Imaging with the Helmholtz equation

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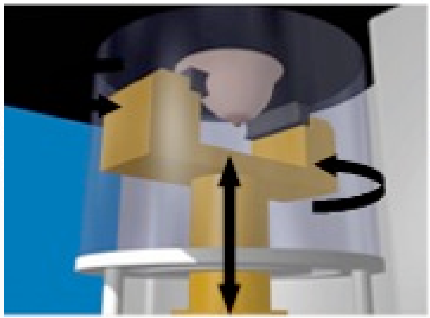
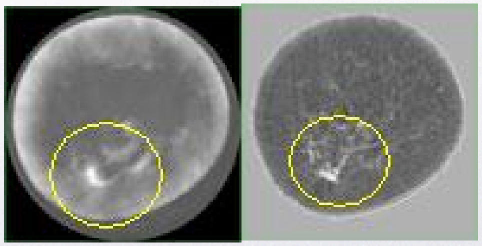
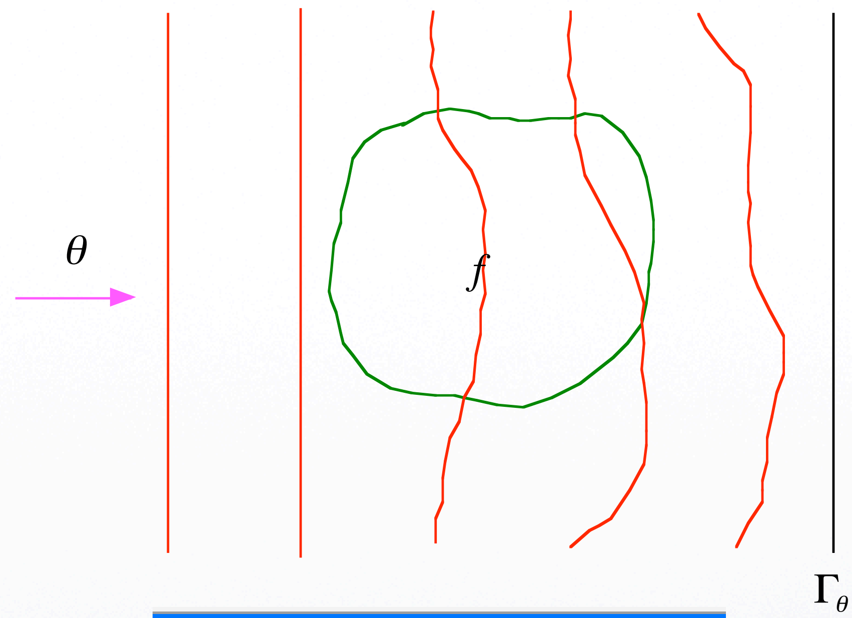
Ultrasound Tomography





Ultrasound tomography

$$\Delta u(x) + k^2(1 + f(x))u(x) = 0,$$
$$u(x) = \exp(ikx\cdot\theta) + u_s(x).$$



Inverse problem: Find f from $u(x)$ for $\Gamma_\theta,\ \theta \in S^1$

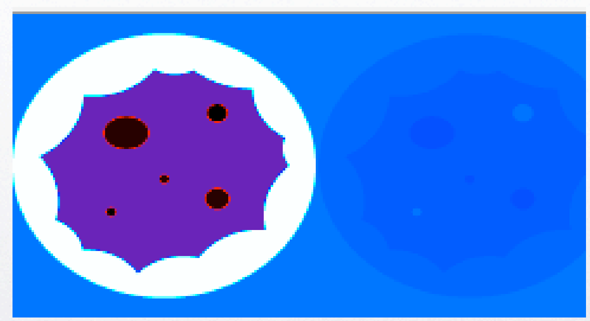
$$f(x) = \frac{c_0^2}{c^2} - 1 - \frac{i}{k}\frac{2\alpha c_0}{c},$$

