Extracting Macroscopic Data from Microscopic Images -

grain boundaries and macroscopic deformations from images on atomic scale

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Extracting macroscopic data from microscopic images

joint work with:

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Overview

**Macroscopic Parameters from Microscopic Observations**

Transmission electron microscopy
(courtesy G.H. Campell, Lawrence Livermore Nat. Lab.)

Phase field crystal simulation
[Rätz, Voigt '06]
Overview

macroscopic parameters from microscopic observations

transmission electron microscopy (courtesy G.H. Campell, Lawrence Livermore Nat. Lab.)

phase field cristal simulation [Rätz, Voigt '06]

aim: identification of grain boundary contours, orientations, and macroscopic deformation fields
Overview

macroscopic parameters from microscopic observations

- transmission electron microscopy (courtesy G.H. Campell, Lawrence Livermore Nat. Lab.)
- phase field crystal simulation
  
  [Rätz, Voigt '06]

aim: identification of grain boundary contours, orientations, and macroscopic deformation fields

→ a generalized Mumford Shah approach
Recall: Mumford Shah free discontinuity problem

Given \( g : \Omega \rightarrow \mathbb{R} \) find an set \( S \subset \Omega \) and a piecewise smooth \( u : \Omega \setminus S \rightarrow \mathbb{R} \) such that
\[
E[u, S] = \int_{\Omega} (u - g)^2 \, dx + \mu \int_{\Omega \setminus S} |\nabla u|^2 \, dx + \nu \mathcal{H}^{d-1}(S)
\]
is minimized.

[Mumford, Shah '86]
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[Muoford, Shah '86]

Shape optimization perspective

Suppose $\Omega$ is partitioned into domains $\mathcal{O}_i$ ($i = 1, \ldots, m$) with $\Omega = \bigcup_{i=1}^{m} \mathcal{O}_i$, $\mathcal{O}_i \cap \mathcal{O}_j = \emptyset$ and consider
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S = \bigcup_{i=1}^{m} \partial O_i , \quad u_i = u[O_i] = u\big|_{O_i}.
\]

Then we ask for a minimizing partition \((O_i)_{i=1,\ldots,m}\) of

\[
E[(O_i)_{i=1,\ldots,m}] = E[(u[O_i], O_i)_{i=1,\ldots,m}]
\]
Mumford Shah model and shape optimization

General structure of the Mumford Shah functional

$$E[(u_i, O_i)_i] = \sum_i (E_{\text{fid}}[u_i, O_i] + E_{\text{prior}, u}[u_i, O_i] + E_{\text{prior}, O}[O_i])$$

where \((O_i)_i\) is a domain partition

\(u_i\) a parameter (function) on \(O_i\)
Mumford Shah model and shape optimization

General structure of the Mumford Shah functional

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“region competition”: different local descriptors \(u_i\) compete for terrain [Zhu, Yuille '96]
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- a macroscopic orientation parameter describes the microscopic lattice anisotropy
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“region competition”: different local descriptors \(u_i\) compete for terrain [Zhu, Yuille '96]

in our generalized Mumford Shah approach:

- a macroscopic orientation parameter describes the microscopic lattice anisotropy
- a macroscopic deformation influences the lattice pattern
Some related work on pattern analysis

- Unsupervised texture segmentation using Markov random fields [Manjunath, Chellappa ’91]
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Identification of grain boundary at atomic scale

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- ...

Identification of grain boundary at atomic scale

An energy based on local pattern classification

description of the lattice:
Identification of grain boundary at atomic scale

An energy based on local pattern classification

description of the lattice:

\[ x + M(\alpha)q_i \quad \text{atom position} \]
\[ x + M(\alpha)q_i \quad \text{neighbour locations} \]
\[ (i = 1, \ldots, m) \]

\[ M(\alpha) := \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \]

\[ \text{e.g. } q_i := d \left( \cos \left( i \frac{\pi}{3} \right), \sin \left( i \frac{\pi}{3} \right) \right) \]
Identification of grain boundary at atomic scale

