# Interior Elastodynamics Inverse Problems I:
## *Basic Properties*

**Joyce McLaughlin**

Rensselaer Polytechnic Institute

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## Interior Elastodynamics Inverse Problems

Data: Propagating Elastic Wave

Characteristics:

- Initially the medium is at rest
- Time and space dependent interior displacement measurements
- Wave has a propagating front

# Our Application: Transient Elastography

Goal: Create image of shear wave speed in tissue

Characteristics of the application:

- Shear wavespeed increases 2-4 times in abnormal tissue
- Shear wavespeed is 1-3 m/sec in normal tissue
- Interior displacement of wave can be measured with ultrasound (or MRI)
- Ultrasound utilizes compression wave whose speed is 1500 m/sec
- Low amplitude $\rightarrow$ linear equation model
- Wave is initiated by impulse on the boundary $\rightarrow$ data has central frequency
- Data supplied by Mathias Fink



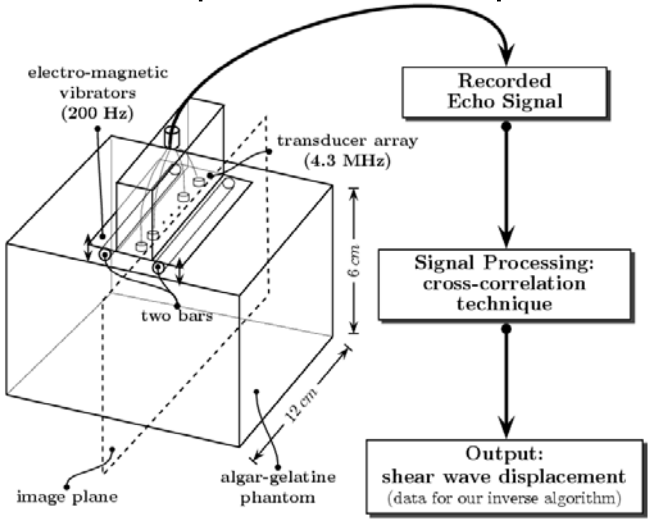
# Compare with

- Static experiment: tissue is compressed (Ophir)
- Dynamic sinusoidal excitation:

  time harmonic boundary source (Parker)
- Our application: Transient elastography

Group members: Lin Ji, Kui Lin, Antoinette Maniatty,
Eunyoung Park, Dan Renzi, Jeong-Rock Yoon

# Experimental Setup




The device that is used by Fink's Lab to excite the target tissue and to measure the shear wave motion at the same time



# Two Bar Transducer

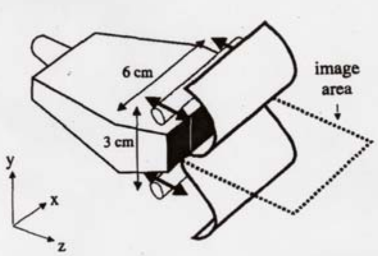



Fig. 12. The shear wave front has a cylindrical shape in the out-of-plane direction.

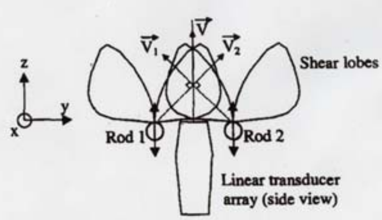



Fig. 7. Shear lobes superimposed in the image area. The transverse waves coming from the rods interfere in the image area and produce a large displacement vector V in the z-direction.

# Cross Correlation

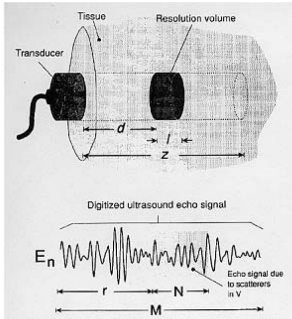



Fig. 1. Digitized RF echo $E_n$ reflected from tissue scatterers within the ultrasound beam. If the digitized echo is $M$ points long, the footprint corresponding to a resolution volume at a distance $d$ from the transducer will be located at a position $r$ within the echo.