$c = c(x)$ local speed of sound

c_0 speed of sound in ambient medium

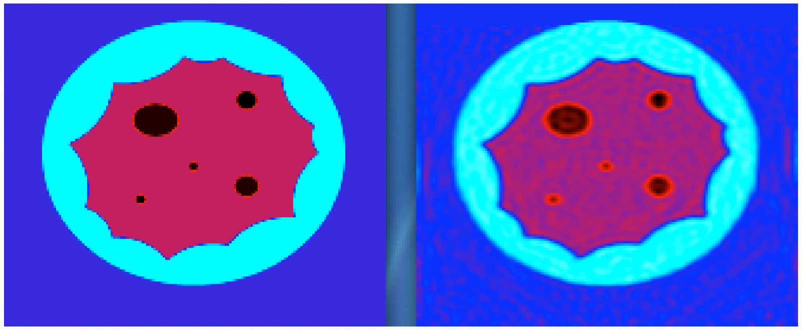
$\alpha = \alpha(x)$ attenuation

$\mathrm{k} = \omega / \mathrm{c}_0$ wavenumber



$\Re f$ $\Im f$

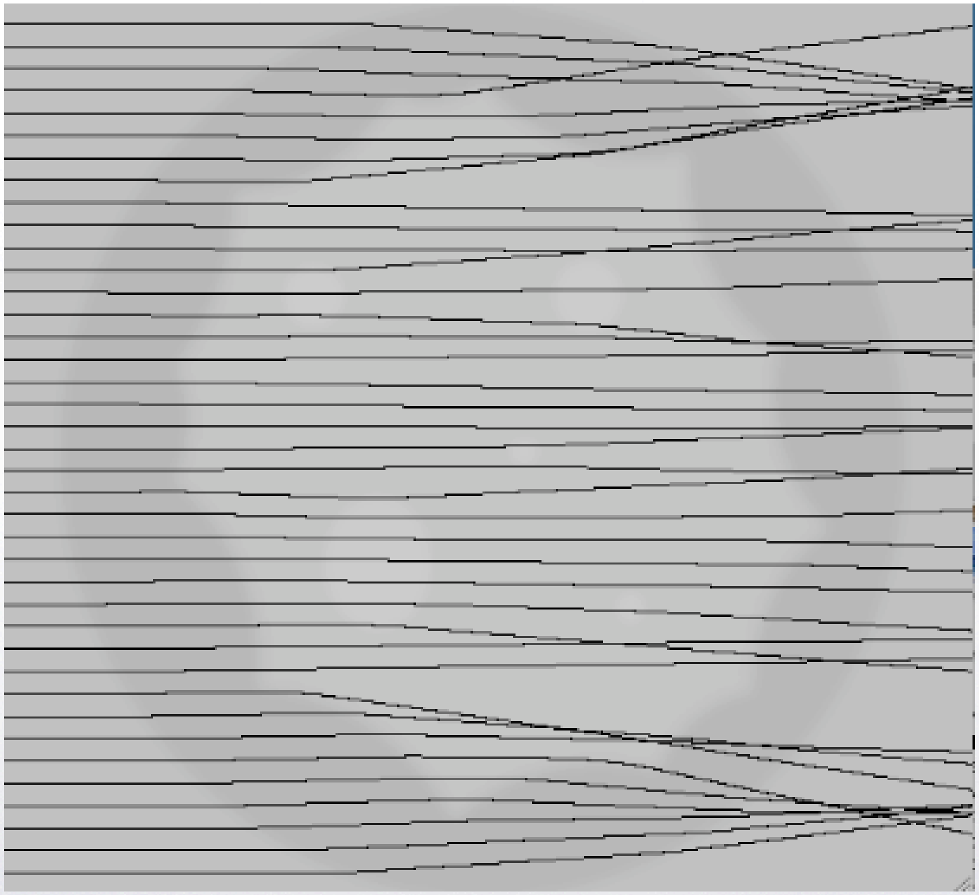
Kaczmarz with plane wave stacking in medical imaging