An energy based on local pattern classification

description of the lattice:

\[ x \]
atom position

\[ x + M(\alpha)q_i \]
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indicator function for atomic dots:

\[ \chi_{[u>\theta]}(x) := \begin{cases} 
1; & u(x) > \theta \\
0; & \text{else}
\end{cases} \]
description of the lattice:

\[ x \] atom position

\[ x + M(\alpha)q_i \] neighbour locations

\[ (i = 1, \cdots, m) \]

local lattice classification function:
description of the lattice:

\[ x \quad \text{atom position} \]
\[ x + M(\alpha)q_i \quad \text{neighbour locations} \]
\[ (i = 1, \cdots, m) \]

local lattice classification function:

\[ f[\alpha](x) = \frac{d^2}{mr^2} \chi_{[u>\theta]}(x) \sum_{i=1}^{m} (1 - \chi_{[u>\theta]}(x + M(\alpha)q_i)) , \]

where \( d \) distance between atom dots,
\( m \) number of neighbouring dots,
\( r \) dot radius.
macroscopic lattice orientation function:

\[ \alpha = \sum_{j=1, \ldots, n} \alpha_j \chi_{O_j} \]
Identification of grain boundary at atomic scale

An energy based on local pattern classification (cont.)

macroscopic lattice orientation function:

\[
\alpha = \sum_{j=1,\ldots,n} \alpha_j \chi_{\mathcal{O}_j}
\]

Mumford Shah type functional \( E_{\text{grain}} \) acting on lattice orientations \( \alpha_j \) and grain domains \( \mathcal{O}_j \):

\[
E_{\text{grain}}[(\alpha_j, \mathcal{O}_j)_{j=1,\ldots,n}] = \sum_{j=1,\ldots,n} \left( \int_{\mathcal{O}_j} f[\alpha_j](x) \, dx + \frac{\nu}{2} \mathcal{H}^1(\partial \mathcal{O}_j) \right),
\]
macroscopic lattice orientation function:

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\]
direct consequences of the variational approach:

- relation between interface curvature on $O_j \cap O_k$ and fidelity term:
  \[
  \nu \kappa = - (f[\alpha_j] - f[\alpha_k])
  \]

- \[
  0 \leq f[\alpha] \leq \frac{d^2}{r^2} \Rightarrow |\kappa| \leq \frac{d^2}{\nu r^2}
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  \[ 0 \leq f[\alpha] \leq \frac{d^2}{r^2} \Rightarrow |\kappa| \leq \frac{d^2}{\nu r^2} \]

- Young's law at triple points, i.e. three grains always meet at equal angles of $\frac{2}{3} \pi$. 
Recall: Chan-Vese approximation

\[ E[u, S] = \int_{\Omega} (u - g)^2\,dx + \mu \int_{\Omega \setminus S} |\nabla u|^2\,dx + \nu H_{d-1}(S) \]

in the piecewise constant case with \( S = \partial O_1 \cup \partial O_2 \), \( u|_{O_i} \equiv \text{const} \)

we consider \( O_1 = [\phi < 0] = [H(\phi) = 0] \), \( O_2 = [H(\phi) = 1] \) (Heaviside function) and reformulate

\[ E[u_1, u_2, \phi] = \int_{\Omega} H(\phi)( (u_2 - g)^2 + \mu |\nabla u_2|^2 ) + (1 - H(\phi)) ( (u_1 - g)^2 + \mu |\nabla u_1|^2 )\,dx + \nu |D_H(\phi)|_H(\Omega) \]

cf. [Vese, Chan '99]
numerical algorithm for grain boundary extraction

Recall: Chan-Vese approximation

for the original Mumford Shah functional

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(\( H \) heavyside fct.)
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\[ E[u_1, u_2, \phi] = \int_{\Omega} H(\phi)((u_2 - g)^2 + \mu |\nabla u_2|^2) + \]

\[ (1 - H(\phi))((u_1 - g)^2 + \mu |\nabla u_1|^2) \, dx + \nu |D H(\phi)|(\Omega) \]

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we consider \( \mathcal{O}_1 = [\phi < 0] = [H(\phi) = 0], \quad \overline{\mathcal{O}_2} = [H(\phi) = 1] \)

