$

The (scaled) random fluctuations $\mu(\mathbf{x})$ are statistically homogeneous, have mean zero and finite correlations. The strength of the fluctuations is σ and the their size is l (simplest model of mono-scale fluctuations).

The probing pulse

The pulse $e^{-i\omega_0 t} f_B(t)$, with Fourier transform $\widehat{f}_B(\omega - \omega_0)$, has bandwidth B (support of $\widehat{f}_B$ is $2B$) and carrier frequency ω_0.

Broadband is important in imaging, as are large aperture arrays.

In a homogeneous background $n(\mathbf{x}) = 1$ and the Green's function $\widehat{G}$ is

$$\widehat{G}_0(\mathbf{x}, \mathbf{y}, \omega) = \frac{e^{ik|\mathbf{y}-\mathbf{x}|}}{4\pi|\mathbf{y} - \mathbf{x}|} = \frac{e^{i\omega\tau(\mathbf{y},\mathbf{x})}}{4\pi|\mathbf{y} - \mathbf{x}|}$$

with

$$\tau(\mathbf{y}, \mathbf{x}) = \frac{|\mathbf{y} - \mathbf{x}|}{c_0}$$

the travel time in the homogeneous medium.

Structure of the inverse problem

Primary goal: To reconstruct the reflectivity $\varrho(\mathbf{y})$, $\mathbf{y} \in \mathcal{D}$. This is often the contrast in propagation speed in the scatterer. When is the Born approximation adequate?

Secondary goal: To reconstruct the background propagation velocity $c(\mathbf{y})$: velocity estimation. This may involve interfaces that are mostly known up to a few undetermined parameters. Needed primarily for an accurate reconstruction of the reflectivity using arrival times.

Secondary goal: To characterize the overall properties of the clutter, that is, the random fluctuations in the propagation speed. Needed for a good statistical stabilization of the data and for assessing blurring.

Synthetic aperture imaging or zero offset migration

Same source/receiver locations (zero offset). Sum over all sources the migrated, or back-propagated, weighted data

$$I^{KM}(\mathbf{y}^S) = \sum_{\mathbf{x}_s} \frac{y_3^S}{|\mathbf{y}^S - \mathbf{x}_s|} P(\mathbf{x}_s, \mathbf{x}_s, 2\tau(\mathbf{x}_s, \mathbf{y}^S))$$

Here $2\tau(\mathbf{x}_s, \mathbf{y}^S)$ is the travel time from $\mathbf{x}_s$ to the search point $\mathbf{y}^S$ and back. This is the Kirchhoff Migration imaging functional, whose Fourier form is

$$I^{KM}(\mathbf{y}^S) = \frac{1}{2\pi} \int d\omega \sum_{\mathbf{x}_s} \frac{y_3^S}{|\mathbf{y}^S - \mathbf{x}_s|} \overline{\widehat{P}(\mathbf{x}_s, \mathbf{x}_s, \omega)} e^{2i\omega\tau(\mathbf{x}_s, \mathbf{y}^S)}$$

If the array is large and the bandwidth is also large then $I^{KM}(\mathbf{y}^S)$ reconstructs well the reflectivity $\varrho(\mathbf{y}^S)$ in deterministic homogeneous media. This is the fundamental theorem of synthetic aperture imaging.

Note that non-zero offset data, if available, are not needed here.

How does SAI perform in clutter?

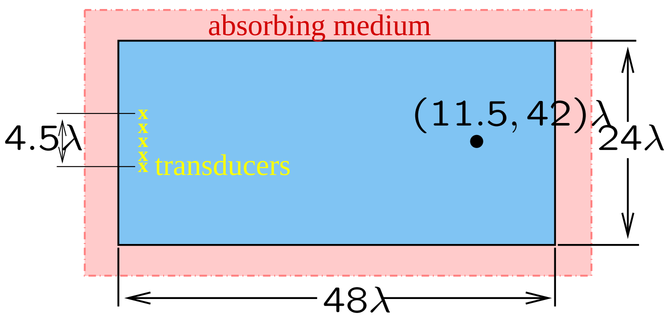
Not well: $I^{KM}(\mathbf{y}^S)$ has large random phases from $\widehat{G}(\mathbf{x}_s, \mathbf{y}, \omega)$ in

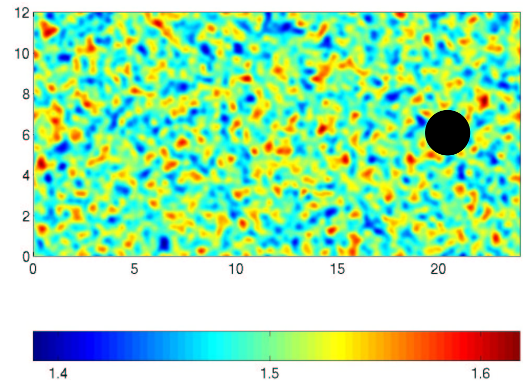
$$I^{KM}(\mathbf{y}^S) \approx \sum_{\mathbf{x}_s} \frac{y_3^S}{|\mathbf{y}^S - \mathbf{x}_s|} \int d\omega \frac{\omega^2 \widehat{f}_B(\omega - \omega_o)}{2\pi c_0^2} e^{-2i\omega\tau(\mathbf{x}_s, \mathbf{y}^S)} \cdot \int_{\mathcal{D}} d\mathbf{y}\, \varrho(\mathbf{y}) \widehat{G}(\mathbf{x}_s, \mathbf{y}, \omega) \widehat{G}(\mathbf{x}_s, \mathbf{y}, \omega)$$

We have a statistically unstable imaging functional that cannot be fixed by summing the data over larger arrays.

We need to use offset data.

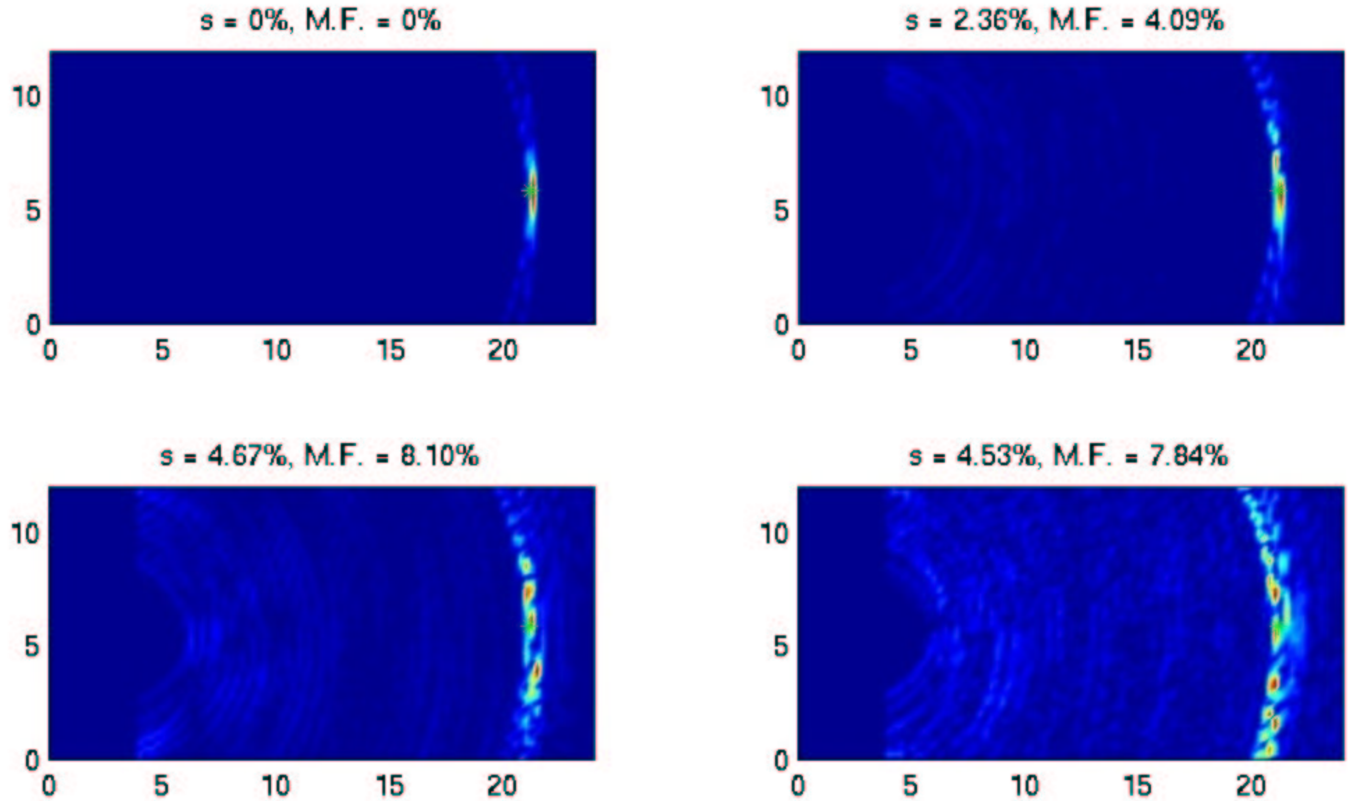
What about background velocity estimation for good travel times? This needs offset data too, as is done extensively in seismic imaging.