original reconstruction

Computing time <1 minute on a 3Ghz double processor PC

Rays for the SLC breast phantom



plane wave

$$e^{ikx_3}$$

object

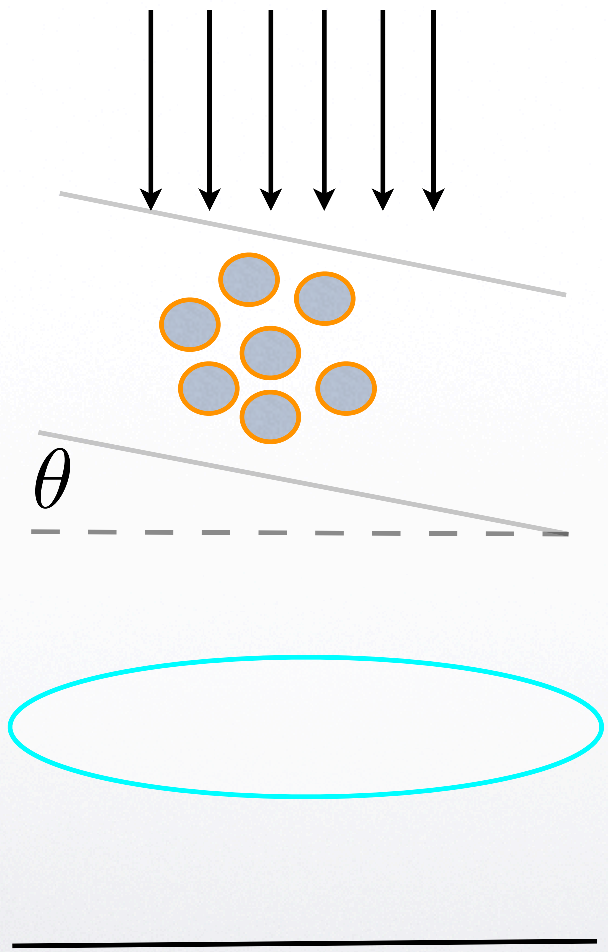
$$f(x) = f_1(x) + i f_2(x)$$

tilt angle

$$\Delta^2 u + k^2 (1 + f) u = 0$$

+radiation condition

lens



detector Γ

$$g_\theta = u|_\Gamma$$

Role of parameter k :

i) k determines the resolution $\lambda = 2\pi/k$

ii) k large makes it difficult to solve the Helmholtz equation numerically

Usual treatment by X-ray transform:

For

$$\gamma r \sup_{|x|<r} |f(x)| < \lambda \qquad \lambda = 2\pi/k$$

$$f(x) = 0 \text{ for } |x| > r$$

we have

$$u(x) \approx \frac{k}{2i} \int f(x + se_3) ds$$

Goal of this talk:

Exact (iterative) solution without any approximation, such as Born, Rytov, WKB,...

Initial Value Problem for the Helmholtz Equation

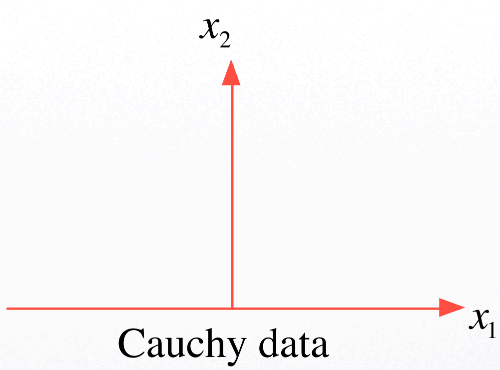
$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + k^2 u = 0 \qquad u(x_1,0) = u_0(x_1),\ \frac{\partial u}{\partial x_2}(x_1,0) = u_1(x_1)$$

Fourier transform with respect to x_1:

$$\hat{u}(\xi_1,x_2) = (2\pi)^{-1/2}\int \exp(-ix_1\xi_1)u(x_1,x_2)dx_1$$

Ordinary differential equation in x_2:

$$\frac{d^2\hat{u}(\xi_1,x_2)}{dx_2^2} + (k^2 - \xi_1^2)\hat{u}(\xi_1,x_2) = 0$$



Solution:

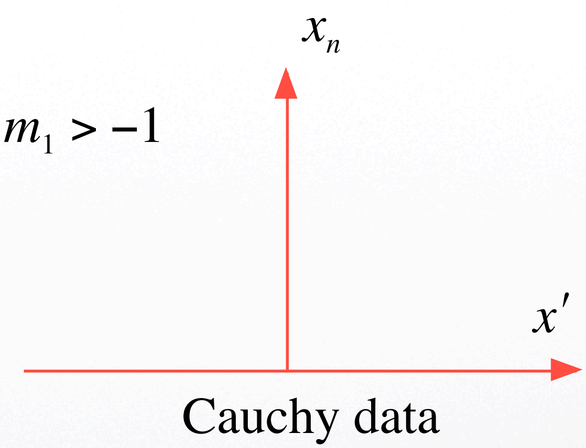
$$\hat{u}(\xi_1,x_2) = \hat{u}_0(\xi_1)\cos(\kappa(\xi_1)x_2) + \frac{\hat{u}_1(\xi_1)}{\kappa(\xi_1)}\sin(\kappa(\xi_1)x_2),\ \kappa(\xi_1) = \sqrt{k^2 - \xi_1^2}$$

Stable as long as $\xi_1^2 \le k^2$

Stability estimates for the Cauchy problem of the inhomogeneous Helmholtz equation

$$\Delta u + k^2(1+f)u = r, \quad x_n > 0, \quad f \geq 1 + m_1, \; m_1 > -1$$

$$u(x',0) = 0, \quad \frac{\partial u(x',0)}{\partial x_n} = 0, \quad x' \in R^{n-1}$$



u_κ = low pass filtered (in x') version with cut-off κ of u

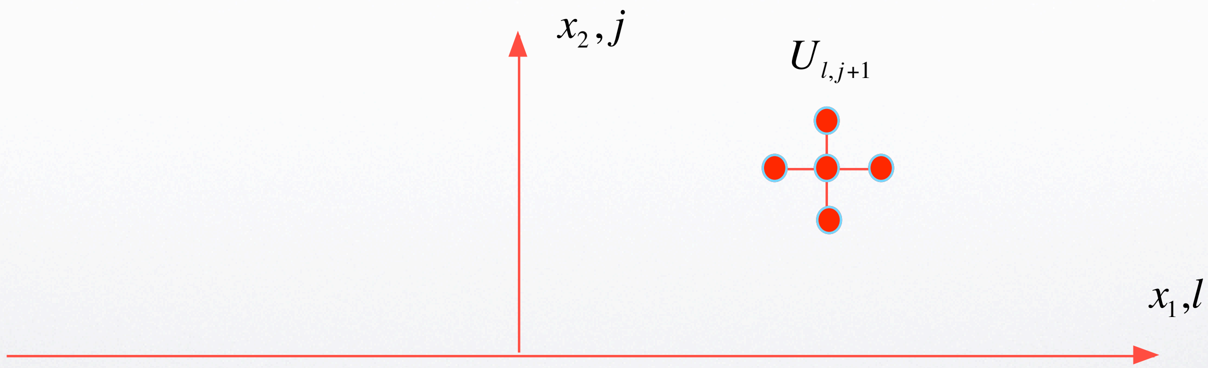
$$\|u_{\kappa\vartheta}(x',x_n)\|_{L^2(R^{n-1})} \leq \frac{c(x_n)}{\kappa\vartheta}\|r\|_{L^2(R^{n-1}\times[0,x_n])}, \quad \kappa = k(1+m_1), \quad 0 < \vartheta < 1$$