(\( H \) heavyside fct.) and approximate

\[ E_\delta[u_1, u_2, \phi] = \int_{\Omega} H_\delta(\phi)(u_2 - g)^2 + \mu |\nabla u_2|^2 + (1 - H_\delta(\phi))(u_1 - g)^2 + \mu |\nabla u_1|^2 + \nu |\nabla H_\delta(\phi)| \, dx \]

where \( H_\delta(s) = \frac{1}{2} + \frac{1}{\pi} \arctan(\frac{x}{\delta}) \).

cf. [Vese, Chan '99]
numerical algorithm for grain boundary extraction

Grain boundary extraction by a Chan-Vese approach

regularized lattice classification function:

\[ f_\epsilon[\alpha](X) = \frac{d^2}{m r^2} H_\epsilon(u(x) - \theta) \sum_{i=1,\ldots,m} (1 - (H_\epsilon(u(x + M(\alpha)q_i) - \theta))) \]

\( \longrightarrow \) to be motivated later
A regularized lattice classification function:

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→ to be motivated later

In the two grain case we obtain the approximate functional:

\[ E_{\delta,\epsilon}[\alpha_1, \alpha_2, \phi] = \int_\Omega H_\delta(\phi)f_\epsilon[\alpha_2] + (1 - H_\delta(\phi))f_\epsilon[\alpha_1] + \nu |\nabla H_\delta(\phi)| \, dx \]
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where two types of regularization are involved:

- regularization parameter \( \delta \) for the macroscopic interfaces
Grain boundary extraction by a Chan-Vese approach

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\[ \rightarrow \text{to be motivated later} \]

in the two grain case we obtain the approximate functional:

\[ E_{\delta,\epsilon}[\alpha_1, \alpha_2, \phi] = \int_{\Omega} H_{\delta}(\phi) f_{\epsilon}[\alpha_2] + (1 - H_{\delta}(\phi)) f_{\epsilon}[\alpha_1] \]

\[ + \nu |\nabla H_{\delta}(\phi)| \, dx \]

where two types of regularization are involved:

- regularization parameter \( \delta \) for the macroscopic interfaces
- regularization parameter \( \epsilon \) for the microscopic interfaces

\( (\epsilon \leq \delta) \)
the functional to be minimized:

\[ E_{\delta, \epsilon}[\alpha_1, \alpha_2, \phi] = \int_{\Omega} H_\delta(\phi) f_\epsilon[\alpha_2] + (1 - H_\delta(\phi)) f_\epsilon[\alpha_1] + \nu |\nabla H_\delta(\phi)| \, dx \]
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Algorithm (in the two grain case):
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Algorithm (in the two grain case):

- for fixed \( \Phi^k \in \mathcal{V}_h \) update \( \alpha_1^k, \alpha_2^k \) via a discrete version of

\[ \alpha_1^{k+1} = \alpha_1^k - \tau \int_{\Omega} H_\delta(\phi) \partial_{\alpha_1} f_\epsilon[\alpha_1^k] \]

- \( \alpha_2^{k+1} = \alpha_2^k - \tau \int_{\Omega} H_\delta(\phi) \partial_{\alpha_2} f_\epsilon[\alpha_2^k] \)
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Numerical relaxation with regularized gradient descent

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  \[
  \alpha^{k+1}_1 = \alpha^k_1 - \tau \int_{\Omega} H_\delta(\phi) \partial_1 f_\epsilon[\alpha^k_1] \\
  \alpha^{k+1}_2 = \alpha^k_2 - \tau \int_{\Omega} (1 - H_\delta(\phi)) \partial_2 f_\epsilon[\alpha^k] 
  \]

- for given \( \alpha^k_{1,2} \) compute \( \Phi^k \in \mathcal{V}_h \) via a discrete time step of:
  \[
  g \left( H'_\delta(\phi)^{-1} \partial_t \phi, \theta \right) = \int_{\Omega} \nu \frac{\nabla \phi}{|\nabla \phi|} \nabla \theta + (f_\epsilon[\alpha^k_1] - f_\epsilon[\alpha^k_2]) \theta \, dx \, ,
  \]
Numerical algorithm for grain boundary extraction

Numerical relaxation with regularized gradient descent

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Algorithm (in the two grain case):