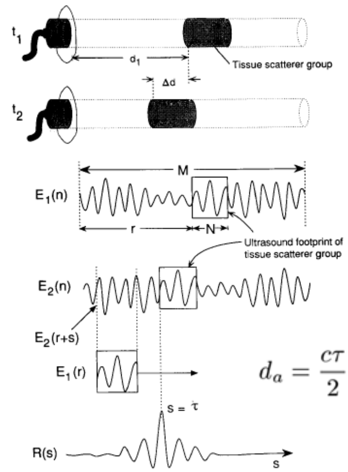



Fig. 4. Digital tracking of a tissue volume by locating the maximum in the normalized correlation function. A window of width $N$ at a desired range $r$ is removed from one echo $E_1$ and is correlated at different locations s along another echo $E_2$. The value of $s$ producing the maximum correlation coefficient $R(s)$ corresponds to $s = \tau$.

Correlation Coefficient:

$$R_{12}(s) = \frac{\sum_{i=0}^{N-1} E_1(r+i)E_2(r+s+i)}{\sqrt{\sum_{j=0}^{N-1}[E_1(r+j)]^2 \sum_{j=0}^{N-1}[E_2(r+s+j)]^2}}$$



# Mathematical Models

$$\begin{aligned}
&\nabla(\lambda\nabla\cdot\vec{u}) + \nabla\cdot\left(\mu(\nabla\vec{u} + (\nabla\vec{u})^T)\right) = \rho\vec{u}_{tt} && (x,t)\in\Omega\times(0,T),\\
&\vec{u} = \vec{u}_t = 0 && x\in\Omega,\quad t=0,\\
\text{either}\ &\left[\lambda(\nabla\cdot\vec{u})I + \mu(\nabla\vec{u} + (\nabla\vec{u})^T)\right]\cdot\vec{n} = \vec{f} && (x,t)\in\partial\Omega\times(0,T),\\
\text{or}\ &\vec{u} = \vec{g} && (x,t)\in\partial\Omega\times(0,T).
\end{aligned}$$

OR

$$\begin{aligned}
&\nabla\cdot(\mu(x)\nabla u) = \rho u_{tt} && (x,t)\in\Omega\times(0,T),\\
&u = u_t = 0 && x\in\Omega,\quad t=0,\\
&\mu\frac{\partial u}{\partial n} = f \quad \text{or} \quad u = g, && (x,t)\in\partial\Omega\times(0,T).
\end{aligned}$$

Data: $f$ or $g$ and $u(x,t)$

Goal: Find $\mu/\rho$ and/or $(\lambda+2\mu)/\rho$ and/or $\rho$

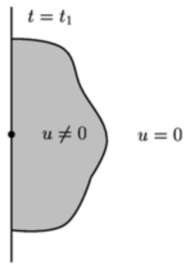
# How rich is the data set?

**Theorem**. Let $\rho_j \in C^0(\overline{\Omega}), \mu_j$ : Lipschitz continuous, $\mu_j, \rho_j \geq \alpha_0 > 0$.
Let $u \in H^2(\Omega \times (0,T))$ satisfy for $f \in H^{3/2}(\partial\Omega \times (0,T))$ :

$$\begin{aligned} \nabla \cdot (\mu_j(x)\nabla u) &= \rho_j u_{tt} && \Omega \times (0,T), \\ u = u_t &= 0 && x \in \Omega, \quad t = 0, \\ \mu_j \frac{\partial u}{\partial n} &= f && \partial\Omega \times (0,T). \end{aligned}$$

Then $(\mu_1, \rho_1) = (\mu_2, \rho_2)$ $\quad \forall x$ where $u(x,t) \neq 0$, some $t$.

*Proof*

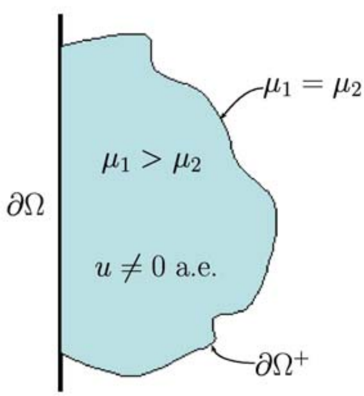



$$\left(\frac{\mu_1}{\rho_1} - \frac{\mu_2}{\rho_2}\right)\Delta u + \left(\frac{\nabla\mu_1}{\rho_1} - \frac{\nabla\mu_2}{\rho_2}\right)\cdot \nabla u = 0$$



# Proof (continued)

Let $\Omega^+ = \{x \in \Omega : \mu_1(x) > \mu_2(x)\}$.




$$c_s^2 = \sqrt{\mu_1/\rho_1} = \sqrt{\mu_2/\rho_2}.$$

$$\begin{aligned} &\int_0^T \int_{\Omega^+} (\mu_1 - \mu_2)\left\{\frac{1}{c_s^2}|u_t|^2 + |\nabla u|^2\right\} dxds \\ =&2\int_0^T \int_{\partial\Omega^+} \int_0^s u_t(\mu_1 - \mu_2)\nabla u \cdot \nu dt dS_x ds \\ =&0. \end{aligned}$$

# Elasticity Case

**Theorem**. Assume $\rho_j \in C^1(\overline{\Omega})$, $\mu_j, \lambda_j \in C^2(\overline{\Omega})$, $\rho_j, \mu_j, \lambda_j \geq \alpha_0 > 0$. Let $\vec{u} \in [H^2(\Omega \times (0,T))]^n$ satisfy:

$$\begin{aligned} \nabla(\lambda_j \nabla \cdot \vec{u}) + \nabla \cdot \left(\mu_j (\nabla \vec{u} + (\nabla \vec{u})^T)\right) &= \rho_j \vec{u}_{tt} && \Omega \times (0,T), \\ \vec{u} = \vec{u}_t &= 0 && x \in \Omega, \quad t = 0, \\ \left[\lambda_j \nabla \cdot \vec{u} I + \mu_j (\nabla \vec{u} + (\nabla \vec{u})^T)\right] \cdot \vec{n} &= \vec{f} && \partial\Omega \times (0,T), \end{aligned}$$

with $\lambda_1/\rho_1 = \lambda_2/\rho_2$ in $\Omega$.

Then $(\mu_1, \rho_1) = (\mu_2, \rho_2)$ $\qquad \forall x$ where $\vec{u}(x,t) \neq 0$, some $t$.

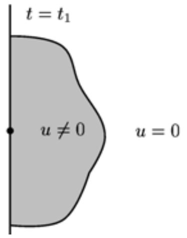


Key Feature: make system of seven equations in $\vec{u}$, $\nabla \cdot \vec{u}$, $\nabla \times \vec{u}$.

Then $(\vec{u}, v, \vec{w}) := (\vec{u}, \nabla \cdot \vec{u}, \nabla \times \vec{u})$ satisfies the following system:

$$\begin{aligned} \vec{u}_{tt} &= \frac{\mu_j}{\rho_j} \Delta \vec{u} + \frac{\lambda_j + \mu_j}{\rho_j} \nabla v + \frac{\nabla \lambda_j}{\rho_j} v + \left(\nabla \vec{u} + \nabla \vec{u}^T\right) \frac{\nabla \mu_j}{\rho_j}, \\ v_{tt} &= \frac{\lambda_j + 2\mu_j}{\rho_j} \Delta v + \frac{1}{\lambda_j + 2\mu_j} \nabla \left( \frac{(\lambda_j + 2\mu_j)^2}{\rho_j} \right) \cdot \nabla v + \nabla \cdot \left( \frac{\nabla \lambda_j}{\rho_j} \right) v \\ &\quad - \frac{1}{\mu_j} \nabla \left( \frac{\mu_j^2}{\rho_j} \right) \cdot \nabla \times \vec{w} + \left(\nabla \vec{u} + \nabla \vec{u}^T\right) \cdot \nabla \left( \frac{\nabla \mu_j}{\rho_j} \right), \\ \vec{w}_{tt} &= \frac{\mu_j}{\rho_j} \Delta \vec{w} + \nabla \left( \frac{\mu_j}{\rho_j} \right) \times (2\nabla v - \nabla \times \vec{w}) + \nabla \left( \frac{1}{\rho_j} \right) \times \nabla (\lambda_j v) \\ &\quad + \nabla \vec{w} \frac{\nabla \mu_j}{\rho_j} + \nabla \left( \frac{\nabla \mu_j}{\rho_j} \right)^T \times \left(\nabla \vec{u} + \nabla \vec{u}^T\right). \end{aligned}$$

# How rich is the data set?




$$\nabla \cdot (\mu(x)\nabla u) = \rho u_{tt}$$

Arrival Time:

$$\hat{T}(x) = \inf\{t \in (0,T) : u(x,t) \neq 0\}$$

Assumptions: $\mu, \rho \in \mathcal{C}^1(\bar{\Omega})$, $0 < \alpha_0 < \sqrt{\mu_1/\rho_1}, \sqrt{\mu_2/\rho_2} < \alpha_1$.

**Theorem**. If $\hat{T}(x)$ is Lipschitz continuous, then

$$|\nabla \hat{T}| = \sqrt{\rho/\mu}, \qquad \text{i.e., } \sqrt{\mu/\rho} = 1/|\nabla \hat{T}|$$

**Theorem**. If $\hat{T}_1(x)$, $\hat{T}_2(x)$ are Lipschitz continuous arrival times corresponding to $\sqrt{\mu_1/\rho_1}$, $\sqrt{\mu_2/\rho_2}$, then

$$\frac{1}{\alpha_1}\int_\Omega \Big||\nabla \hat{T}_1| - |\nabla \hat{T}_2|\Big| dx \leq \int_\Omega |\sqrt{\mu_1/\rho_1} - \sqrt{\mu_2/\rho_2}| dx \leq \frac{1}{\alpha_0}\int_\Omega |\nabla(\hat{T}_1 - \hat{T}_2)| dx.$$

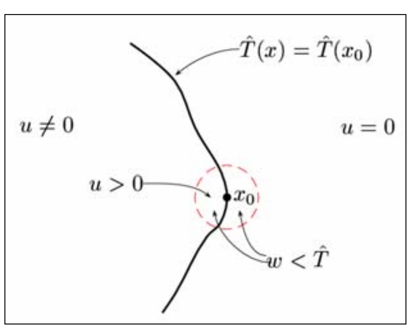


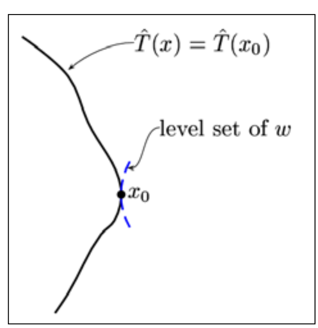
*Proof* Assume $u \in H^2(\Omega \times (0,T))$ for $n = 2$,

$u \in H^{2+\delta}(\Omega \times (0,T))$ for $n = 3$,

$\Rightarrow \quad u \in \mathcal{C}(\bar{\Omega} \times [0,T])$.

Assume $\hat{T}(x)$ is Lipschitz continuous.




$$w \in \mathcal{C}^1, \quad w(x_0) = \hat{T}(x_0), \quad \nabla w(x_0) = \nabla \hat{T}(x_0).$$

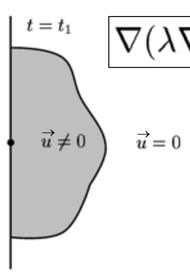
# Theorems we use

**Theorem**: (Evans) Let $\hat{\Omega} \subset \mathbb{R}^n$ be open and $v \in \mathcal{C}^0(\hat{\Omega})$ be differentiable at $x_0 \in \hat{\Omega}$. Then there exists $w \in \mathcal{C}^1(\hat{\Omega})$ with $w(x_0) = v(x_0)$ and $w < v$ in a punctured neighborhood of $x_0$, which implies $\nabla w(x_0) = \nabla v(x_0)$.

**Theorem**: (Eller, Isakov, Nakamura, Tataru) Let $\hat{\Omega} \subset \Omega \subset \mathbb{R}^n$ be open and $(t_1, t_2) \subset (0, T)$. Let $0 < \rho, \mu \in \mathcal{C}^1(\bar{\Omega})$ and $u \in H^2(\Omega \times (0,T))$ be a solution to the wave equation satisfying homogeneous initial condition and one of the Dririchlet/Neumann boundary conditions. Let $S$ be an $n$-dimensional $\mathcal{C}^\infty$ surface in $\hat{\Omega} \times (t_1, t_2)$ defined as in the above remark and assume that $S$ is noncharacteristic with respect to the operator $\nabla \cdot \mu \nabla - \rho \partial_t^2$. Then $u$ satisfies the unique continuation principle in the sense that if $u(x,t) = 0$ for $(x,t)$ near $S$ satisfying $\phi(x,t) > 0$ then also $u(x,t) = 0$ for $(x,t)$ near $S$ satisfying $\phi(x,t) < 0$.



# Equations of Elasticity




$$\nabla(\lambda \nabla \cdot \vec{u}) + \nabla \cdot \left(\mu(\nabla \vec{u} + (\nabla \vec{u})^T)\right) = \rho \vec{u}_{tt}$$

Arrival Time:

$\hat{T}(x) = \inf\{t \in (0,T) : \vec{u}(x,t) \neq 0\}$

Assumptions: $\lambda, \mu, \rho \in \mathcal{C}^1(\bar{\Omega})$, $0 < \alpha_0 < \sqrt{(\lambda + 2\mu)/\rho}, \sqrt{\mu/\rho} < \alpha_1$.

**Theorem**. If $\hat{T}(x)$ is Lipschitz continuous, then

$$|\nabla \hat{T}| = \sqrt{\rho/(\lambda + 2\mu)}, \qquad \text{i.e., } \sqrt{(\lambda + 2\mu)/\rho} = 1/|\nabla \hat{T}|$$

# Nonuniqueness in Anisotropic Media

**Theorem** *Let $U(s) = 0$ for $s < 0$, $\varphi$ be fixed, and $(\rho, \eta, \omega)$ satisfy*

$$\nabla\eta \cdot \nabla^\perp\varphi = -\nabla \cdot \left(\frac{\rho\nabla\varphi}{|\nabla\varphi|^2}\right) \quad \text{and} \quad \rho\omega > \eta^2|\nabla\varphi|^2.$$

*Let $M$ be the symmetric positive-definite matrix represented by*

$$M = \begin{pmatrix} \rho|\nabla\varphi|^{-2} & \eta \\ \eta & \omega \end{pmatrix} \quad \text{with respect to} \quad \left\{\frac{\nabla\varphi}{|\nabla\varphi|}, \frac{\nabla^\perp\varphi}{|\nabla\varphi|}\right\}.$$

*Then the traveling wave $u(x,t) = U(t - \varphi(x))$ solves*

$$\begin{cases} \nabla \cdot \Big(M(x)\nabla u(x,t)\Big) = \rho(x)\partial_t^2 u(x,t), \\ u(\cdot,0) = \partial_t u(\cdot,0) = 0 \quad \text{on } \Omega, \\ u(x,t) = U(t-\varphi(x)) \quad \text{on } \partial\Omega \times (0,T), \\ M\nabla u \cdot \nu = -\dot{U}\left(\rho\frac{\nabla\varphi\cdot\nu}{|\nabla\varphi|^2} + \eta\nabla^\perp\varphi\cdot\nu\right) \quad \text{on } \partial\Omega\times(0,T). \end{cases}$$

- The Neumann boundary data is determined by $(\rho,\eta)|_{\partial\Omega}$ (independent of $\omega$).



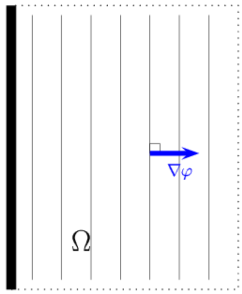
# Sketch of Proof

Find conditions for symmetric positive matrix $M$ so that $u(x,y,t) := U(t-\varphi(x,y)) \in \mathcal{C}^2(\bar{\Omega}\times[0,T])$ solves

$$\nabla \cdot \Big(M(x,y)\nabla u(x,y,t)\Big) = \rho\partial_t^2 u(x,y,t).$$

- $\underbrace{\nabla\varphi\cdot M\nabla\varphi}_{=\rho\ (\star)}\ddot{U} - \underbrace{\nabla\cdot(M\nabla\varphi)}_{=0\ (\star)}\dot{U} = \rho\ddot{U}.$
- $\widehat{M}(x,y) = \begin{pmatrix} \rho(x,y)|\nabla\varphi|^{-2} & \eta(x,y) \\ \eta(x,y) & \omega(x,y) \end{pmatrix}.$
- $(\star) \Longrightarrow -\nabla\cdot(\rho|\nabla\varphi|^{-2}\nabla\varphi) = \nabla\cdot(\eta\nabla^\perp\varphi) = \nabla\eta\cdot\nabla^\perp\varphi.$
- Positive definite $\Longrightarrow \omega > \eta^2|\nabla\varphi|^2.$

# Nonuniqueness Example (The simplest)




- $\Omega = \mathbb{R}^2_+$, $\varphi(x, y) = x$.
- Fix $\rho = 1$, $\eta = 0$.

For any $\omega > 0$, $u(x, y, t) := U(t - x)$ satisfies

$$\nabla \cdot \left( \begin{pmatrix} 1 & 0 \\ 0 & \omega(x, y) \end{pmatrix} \nabla u(x, y, t) \right) = \rho \partial_t^2 u(x, y, t).$$



# Conclusion

1. Data set is rich.
2. Data identifies more than one physical property.
3. Arrival Time is a particularly rich data set.

# Open Questions

1. In the elastic case, how is shear wave front defined and used for identification?
2. What if not all the components of the elastic displacement vector are measured?
3. When additional physical properties are to be determined what are the continuous dependence results?
4. When there is a large discrepancy in wavespeeds an incompressible model may be appropriate. What are the uniqueness and continuous dependence results in this case?