Stable marching for the Helmholtz equation

Compute a preliminary value $U_{l,j+1}$ from

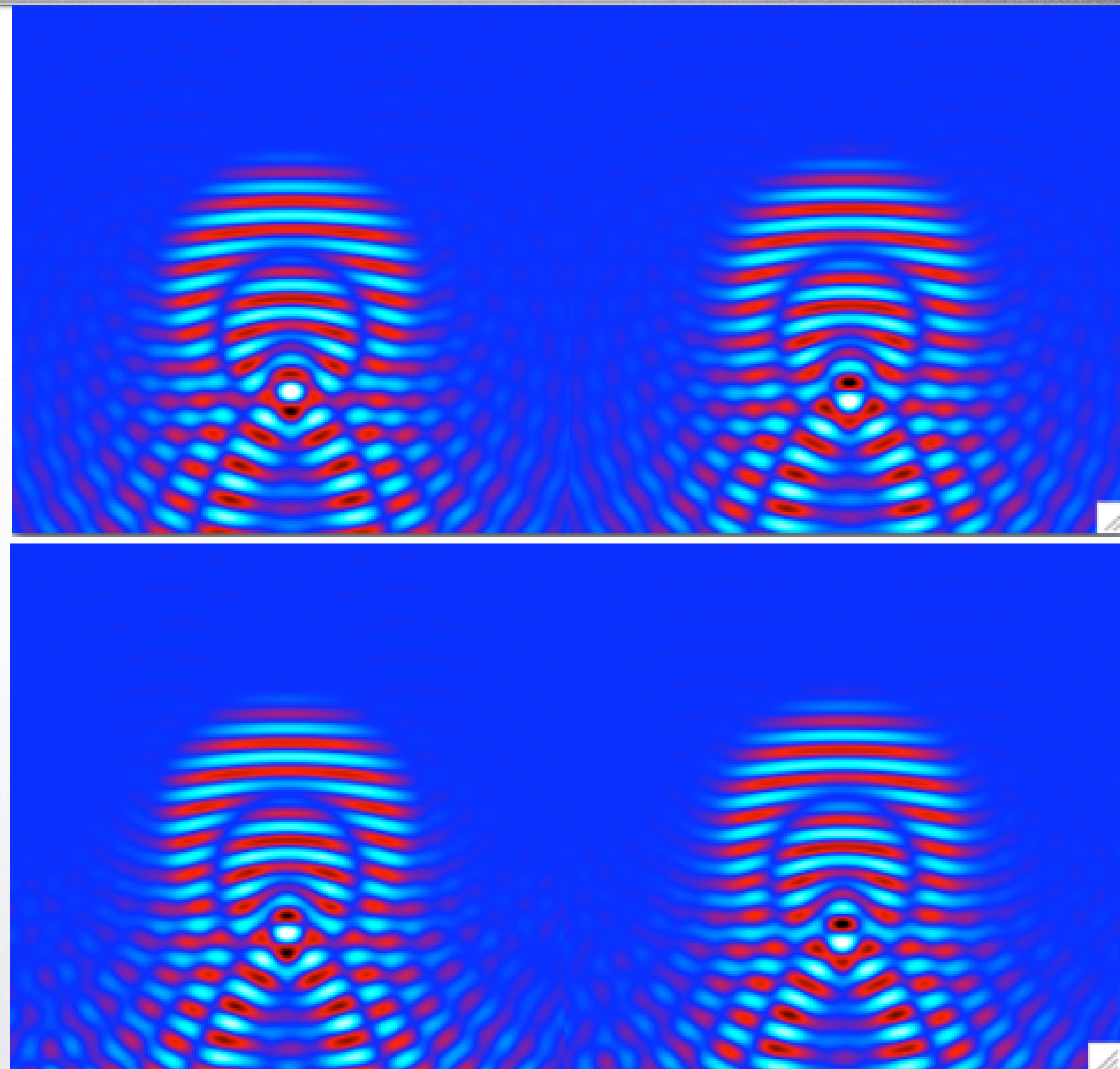
$$-4u_{l,j} + U_{l,j+1} + u_{l,j-1} + u_{l+1,j} + u_{l-1,j} + h^2k^2(1+f_{l,j}) = 0$$

Compute $u_{l,j+1}$ by low pass filtering of $U_{l,j+1}$ with respect to l



$$\frac{\partial u}{\partial x_2}(x_1,0) = i(2\pi)^{-1/2}\int \sqrt{k^2-\xi_1^2}\,\hat{u}(\xi_1,0)\exp(-ix_1\xi_1)\,d\xi_1$$

Exact (finite difference time domain, followed by Fourier transform)



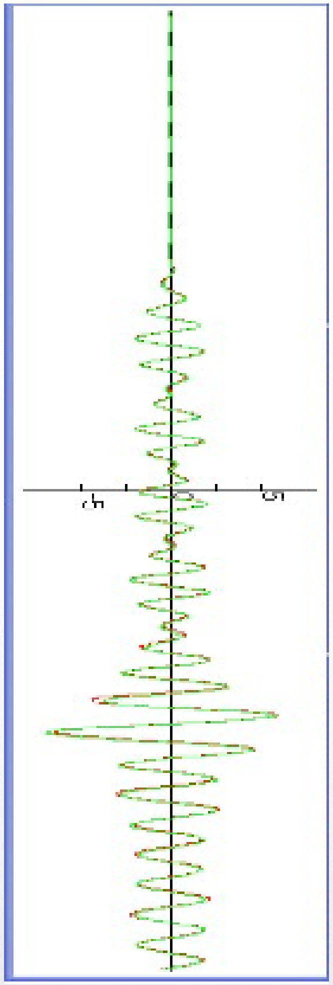
LUNEBERG LENSE

$$f(x) = (1 - |x|^2)_+$$

Initial value technique

Vertical cross section

through real part



green: exact

red: initial value technique

focal point

Kaczmarz' method for Helmholtz equation in frequency domain

Define $R_\theta(f) = u|_{\Gamma_\theta^+}$ where u solution of

$$\Delta u + k^2(1+f)u = 0,\ \ u = g_\theta^-,\ \ \frac{\partial u}{\partial \theta} = \frac{\partial g_\theta^-}{\partial \theta} \text{ on } \Gamma_\theta^-$$

Solve $R_\theta(f) = g_\theta$ for all measured directions θ iteratively:

Update: $f \leftarrow f - \alpha(R_\theta'(f))^*(R_\theta(f) - g_\theta)$

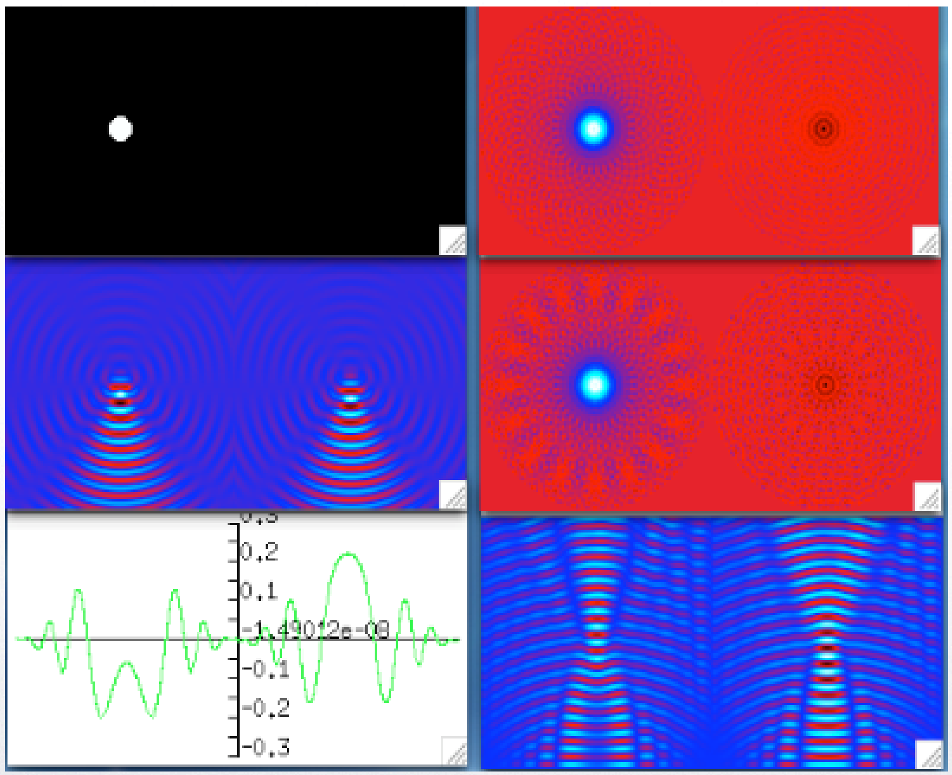
Evalutation of $(R_\theta'(f))^* r$: $\quad (R_\theta'(f))^* r = \overline{z}\overline{u}$

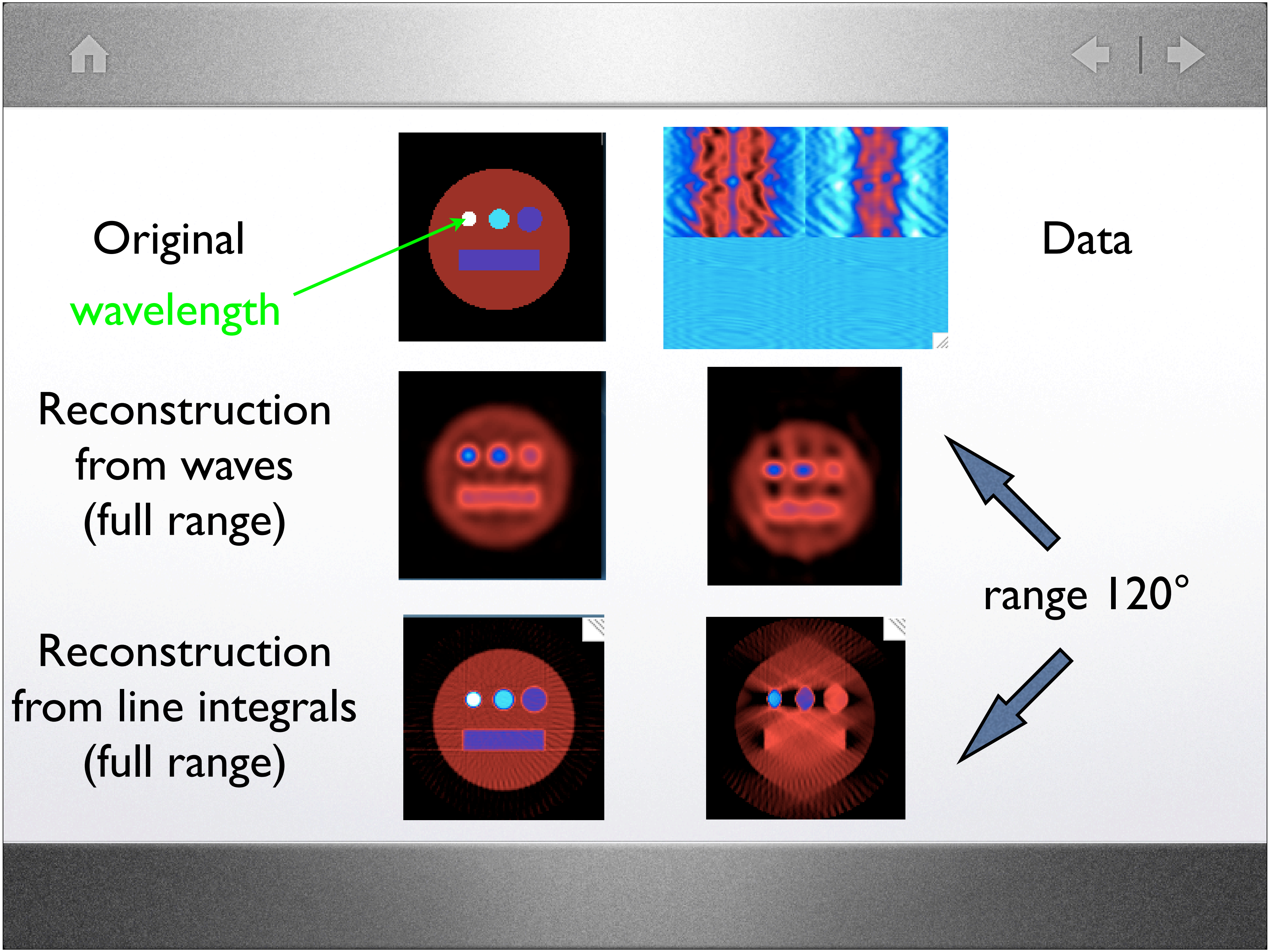
$$\Delta z + k^2(1+f)z = 0$$

$$z = 0,\ \frac{\partial z}{\partial \nu} = \overline{r} \text{ on } \Gamma_\theta^+$$

time reversal
backpropagation
backprojecton
adjoint differentiation
migration

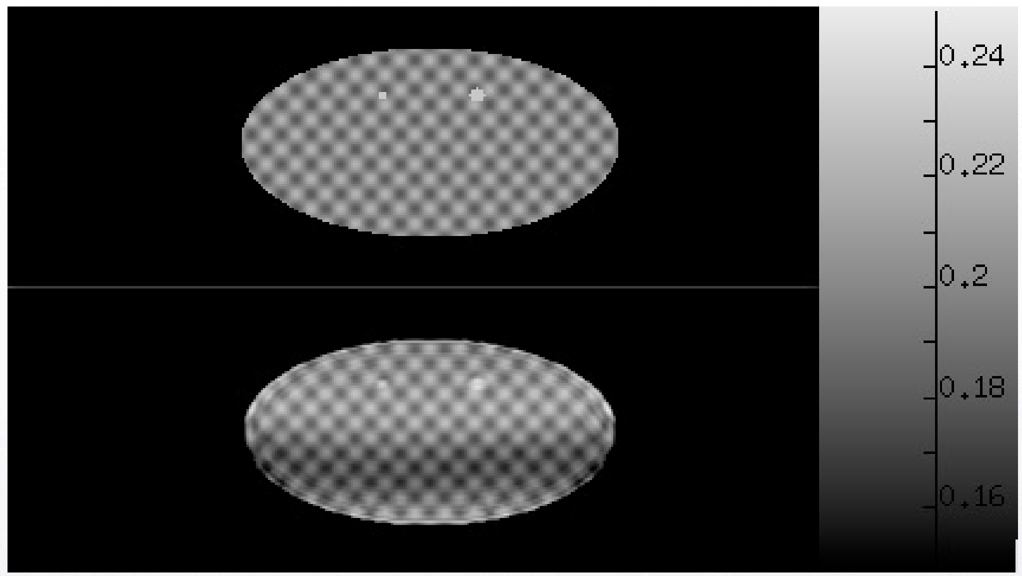
Example for Kaczmarz‘ method for simple object



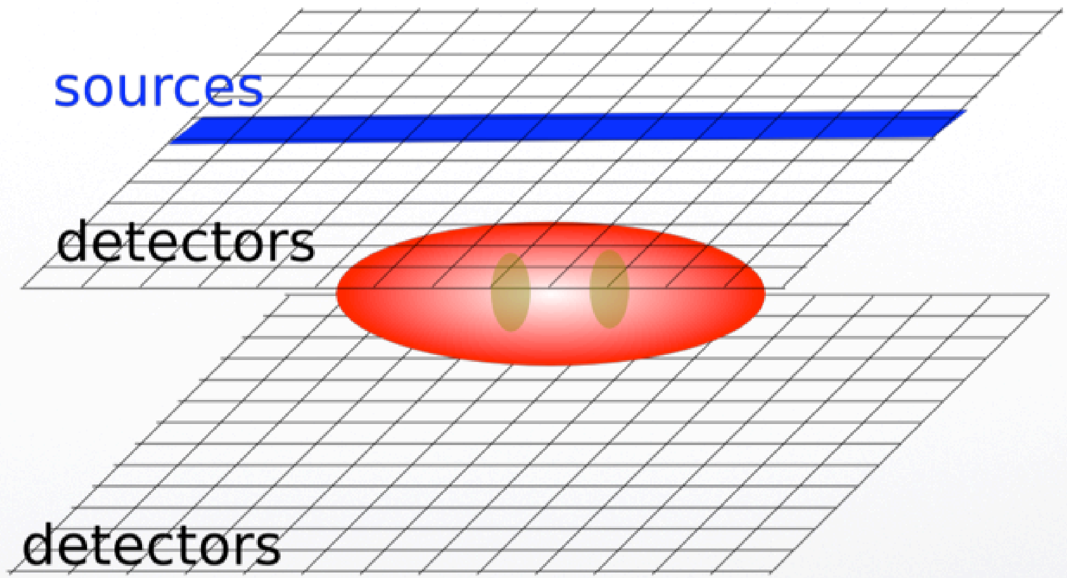

Original
wavelength
Data
Reconstruction
from waves
(full range)
range 120°
Reconstruction
from line integrals
(full range)

Original

Reconstruction



Future:
Going 3D?
Backscatter?



Conclusions

Imaging from waves is possible

Algorithms roughly as efficient as ART in ET

Resolution corresponds to wavelength

Limited angle for waves easier than for ET