- for fixed $\Phi^k \in \mathcal{V}_h$ update $\alpha_1^k$, $\alpha_2^k$ via a discrete version of

  $$
  \begin{align*}
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  \end{align*}
  $$

- for given $\alpha_1^k, \alpha_2^k$ compute $\Phi^k \in \mathcal{V}_h$ via a discrete time step of:

  $$
  g \left( H'_\delta(\phi)^{-1} \partial_t \phi, \theta \right) = \int_{\Omega} \nu \frac{\nabla \phi}{|\nabla \phi|} \nabla \theta + (f_\epsilon[\alpha_1^k] - f_\epsilon[\alpha_2^k]) \theta \, dx,
  $$

  where $g(\xi_1, \xi_2) = \int_{\Omega} \xi_1(x) \xi_2(x) + \frac{\sigma^2}{2} \nabla \xi_1(x) \cdot \nabla \xi_2(x) \, dx$
numerical algorithm for grain boundary extraction

Grain boundary extraction on TEM images

[Berkels, Rätz, R., Voigt ’06]
energy decay plot:

The *crosses* mark refinements of the scale parameter $\sigma$. 

The graph shows the energy decay over iterations.
numerical algorithm for grain boundary extraction

Multi-phase grain boundary extraction

Segmentation with multiple grains using three level set functions

[Vese, Chan ’02]
A first generalization: liquid–solid interfaces

An additional liquid–solid interface

Liquid / crystal and grain interface evolution [Rätz, Voigt '06]
A first generalization: liquid–solid interfaces

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Liquid / crystal and grain interface evolution [Rätz, Voigt '06]

Local classification based on the image:

Description of the liquid phase: $u(x) \in [\theta_1, \theta_2]$ and $|\nabla u(x)| < \gamma$
liquid–solid interfaces

An additional liquid–solid interface

liquid / crystal and grain interface evolution [Rätz, Voigt ’06]

local classification based on the image:

description of the liquid phase: $u(x) \in [\theta_1, \theta_2]$ and $|\nabla u(x)| < \gamma$
Incorporating a liquid–solid interface (cont)

local liquid classification function:

\[ g(x) = 1 - \chi_{[u>\theta_1]}(x) \chi_{[u<\theta_2]}(x) \chi_{[|\nabla u|<\gamma]}(x) \]
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Mumford Shah type functional \( E_{\text{phase}} \) acting on liquid domain \( \mathcal{O}_L \):

\[
E_{\text{phase}}[\mathcal{O}_L] = \int_{\mathcal{O}_L} 1 - g(x) \, dx + \int_{\Omega \setminus \mathcal{O}_L} g(x) \, dx + \nu \mathcal{H}^1(\mathcal{O}_L)
\]
Incorporating a liquid–solid interface (cont)

Local liquid classification function:

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Application for a test case
a first generalization: liquid–solid interfaces

Combined model

\[ E[\mathcal{O}_L, (\alpha_j, \mathcal{O}_j)_j] = \int_{\mathcal{O}_L} g(x) \, dx + \int_{\Omega \setminus \mathcal{O}_L} (1 - g(x)) \, dx + \nu \mathcal{H}^1(\partial \mathcal{O}_L) \]

\[ + \eta \sum_{j=1,\ldots,n} \left( \int_{\mathcal{O}_j} f[\alpha_j](x) \, dx + \frac{\nu}{2} \mathcal{H}^1(\partial \mathcal{O}_j) \right). \]
a first generalization: liquid–solid interfaces

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\]

Result on PFC data

[Berkels, Rätz, R., Voigt '06]
description of a deformed lattice:

\[ \psi(x) \] atom position

\[ \psi \] elastic deformation

\[ \psi(x + M(\alpha)q_i) \] neighbor position

local lattice classification function:

\[ f[\alpha, \psi](x) = \frac{d^2}{m r^2} \chi_{[u>\theta]}(\psi(x)) \]
\[ \cdot \sum_{i=1}^{m} (1 - \chi_{[u>\theta]} \psi(x + M(\alpha)q_i)) \]
we have to deal with two types of deformations:

- the observer transformation induced by $M(\alpha)$
The single crystal case

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We are interested in the elastic deformation up to rigid body motions, hence we consider the following functional to be minimized for a single crystal:

$$E_{\text{single}}[\alpha, \psi] = \int_{\Omega} f[\alpha, \psi](x) \, dx$$
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We are interested in the elastic deformation up to rigid body motions, hence we consider the following functional to be minimized for a single crystal:

$$E_{\text{single}}[\alpha, \psi] = \int_{\Omega} f[\alpha, \psi](x) \, dx + \mu E_{\text{elast}}[\psi],$$

with

$$E_{\text{elast}}[\alpha, \psi] = \frac{1}{2} \int_{D} \left| D\psi(x) + D\psi(x)^T - 2 \mathbb{I} \right|^2 \, dx$$

under the constraint for the angular momentum

$$\int_{\Omega} \psi_2(x)x_1 - \psi_1(x)x_2 \, dx = 0.$$

[Berkels, Rätz, R., Voigt '07]
Concentration of energy at the minimizer

consider the Euler Lagrange equations:

\[-2\mu (\Delta \psi(x) + \nabla \text{div} \psi(x)) = \lambda (0 - 1)
\]

where \(\lambda\) is a Lagrange multiplier reflecting the constraint.

For \(\psi\) on \(u \circ \psi \neq 0\):

\[
\left( D\psi^T + D\psi \right) \cdot \nu = d_2 \nabla u \circ \psi^2 m \mu r^2 m \sum_{i=1} \left( 1 - H(u \circ \psi(\cdot) + M(\alpha)q_i - \theta) - H(u \circ \psi(\cdot) - M(\alpha)q_i - \theta) \right)
\]

For the orientation \(\alpha\):

\[
0 = m \sum_{i=1} M'(\alpha)q_i \int \left[ u \circ \psi = \theta \right] H(u \circ \psi(\cdot) - M(\alpha)q_i - \theta) D\psi^T \nabla u \circ \psi dH_1,
\]

where \(M'(\alpha) = 
\[
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]
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Consider the Euler-Lagrange equations:

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For $\psi$ on $[u \circ \psi = 0]$:

$$\left[(D\psi^T + D\psi) \cdot \nu\right] = \frac{d^2 \nabla u \circ \psi}{2m\mu r^2} \sum_{i=1}^{m} \left( 1 - H(u \circ \psi(\cdot + M(\alpha)q_i) - \theta) - H(u \circ \psi(\cdot - M(\alpha)q_i) - \theta) \right)$$
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where $M'(\alpha) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} M(\alpha)$. 
regularized lattice classification function:

\[ f_\epsilon[\alpha, \psi] = \frac{d^2}{mr^2} H_\epsilon(u \circ \psi - \theta) \sum_{i=1}^{m} (1 - (H_\epsilon(u(\psi(\cdot + M(\alpha)q_i)) - \theta))) \]

regularized single grain energy:

\[ E_{\text{single}}^\epsilon[\alpha, \psi] = \int_{\Omega} f_\epsilon[\alpha, \psi](x) \, dx + \mu E_{\text{elast}}[\psi] \]
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and again a regularized descent now in the deformation \( \psi \):

\[ g(\tilde{\psi}^{k+1} - \psi^k, \zeta) = -\tau_{\psi}^k \partial_\psi E_{\text{single}}^\epsilon[\alpha^k, \psi^k](\zeta) \quad \forall \text{variations } \zeta, \]
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\[ \psi^{k+1} = \tilde{\psi}^{k+1} - S(\cdot - x_\Omega), \]

where

\[ S = \frac{1}{2|\Omega|} \int_\Omega D\tilde{\psi}^{k+1} - (D\tilde{\psi}^{k+1})^T \, dx, \quad x_\Omega = \frac{1}{|\Omega|} \int_\Omega \, dx, \]
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and \( g(\zeta_1, \zeta_2) = \int_{\mathcal{D}} \zeta_1(x) \cdot \zeta_2(x) + \frac{\sigma^2}{2} D\zeta_1(x) : D\zeta_2(x) \, dx \)
test case (first row) and real data:
generalization for elastically stressed lattices

Application for the single grain functional

test case (first row) and real data:

courtesy: N. Schryvers (Antwerpen University)
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generalization for elastically stressed lattices