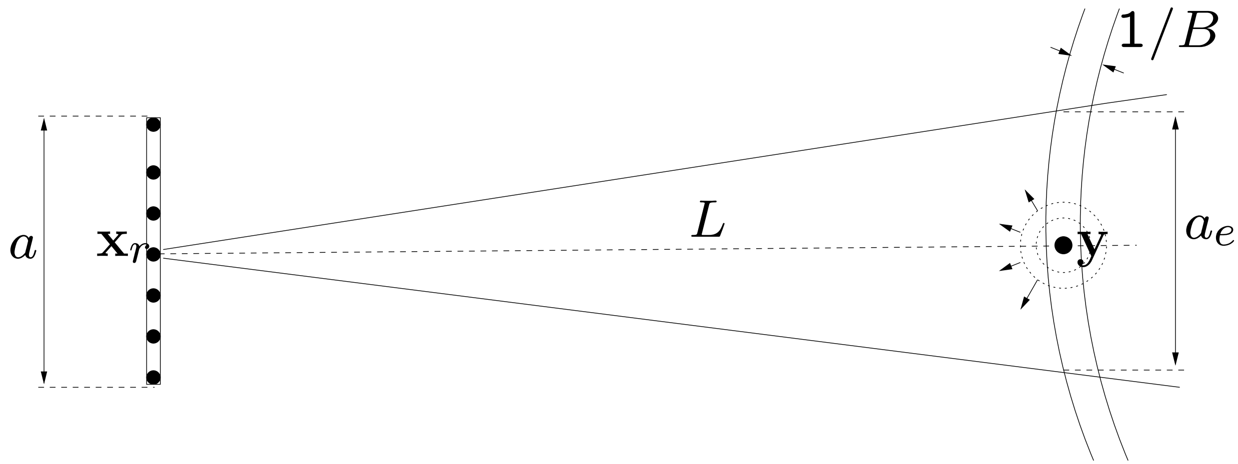


On the left, the dimensions of the problem are in terms of the central wavelength $\lambda = 0.5$mm. The medium is infinite in all directions so in the numerical computations an absorbing layer surrounds the domain.

On the right is a typical realization of the random sound speed $c(\mathbf{x})$. The target is shown as a large black dot ●. The units in the horizontal and vertical axes are mm and, in the color bar, km/s. The standard deviation is $s = 4.95\%$



Imaging a passive point scatterer with the SAI functional I^{KM}. The standard deviation is s Note the statistical instability of the imaging functional.



A source at $\mathbf{y}$ emits a pulse received at $\mathbf{x}_r$. Recorded data is:

$$\widehat{P}(\mathbf{x}_r, \omega) = \widehat{f}_B(\omega - \omega_0)\widehat{G}(\mathbf{x}_r, \mathbf{y}, \omega)$$

If we knew the Greens's function (travel time) of the medium then we would consider the imaging functional

$$\Gamma^{TR}(\mathbf{y}^S) = \int d\omega \sum_{\mathbf{x}_r} \overline{\widehat{P}(\mathbf{x}_r, \omega)}\widehat{G}(\mathbf{x}_r, \mathbf{y}^s, \omega)$$

This is TIME REVERSAL: Spot size around the source location is $\frac{\lambda_0 L}{a_e}$, with $a_e >> a$ the effective aperture. Super-resolution.

Mimic time reversal by computing cross-correlations of data traces, the interferograms

$$I^{INT}(\mathbf{y}^S) = \sum_{\mathbf{x}_r, \mathbf{x}_{r'}} P(\mathbf{x}_r, \cdot) *_t P(\mathbf{x}_{r'}, -\cdot)\big|_{\tau(\mathbf{x}_r, \mathbf{y}^S) - \tau(\mathbf{x}_{r'}, \mathbf{y}^S)}$$

The interferograms are self-averaging. We are doing Differential Kirchhoff Migration on the lag of the interferograms.

In the frequency domain we have

$$I^{INT}(\mathbf{y}^S) = \int d\omega \left| \sum_{\mathbf{x}_r} \widehat{P}(\mathbf{x}_r, \omega) e^{-i\omega\tau(\mathbf{x}_r, \mathbf{y}^S)} \right|^2$$

But this is almost Matched Field Imaging

$$I^{MF}(\mathbf{y}^S) = \int d\omega \left| \sum_{\mathbf{x}_r} \overline{\widehat{P}(\mathbf{x}_r, \omega)} \widehat{G}_0(\mathbf{x}_r, \mathbf{y}^S, \omega) \right|^2$$

This is because

$$\widehat{G}_0(\mathbf{x}_r, \mathbf{y}^S, \omega) = \frac{e^{i\omega\tau(\mathbf{x}_r, \mathbf{y}^S)}}{4\pi|\mathbf{x}_r - \mathbf{y}^S|}$$

So to image, we use the Green's function of the homogeneous reference medium because we do not know the random Green's function $\widehat{G}(\mathbf{x}_r, \mathbf{y}, \omega)$. We back-propagate with the best available information: in a homogeneous medium. But we back-propagate interferograms of the data.

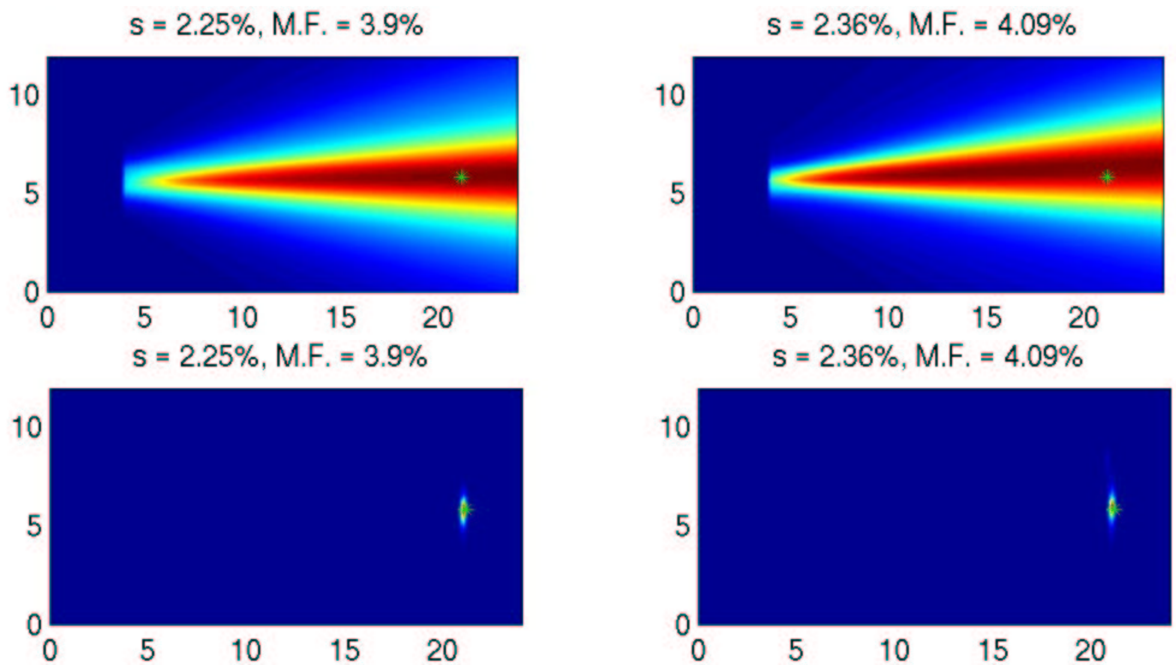
We can calculate approximately the form of this imaging functional. If $\mathbf{y} = (\mathbf{0}, L)$ and $\mathbf{y}^S = (\xi^S, L + \eta^S)$ then

$$I^{INT} \approx C e^{-\frac{|\xi^S|^2}{2(L+\eta^S)^2}\left(\frac{L}{a_e(L)}\right)^2}$$

This gives the cross-range in the cone with slope a_e/L. Range is lost and must be estimated with windowing of interferograms, and partly by triangulation.

(a)

(b)

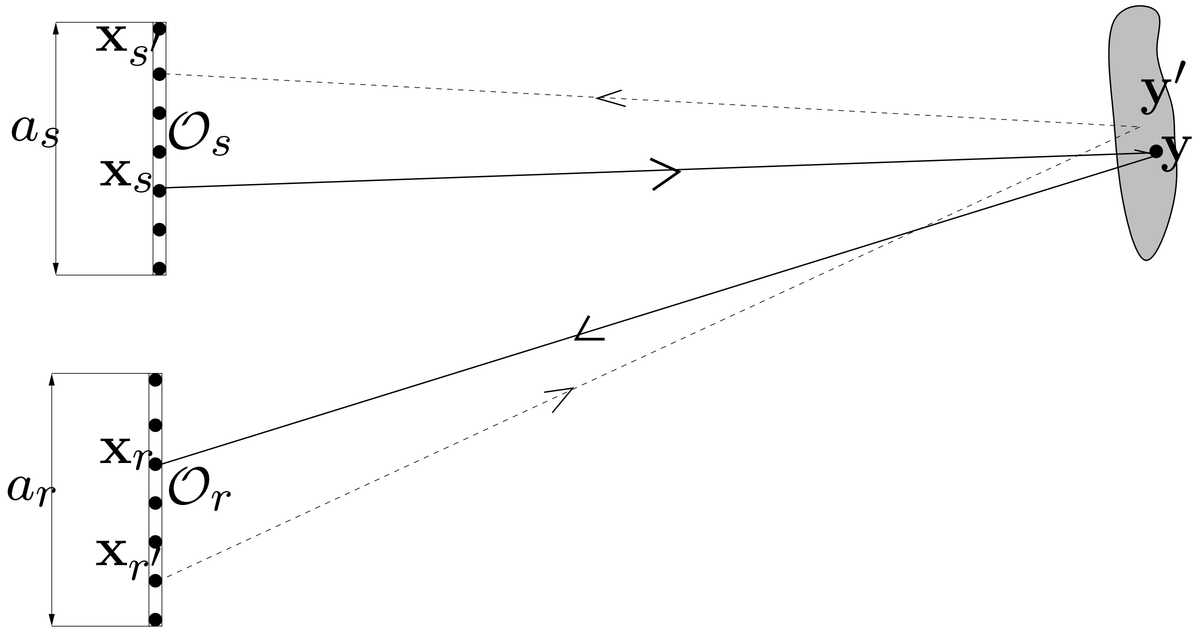


In (a) imaging with the matched field functional for two realizations of the random medium with standard deviation $s = 2.25\%$ (left) and $s = 2.36\%$ (right). In (b) we combine the matched field functional with arrival time estimation for the same realizations of the random medium.

Note the statistical stability of the imaging functionals.

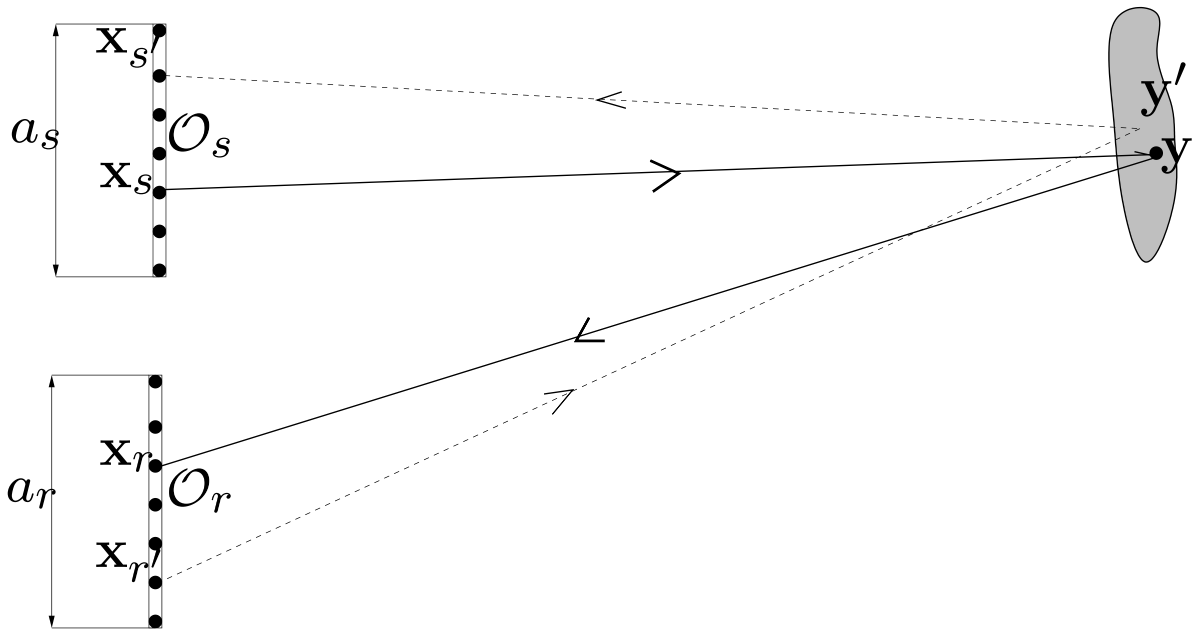
Loss of resolution from clutter: how can we control it?

- We need to use large arrays
- We should use only a small part of the (possibly synthetic) array data to stabilize statistically the information they carry
- Introduce a new method for processing array data to stabilize clutter effects: Coarse-grain the data with a Local Data Covariance (LDC) reduction
- TRANSFORM ARRAY IMAGING IN CLUTTER TO A DETERMINISTIC IMAGE DEBLURRING PROBLEM



Construct the local matched field functional of the data for small arrays around $\mathcal{O}_s, \mathcal{O}_r$. That is, compute

$$I^{LDC}(\mathcal{O}_s, \mathcal{O}_r, \mathbf{y}^S) = \sum_{s,s';r,r'} \frac{1}{2\pi} \int d\omega e^{-i\omega(\tau_{sr}(\mathbf{y}^S) - \tau_{s'r'}(\mathbf{y}^S))} \cdot \widehat{P}(\mathbf{x}_s, \mathbf{x}_r, \omega) \overline{\widehat{P}(\mathbf{x}'_s, \mathbf{x}'_r, \omega)}$$



The theoretical model for this is

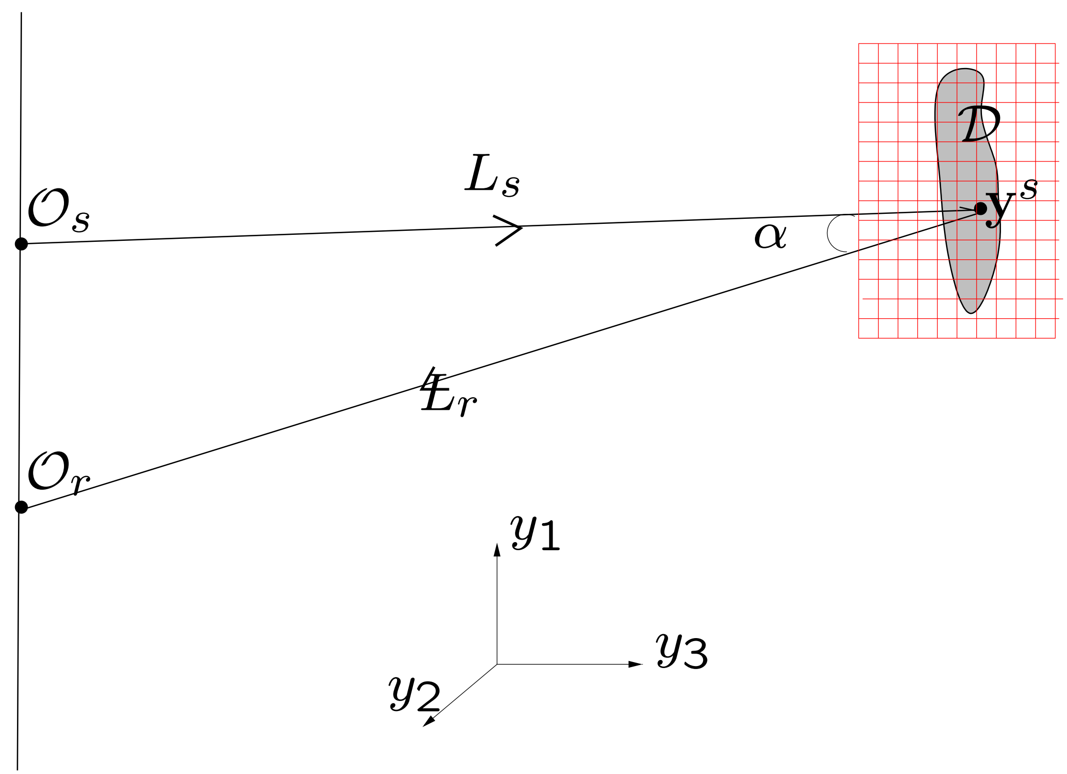
$$I^{LDC}(\mathcal{O}_s, \mathcal{O}_r, \mathbf{y}^S) = \sum_{s,s';r,r'} \int d\omega \frac{\omega^4}{2\pi c_0^4} |\widehat{f}_B(\omega - \omega_0)|^2 e^{-i\omega(\tau_{sr}(\mathbf{y}^S) - \tau_{s'r'}(\mathbf{y}^S))}$$

$$\cdot \int_{\mathcal{D}\times\mathcal{D}} d\mathbf{y} d\mathbf{y}' \varrho(\mathbf{y}) \varrho(\mathbf{y}') \widehat{G}(\mathbf{x}_s, \mathbf{y}, \omega) \overline{\widehat{G}(\mathbf{x}'_s, \mathbf{y}', \omega)} \widehat{G}(\mathbf{x}_r, \mathbf{y}, \omega) \overline{\widehat{G}(\mathbf{x}'_r, \mathbf{y}', \omega)}$$

In a remote sensing regime $a_s, a_r << L$, in random media the coarse-grained LDC imaging functional is

$$I^{LDC}(\mathcal{O}_s, \mathcal{O}_r, \mathbf{y}^S) \approx C(\mathcal{O}_s, \mathcal{O}_r; f) \cdot$$

$$\int_{\mathcal{D}} d\mathbf{y} (\varrho(\mathbf{y}))^2 e^{-\frac{(y_2 - y_2^S)^2}{2}\left(\frac{1}{(a_e^s)^2} + \frac{1}{(a_e^r)^2}\right) - \frac{(y_1 - y_1^s)^2}{2(a_e^s)2} - \frac{((y_1 - y_1^S)\cos\alpha - (y_3 - y_3^S)\sin\alpha)^2}{2(a_e^r)^2}}$$



- The LDC reduction (local MF's) does not image the reflectivity $\varrho(\mathbf{y})$. It reduces in a precise manner the imaging-in-clutter to an image-deblurring with noise problem.

$$I^{LDC}(\mathcal{O}_s, \mathcal{O}_r, \mathbf{y}^S) = K^{\ominus}_{s,r} * \varrho^2(\mathbf{y}^S) + N_{s,r}$$

- For LDC coarse-graining we need some prior estimate of the effective apertures (clutter effects). The theory tells us: how to estimate effective apertures, how to collect good array imaging data in clutter, and how to do a good LDC coarse-graining

- The key to resolution enhancement in deblurring is having multiple views of the reflectivity from the coarse-grained source-receiver locations $\mathcal{O}_s$ and $\mathcal{O}_r$, a subset of R^4.

Conclusions

- The well-established theory of the random Schrödinger equation is largely sufficient for a careful analysis of the I-LDC course-graining.

- The Local Data Covariance coarse-graining is a controlled and precise method for minimizing the effects of clutter when highly resolved array imaging data is available.

- Research directions: Explore imaging problems that are in-between clean media (coherent array data) and very cluttered media (with only incoherent array data after LDC coarse-graining). Windowing. Sparse data. Numerics.