Combined model for elastically deformed grains

joint functional for \((\alpha_j, \Omega_j)_{j=1,\ldots,n}\) and \(\psi\):

\[
E_{\text{joint}}[(\alpha_j, \Omega_j)_{j=1,\ldots,n}, \psi] := \sum_{j=1,\ldots,n} (E_{\Omega_j}[\alpha_j, \psi] + \eta \text{Per}(\Omega_j)) + \mu E_{\text{elast}}[\psi]
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\]

regularized functional in the two grain case:

\[
E_{\text{joint}}^{\delta, \epsilon}[\alpha_1, \alpha_2, \phi, \psi] := \int_{\Omega} H_\delta(\phi) f_\epsilon[\alpha_2, \psi] + (1 - H_\delta(\phi)) f_\epsilon[\alpha_1, \psi] + \nu |\nabla H_\delta(\phi)| \, dx
+ \mu \int_{D} |D\psi(x) + D\psi(x)^T - 2 \mathbb{I}|^2 \, dx.
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Applications for a test case and for real data:
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courtesy: N. Schryvers (Antwerpen University)
Improving the model:
**Outlook**

**Improving the model:**

considering the proper anisotropic elastic regularization:

\[ C_{ijkl}(\alpha) = \sum_{\beta,\gamma,\delta,\eta} C_{ijkl}^{\text{ref}} M(\alpha)_{i\beta} M(\alpha)_{j\gamma} M(\alpha)_{k\delta} M(\alpha)_{l\eta}, \]

where the \( C_{ijkl}^{\text{ref}} \)’s are priori known material parameters.
**Outlook**

- **Improving the model:**
  - considering the proper anisotropic elastic regularization:

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C_{ijkl}(\alpha) = \sum_{\beta,\gamma,\delta,\eta} C_{ijkl}^{ref} M(\alpha)_{i\beta} M(\alpha)_{j\gamma} M(\alpha)_{k\delta} M(\alpha)_{l\eta},
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where the \( C_{ijkl}^{ref} \)'s are priori known material parameters.
  - evaluation of realistic macroscopic stresses

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C_{ijkl}(\alpha) \frac{\nabla \psi + (\nabla \psi)^T}{2}
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    - studying the dynamics of grain boundaries
Improving the model:

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- evaluation of realistic macroscopic stresses

\[ C_{ijkl}(\alpha) \frac{\nabla \psi + (\nabla \psi)^T}{2} \]

- studying the dynamics of grain boundaries

- joining image acquisition and image processing
Anisotropic Cartoon Extraction

A related two scale problem

Given: image $u_0$ dominated by right angle structures, we ask for a cartoon $u$ and an anisotropic classification $\alpha$. 
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A related two scale problem

**Given:** image $u_0$ dominated by right angle structures we ask for a cartoon $u$ and an anisotropic classification $\alpha$. 

![Image](image.png)
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**Anisotropic Cartoon Extraction**

**A related two scale problem**

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$\rightarrow$ Joint extraction and orientation classification
Recall: the classical ROF model

Minimizing

\[ E[u] := \frac{\lambda}{2} \int_{\Omega} (u_0 - u)^2 \, dx + \int_{\Omega} |\nabla u|_2 \, dx \]

gives a cartoon of \( u_0 \). Here \( |x|_2 = \sqrt{x_1^2 + x_2^2} \).

[Rudin, Osher, Fatemi '92]
Anisotropic Cartoon Extraction

Recall: the classical ROF model

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Example

Original

Reconstruction
The anisotropic ROF model

Given an anisotropy $\gamma$, minimizing

$$E_{\gamma}[u] := \frac{\lambda}{2} \int_{\Omega} (u_0 - u)^2 \, dx + \int_{\Omega} \gamma(\nabla u) \, dx$$

[Clarenz, Dziuk, R. ’02], [Esedoglu, Osher ’03]
Given an anisotropy $\gamma$, minimizing

$$E_\gamma[u] := \frac{\lambda}{2} \int_\Omega (u_0 - u)^2 \, dx + \int_\Omega \gamma(\nabla u) \, dx$$

[Clarenz, Dziuk, R. '02], [Esedoglu, Osher '03]

Example

$$\gamma = |\cdot|_1 \quad \text{and} \quad \gamma = |\cdot|_1 \text{ rot. by } \frac{\pi}{4}$$
The anisotropic ROF model

Given an anisotropy $\gamma$, minimizing

$$E_\gamma[u] := \frac{\lambda}{2} \int_{\Omega} (u_0 - u)^2 \, dx + \int_{\Omega} \gamma(\nabla u) \, dx$$

[Clarenz, Dziuk, R. '02], [Esedoglu, Osher '03]

Example

$\gamma = |\cdot|_1$

$\gamma = |\cdot|_1 \text{ rot. by } \frac{\pi}{4}$

$\rightarrow$ let $\gamma$ depend on a coarse scale orientation $\alpha$
Anisotropic Cartoon Extraction

Defining the anisotropic energy

\[ E_{\gamma}[u, \alpha] := \frac{\lambda}{2} \int_{\Omega} |u_0(x) - u(x)|^2 \, dx + \int_{\Omega} |M(\alpha(x)) \nabla u(x)|_1 \, dx, \]

where \[ M(\alpha) := \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \] (rotation by \(-\alpha\))
Anisotropic Cartoon Extraction

Defining the anisotropic energy

\[ E_\gamma[u, \alpha] := \frac{\lambda}{2} \int_\Omega |u_0(x) - u(x)|^2 \, dx + \int_\Omega |M(\alpha(x)) \nabla u(x)|_1 \, dx, \]

where \( M(\alpha) := \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \) (rotation by \(-\alpha\))

Sample result of the final method

\[ u_0 \quad u \quad \alpha \]
Recall: The method shall be able to reconstruct corners
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→ corners are co-dimension two objects
→ Simple Dirichlet type regularization not sufficient.

\[ E_\alpha[\alpha] := \frac{1}{2} \int_\Omega \left( \mu_1 |\nabla \alpha|^2 + \mu_2 |\Delta \alpha|^2 \right) \, dx. \]
Recall: The method shall be able to reconstruct corners
→ corners are co-dimension two objects
→ Simple Dirichlet type regularization not sufficient.

\[ E_\alpha[\alpha] := \frac{1}{2} \int_\Omega (\mu_1|\nabla \alpha|^2 + \mu_2|\Delta \alpha|^2) \, dx. \]

Final model:

\[ E[u, \alpha] = \int_\Omega \frac{1}{2}|u_0 - u|^2 + |M(\alpha) \nabla u|_1 \, dx + E_\alpha[\alpha]. \]
Postprocessing by Bregman iteration

Reconstruction with zero to two Bregman iterations:
Reconstruction with zero to two Bregman iterations:
Reconstruction with zero to two Bregman iterations:
Postprocessing by Bregman iteration

Reconstruction with zero to two Bregman iterations:
Reconstruction with zero to two Bregman iterations:
Reconstruction with zero to two Bregman iterations:
Reconstruction of a corner test data set

From left to right: Original images, isotropic reconstruction, anisotropic reconstruction with zero/two Bregman iterations
Anisotropic Cartoon Extraction

Application on aerial images

\[ u_0 \]

\[ u \]

\[ \alpha \]
Anisotropic Cartoon Extraction

Application on aerial images (cont.)

$u_0$

$u$

$\alpha$
include a shearing transformation:

\[ M_S(\beta) = \begin{pmatrix} \cos \beta & 1 \\ \sin \beta & 0 \end{pmatrix} \]
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\[ M_S(\beta) = \begin{pmatrix} \cos \beta & 1 \\ \sin \beta & 0 \end{pmatrix} \]

\[ M(\alpha, \beta) = M(\alpha)M_S(\beta) \]

\[ = \begin{pmatrix} \frac{\cos \alpha \cos \beta + \sin \alpha}{\sin \beta} & \cos \alpha \\ \frac{\cos \alpha - \sin \alpha \cos \beta}{\sin \beta} & -\sin \beta \end{pmatrix} \]

generalized model:

\[ E[u, \alpha, \beta] = \int_{\Omega} \frac{\lambda}{2} |u_0 - u|^2 + |M(\alpha, \beta)\nabla u|_1 \, dx + E_\alpha[\alpha] + E_\beta[\beta]. \]
first numerical results: