Reversibility of Backward SLE Lamination

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Joint work with Steffen Rohde
The Schramm-Loewner Evolution (SLE) is a stochastic process of random conformal maps that has received a lot of attention over the last decade. A number of two-dimensional lattice models have been proved to converge to SLE with different parameters, thanks to the work by Schramm, Lawler, Werner, Smirnov, Sheffield, and many others.
SLE = Loewner’s differential equation + random driving function. We are mainly concerned with the chordal Loewner equation:

\[
\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \quad g_0(z) = z,
\]

where \( z \in \mathbb{C} \), and \( \lambda \in C([0, T], \mathbb{R}) \) is called the driving function. Fix \( \kappa > 0 \), and let \( B(t) \) be a standard Brownian motion. The solution of the chordal Loewner equation with \( \lambda(t) = \sqrt{\kappa}B(t) \) is called chordal SLE\(_\kappa\).
Rohde and Schramm showed that chordal SLE\( \kappa \) generates a random curve called the trace: \( \beta(t), 0 \leq t < \infty \), in the closure of the upper half plane, which satisfies \( \beta(0) = 0 \) and \( \lim_{t \to \infty} \beta(t) = \infty \).

The simplest case is \( \kappa \in (0, 4] \), in which \( \beta \) is a simple curve with \( \beta(t) \in \mathbb{H} = \{ \text{Im} z > 0 \} \) for \( t > 0 \), and for every \( t > 0 \),
\[
g_t : \mathbb{H} \setminus \beta(0, t) \xrightarrow{\text{Conf}} \mathbb{H}.
\]
Adding a minus sign to the (forward) chordal Loewner equation, we get the backward chordal Loewner equation:

$$\partial_t f_t(z) = \frac{-2}{f_t(z) - \lambda(t)}, \quad f_0(z) = z.$$  

Setting $\lambda(t) = \sqrt{\kappa} B(t)$, we then get the backward chordal SLE$_{\kappa}$.

The backward and forward Loewner equations are related as follows. Fix $T_0$ such that $\lambda$ is defined on $[0, T_0]$. Let $\lambda_{T_0}(t) = \lambda(T_0 - t), 0 \leq t \leq T_0$. It is easy to check

$$f_{\lambda_{T_0-t}} \circ (f_{\lambda_{T_0}})^{-1} = g_{\lambda T_0}^t, \quad 0 \leq t \leq T_0.$$  

Taking $t = T_0$, we get

$$(f_{\lambda_{T_0}})^{-1} = g_{T_0}^{\lambda T_0}.$$
A half-open simple curve (as a set) in \( \mathbb{H} \) is called an \( \mathbb{H} \)-simple curve, if its open side approaches a single point on \( \mathbb{R} \). If \( \beta \) is an \( \text{SLE}_\kappa (\kappa \leq 4) \) trace, then \( \beta(0, t] \) is an \( \mathbb{H} \)-simple curve for every \( t \).

Suppose \( \kappa \in (0, 4] \) and \( \lambda(t) = \sqrt{\kappa} B(t) \). Then \( \lambda_{T_0}(t) - \lambda(T_0) \) has the same distribution as \( \lambda(t) \), \( 0 \leq t \leq T_0 \). This together with \( (f^\lambda_{T_0})^{-1} = g^{\lambda_{T_0}} \) and the property of the forward \( \text{SLE}_\kappa \) trace shows that the backward chordal \( \text{SLE}_\kappa \) generates a family of \( \mathbb{H} \)-simple curves \( \beta_t \) such that, for every \( t \), \( f_t : \mathbb{H} \overset{\text{Conf}}{\rightarrow} \mathbb{H} \setminus \beta_t \).
Fix \( t_0 \geq 0 \), and let \( f_{t,t_0}, \; t \geq t_0 \), be the solution of

\[
\partial_t f_{t,t_0}(z) = \frac{-2}{f_{t,t_0}(z) - \lambda(t)}, \quad f_{t_0,t_0}(z) = z.
\]

If \( t_2 > t_1 \), then \( f_{t_2,t_1} : \mathbb{H}^{\text{Conf}} \to \mathbb{H} \setminus \beta_{t_2,t_1} \), where \( \beta_{t_2,t_1} \) is an \( \mathbb{H} \)-simple curve. We have \( f_{t_2,t_1} \circ f_{t_1} = f_{t_2} \), and so \( \beta_{t_2} = \beta_{t_2,t_1} \cup f_{t_2,t_1}(\beta_{t_1}) \).
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\partial_t f_{t,t_0}(z) = \frac{-2}{f_{t,t_0}(z) - \lambda(t)}, \quad f_{t_0,t_0}(z) = z.
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When a forward Loewner process and a backward Loewner process both generate \( \mathbb{H} \)-simple curves, they look very similar at any fixed time. However, if we let time evolve, the difference will be clear.
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The backward chordal SLE$_{\kappa}$ does not naturally generate a single curve because $(\beta_t)$ is not an increasing family. We will study a different object: the conformal lamination.
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Every $f_t$ has a continuous extension from $\mathbb{H}$ to $\overline{\mathbb{H}}$, which maps two real intervals with common end point 0 onto the two sides of $\beta_t$. If $f_t(x_1) = f_t(x_2) \in \beta_t$, then we write $x_1 \sim_t x_2$. If $t_1 < t_2$, from $f_{t_2,t_1} \circ f_{t_1} = f_{t_2}$ we see that $x_1 \sim_{t_1} x_2$ implies that $x_1 \sim_{t_2} x_2$. Thus, we may define a global relation: $x_1 \sim x_2$ if there exists $t > 0$ such that $x_1 \sim_t x_2$. In fact, $x_1 \sim x_2$ iff that the solutions $f_t(x_1)$ and $f_t(x_2)$ blow up at the same time, i.e., $\tau(x_1) = \tau(x_2)$. 
It holds that almost surely, \( \tau(x) < \infty \) for every \( x \in \mathbb{R} \). So we get a random self-homeomorphism \( \phi \) of \( \mathbb{R} \) such that \( \phi(0) = 0 \), \( \phi(\pm \infty) = \mp \infty \), and \( y = \phi(x) \) implies \( x \sim y \). We call such \( \phi \) a backward chordal SLE\(_{\kappa} \) lamination.
It holds that almost surely, $\tau(x) < \infty$ for every $x \in \mathbb{R}$. So we get a random self-homeomorphism $\phi$ of $\mathbb{R}$ such that $\phi(0) = 0$, $\phi(\pm\infty) = \mp\infty$, and $y = \phi(x)$ implies $x \sim y$. We call such $\phi$ a backward chordal SLE$_\kappa$ lamination.

A fundamental property of the forward chordal SLE is reversibility. For $\kappa \leq 8$, the law of the SLE$_\kappa$ trace is invariant under the automorphism $z \mapsto -1/z$ of $\mathbb{H}$, modulo time parametrization. This was first proved for $\kappa \leq 4$ (Z, 2007), and later for $4 \leq \kappa \leq 8$ (Miller and Sheffield, 2012). It is false for $\kappa > 8$. 
Our main theorem is that the backward chordal SLE$_\kappa$ lamination has the following reversibility property.

**Theorem**

Let $\kappa \in (0, 4]$, and $\phi$ be a backward chordal SLE$_\kappa$ lamination. Then $\psi(x) := -1/\phi^{-1}(-1/x)$ has the same distribution as $\phi$. 
Sheffield recently proved that, for $\kappa \in (0, 4)$, there is a coupling of a backward chordal SLE$_{\kappa}$ with a free boundary Gaussian free field in $\mathbb{H}$, such that the GFF determines the backward SLE and a quantum length on $\mathbb{R}$, and for $x < 0 < y$, $\phi(x) = y$ iff $[x, 0]$ and $[0, y]$ have the same quantum length.

Sheffield’s theorem seems to be closely related to our main theorem. However, so far we have not found a way to connect these two results. Instead, the proof of our theorem uses an idea in the proof of the reversibility of forward chordal SLE$_{\kappa}$ for $\kappa \in (0, 4)$.
Let $\kappa \in (0, 4]$. Although a backward chordal SLE$_{\kappa}$ process does not naturally generate a single trace, we may still define a normalized global backward SLE$_{\kappa}$ trace as follows.
Let $\kappa \in (0, 4]$. Although a backward chordal SLE$_{\kappa}$ process does not naturally generate a single trace, we may still define a normalized global backward SLE$_{\kappa}$ trace as follows.

Recall that, for each $t$, $f_t : \mathbb{H} \overset{\text{Conf}}{\longrightarrow} \mathbb{H} \setminus \beta_t$ and $f_t(0)$ is the tip of $\beta_t$. We may find $a_t, b_t \in \mathbb{C}$ such that $F_t = a_t f_t + b_t$ fixes both 0 and $i$. As $t \to \infty$, $F_t$ converges to a conformal map $F_\infty$ defined on $\mathbb{H}$, which also fixes 0 and $i$. It turns out that $F_\infty(\mathbb{H}) = \mathbb{C} \setminus \beta$, where $\beta$ is a simple curve, which joins 0 with $\infty$, and avoids $i$, and $F_\infty$ is a realization of the lamination $\phi$ in the sense that $y = \phi(x)$ implies that $F_\infty(x) = F_\infty(y) \in \beta$. We call this $\beta$ a normalized global backward SLE$_{\kappa}$ trace. We have the following reversibility of $\beta$
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$F_t$

$0$

$i$

$0$

$i$

$f_t$

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Theorem

Let \( \kappa \in (0, 4) \), and \( \beta \) be a normalized global backward chordal SLE\( \kappa \) trace. Let \( h(z) = -1/z \). Then \( h(\beta) \) has the same distribution as \( \beta \).
Theorem

Let $\kappa \in (0, 4)$, and $\beta$ be a normalized global backward chordal SLE$_{\kappa}$ trace. Let $h(z) = -1/z$. Then $h(\beta)$ has the same distribution as $\beta$.

This theorem follows from the main theorem and the fact that the SLE$_{\kappa}$ trace is conformally removable, thank to the work by Jones-Smirnov (a Hölder curve is conformally removable) and Rohde-Schramm (an SLE$_{\kappa}$ trace is a Hölder curve).
We now define the conformal transformation of a backward Loewner process via a conformal map with suitable conditions. For this purpose, we introduce some definitions.

- A relatively closed subset $K$ of $\mathbb{H}$ is called an $\mathbb{H}$-hull, if $K$ is bounded and $\mathbb{H} \setminus K$ is simply connected.

Now assume $K$ is an $\mathbb{H}$-hull. Let $I_{\mathbb{R}}(z) = \overline{z}$ be the reflection about $\mathbb{R}$.

- The base of $K$: $B_K = \overline{K} \cap \mathbb{R}$.
- The double of $K$: $K^{\text{doub}} = K \cup I_{\mathbb{R}}(K) \cup B_K$. 
\( g_K \) is the unique \( g_K : \mathbb{H} \setminus K \xrightarrow{\text{Conf}} \mathbb{H} \) such that 
\( g_K(z) = z + o(1/z) \) as \( \mathbb{H} \ni z \to \infty \).

\( g_K \) extends to a conformal map defined on \( \mathbb{C} \setminus K^{\text{doub}} \).

The support of \( K \): \( S_K = \mathbb{C} \setminus g_K(\mathbb{C} \setminus K^{\text{doub}}) \subset \mathbb{R} \).

The \( \mathbb{H} \)-capacity of \( K \): \( \text{hcap}(K) = \lim_{z \to \infty} z(g_K(z) - z) \geq 0 \).

\( f_K = g_K^{-1} \) is defined on \( \mathbb{C} \setminus S_K \) or its subset \( \mathbb{H} \).
Every $\mathbb{H}$-simple curve is an $\mathbb{H}$-hull, whose base is a single point, and whose support is a real interval. An $\mathbb{H}$-simple curve $\beta$ induces a lamination $\phi_\beta$, which is a self-homeomorphism of $S_\beta$ swapping its two end points, such that $y = \phi_\beta(x)$ implies that $f_\beta(x) = f_\beta(y)$. Note that $f_\beta$ maps the two end points of $S_\beta$ to the base of $\beta$: $\overline{\beta} \cap \mathbb{R}$, and maps the only fixed point of $\phi_\beta$ to the tip of $\beta$.

Let $\kappa \in (0, 4]$, and $(\beta_t)$ be the $\mathbb{H}$-simple curves generated by a backward SLE$_\kappa$ process. Then $f_t = f_{\beta_t}$ for every $t$ and $\bigcup S_{\beta_t} = \mathbb{R}$. The SLE$_\kappa$ lamination $\phi$ satisfies $\phi|_{S_{\beta_t}} = \phi_{\beta_t}$ for each $t$. 
Let $K$ and $L$ be two $\mathbb{H}$-hulls. If $L \subset K$, we define another $\mathbb{H}$-hull: $K/L = g_L(K \setminus L)$, call it a quotient hull of $K$, and write $K/L \preceq K$.

Fact: If $M \preceq K$, then $\text{hcap}(M) \leq \text{hcap}(K)$ and $S_M \subset S_K$. 
Let $K$ and $L$ be two \mathbb{H}\text{-hulls}. If $L \subset K$, we define another $\mathbb{H}\text{-hull}$: $K/L = g_L(K \setminus L)$, call it a quotient hull of $K$, and write $K/L \prec K$.

Fact: If $M \prec K$, then $h\text{cap}(M) \leq h\text{cap}(K)$ and $S_M \subset S_K$. 
Proposition

Let $(\beta_t)$ be a family of $\mathbb{H}$-simple curves. Then they are generated by a backward chordal Loewner equation if and only if

(i) $t_1 < t_2$ implies that $\beta_{t_1} \prec \beta_{t_2}$;

(ii) $(\beta_t)$ is normalized such that $\text{hc}(\beta_t) = 2t$ for each $t$.

Moreover, if (i) holds, then $\phi_{\beta_{t_2}}$ extends $\phi_{\beta_{t_1}}$ if $t_2 > t_1$, so $(\beta_t)$ induces a lamination $\phi$, which is a self-homeomorphism of $\bigcup S_{\beta_t}$, and satisfies that $\phi|_{S_{\beta_t}} = \phi_{\beta_t}$ for each $t$. 

Remark. From $\beta_2(\cdot_1) = \beta_2(\cdot_2)$, we get $\beta_1(\cdot_2) = \beta_2(\cdot_2)$. Therefore, $\phi_{\beta_{t_2}}$ extends $\phi_{\beta_{t_1}}$ if $t_2 > t_1$, so $(\beta_t)$ induces a lamination $\phi$, which is a self-homeomorphism of $\bigcup S_{\beta_t}$, and satisfies that $\phi|_{S_{\beta_t}} = \phi_{\beta_t}$ for each $t$. 
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Moreover, if (i) holds, then \(\phi_{\beta_{t_2}}\) extends \(\phi_{\beta_{t_1}}\) if \(t_2 > t_1\), so \((\beta_t)\) induces a lamination \(\phi\), which is a self-homeomorphism of \(\bigcup S_{\beta_t}\), and satisfies that \(\phi|_{S_{\beta_t}} = \phi_{\beta_t}\) for each \(t\).

Remark. From \(\beta_{t_2} = \beta_{t_2,t_1} \cup f_{t_2,t_1}(\beta_{t_1})\), we get \(\beta_{t_1} = \beta_{t_2}/\beta_{t_2,t_1}\).
Theorem

Let $\mathcal{W}$ be a conformal map with domain $\Omega$. Suppose $\Omega$ and $\mathcal{W}$ are symmetric in the sense that $I_{\mathbb{R}}(\Omega) = \Omega$ and $\mathcal{W} \circ I_{\mathbb{R}} = I_{\mathbb{R}} \circ \mathcal{W}$. Let $K$ be an $\mathbb{H}$-hull such that $S_K \subset \Omega$. Then there is a unique symmetric conformal map $\mathcal{W}^K$ defined on $\Omega^K := f_K(\Omega \setminus S_K) \cup K_{\text{doub}}$ such that $\mathcal{W}^K \circ f_K = f_{\mathcal{W}^K(K)} \circ \mathcal{W}$ holds in $\Omega \setminus S_K$, and $S_{\mathcal{W}^K(K)} = \mathcal{W}(S_K)$. Moreover, if $K_1 \prec K_2$ and $S_{K_2} \subset \Omega$, then $\mathcal{W}^{K_1}(K_1) \prec \mathcal{W}^{K_2}(K_2)$. 

We use $\mathcal{W}^*(K)$ to denote $\mathcal{W}^K(K)$, which is also an $\mathbb{H}$-hull.
Let $W$ be a conformal map with domain $\Omega$. Suppose $\Omega$ and $W$ are symmetric in the sense that $I_R(\Omega) = \Omega$ and $W \circ I_R = I_R \circ W$. Let $K$ be an $\mathbb{H}$-hull such that $S_K \subset \Omega$. Then there is a unique symmetric conformal map $W^K$ defined on $\Omega^K := f_K(\Omega \setminus S_K) \cup K^{doub}$ such that $W^K \circ f_K = f_{W^K(K)} \circ W$ holds in $\Omega \setminus S_K$, and $S_{W^K(K)} = W(S_K)$. Moreover, if $K_1 \prec K_2$ and $S_{K_2} \subset \Omega$, then $W^{K_1}(K_1) \prec W^{K_2}(K_2)$.

We use $W^*(K)$ to denote $W^K(K)$, which is also an $\mathbb{H}$-hull.
To prove the theorem, we first consider the case that $K$ is an analytic $\mathbb{H}$-simple curve. Some result on conformal welding is used in this case. Then we use analytic $\mathbb{H}$-simple curves to approximate a general $\mathbb{H}$-hull in the Carathéodory topology.
To prove the theorem, we first consider the case that $K$ is an analytic $\mathbb{H}$-simple curve. Some result on conformal welding is used in this case. Then we use analytic $\mathbb{H}$-simple curves to approximate a general $\mathbb{H}$-hull in the Carathéodory topology.

Now we explain why the theorem is useful. If $K = \beta$ is an $\mathbb{H}$-simple curve, then so is $W^*(\beta)$. Now $\beta$ and $W^*(\beta)$ induce laminations $\phi_\beta$ and $\phi_{W^*(\beta)}$, which are self-homeomorphisms of $S_\beta$ and $S_{W^*(\beta)} = W(S_\beta)$, respectively. From $W_\beta \circ f_\beta = f_{W^*(\beta)} \circ W$ we get $\phi_{W^*(\beta)} = W \circ \phi_\beta \circ W^{-1}$. 
Suppose \((\beta_t)\) are generated by a backward Loewner equation such that \(S_{\beta_t} \subset \Omega\) for every \(t\). If \(t_1 < t_2\), then \(\beta_{t_1} < \beta_{t_2}\), so \(W^*(\beta_{t_1}) \prec W^*(\beta_{t_2})\). But \((W^*(\beta_t))\) may not be normalized by \(\text{hcap}(W^*(\beta_t)) = 2t\). This can be handled with a time-change. Let \(u(t) = \text{hcap}(W^*(\beta_t))/2\). Then \(u\) is continuous and increasing with \(u(0) = 0\), and \((W^*(\beta_u^{-1}(t)))\) is normalized, and so are generated by a backward Loewner equation. We call \((W^*(\beta_u^{-1}(t)))\) the conformal transformation of \((\beta_t)\) via \(W\).
Suppose \((\beta_t)\) are generated by a backward Loewner equation such that \(S_{\beta_t} \subset \Omega\) for every \(t\). If \(t_1 < t_2\), then \(\beta_{t_1} \prec \beta_{t_2}\), so \(W^*(\beta_{t_1}) \prec W^*(\beta_{t_2})\). But \((W^*(\beta_t))\) may not be normalized by \(\text{hcap}(W^*(\beta_t)) = 2t\). This can be handled with a time-change. Let \(u(t) = \frac{\text{hcap}(W^*(\beta_t))}{2}\). Then \(u\) is continuous and increasing with \(u(0) = 0\), and \((W^*(\beta_{u^{-1}(t)})\) is normalized, and so are generated by a backward Loewner equation. We call \((W^*(\beta_{u^{-1}(t)})\) the conformal transformation of \((\beta_t)\) via \(W\).

Let \(\phi\) and \(\phi_W\) be the laminations induced by \((\beta_t)\) and \((W^*(\beta_{u^{-1}(t)})\), respectively. Then they are self-homeomorphisms of \(S := \bigcup S_{\beta_t}\) and \(S_W = \bigcup S_{W^*(\beta_t)}\), respectively, and we have \(S_W = W(S)\) and \(\phi_W = W \circ \phi \circ W^{-1}\).
Now we define backward chordal SLE($\kappa; \rho$) process, where $\rho \in \mathbb{R}$. Let $x \neq y \in \mathbb{R}$. Suppose $\lambda(t)$ and $p(t)$ solve the equations

$$
\begin{align*}
    d\lambda(t) &= \sqrt{\kappa} dB(t) + \frac{-\rho}{\lambda(t) - p(t)} \, dt, \quad \lambda(0) = x; \\
    dp(t) &= \frac{-2}{p(t) - \lambda(t)} \, dt, \quad p(0) = y.
\end{align*}
$$

Then we call the backward chordal Loewner process driven by $\lambda$ the backward chordal SLE($\kappa; \rho$) process started from $(x; y)$. 

Proposition

Let $W$ be a conformal automorphism of $\mathbb{H}$ such that $W(0) \neq \infty$. Let $\kappa \in (0, 4]$ and $(\beta_t)$ be backward chordal SLE$_\kappa$ traces. Suppose $W^{-1}(\infty) \notin S_{\beta_t}$ for $0 \leq t < T$. Then the conformal transformation of $(\beta_t)_{0 \leq t < T}$ via $W$ is a backward chordal SLE($\kappa; -\kappa - 6$) process.
This theorem is similar to the work by Schramm and Wilson, who showed that the image of a forward chordal SLE$_\kappa$ process under a conformal automorphism of $\mathbb{H}$ is an SLE($\kappa; \kappa - 6$) process. The resemblance makes us to believe that the backward SLE$_\kappa$ can be understood as SLE with negative parameter $-\kappa$. It is known that the central charge of SLE$_\kappa$ is $\frac{(8 - 3\kappa)(\kappa - 6)}{2\kappa} \in (-\infty, 1]$, so we guess that backward SLE$_\kappa$ has central charge

$$\frac{(8 - 3(-\kappa))(-\kappa - 6)}{2(-\kappa)} = \frac{(8 + 3\kappa)(\kappa + 6)}{2\kappa} \in [25, \infty).$$
Radial SLE is another important version of SLE. For radial SLE, the unit disc $\mathbb{D} = \{|z| < 1\}$ plays the role of $\mathbb{H}$, the center 0 plays the role of $\infty$, and the unit circle $\mathbb{T} = \{|z| = 1\}$ plays the role of $\mathbb{R}$. We have a very similar theory.
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The forward radial Loewner equation is

$$\partial_t g_t(z) = g_t(z) \cdot \frac{e^{i\lambda(t)} + g_t(z)}{e^{i\lambda(t)} - g_t(z)}, \quad g_0(z) = z.$$ 

If $\lambda(t) = \sqrt{\kappa} B(t)$, we get the radial SLE$_\kappa$ process. In the case $\kappa \in (0, 4]$, there is a random simple curve $\beta$, called the radial SLE$_\kappa$ trace, with $\beta(0) = 1$, $\beta(t) \in \mathbb{D} \setminus \{0\}$ for $t > 0$, and

$$\lim_{t \to \infty} \beta(t) = 0,$$

such that for every $t$, $g_t : \mathbb{D} \setminus \beta(0, t) \xrightarrow{\text{Conf}} \mathbb{D}$. 
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Adding a minus sign, we get the backward radial Loewner equation

$$\partial_t f_t(z) = -f_t(z) \cdot \frac{e^{i\lambda(t)} + f_t(z)}{e^{i\lambda(t)} - f_t(z)}, \quad f_0(z) = z.$$ 

If \( \lambda(t) = \sqrt{\kappa} B(t) \), we get the backward radial \( \text{SLE}_\kappa \) process. In the case \( \kappa \in (0, 4] \), the process generates a family of \( \mathbb{D} \)-simple curves \( (\beta_t) \) such that for each \( t \), \( f_t : \mathbb{D} \xrightarrow{\text{Conf}} \mathbb{D} \setminus \beta_t \). Here a \( \mathbb{D} \)-simple curve is a half-open simple curve in \( \mathbb{D} \setminus \{0\} \), whose open end approaches a single point on \( \mathbb{T} \).
A relatively closed subset $K$ of $\mathbb{D}$ is called a $\mathbb{D}$-hull, if $0 \not\in K$ and $\mathbb{D} \setminus K$ is simply connected. Let $K$ be a $\mathbb{D}$-hull. Let $B_K = \overline{K} \cap \mathbb{T}$ be the base of $K$. Let $K^\text{doub} = K \cup I_\mathbb{T}(K) \cup B_K$ be the double of $K$, where $I_\mathbb{T}(z) = 1/\overline{z}$ is the reflection of $\mathbb{T}$. There is a unique $g_K : \mathbb{D} \setminus K \overset{\text{Conf}}{\rightarrow} \mathbb{D}$ such that $g_K(0) = 0$ and $g_K'(0) > 0$, and $g_K$ extends to $g_K : \mathbb{C} \setminus K^\text{doub} \overset{\text{Conf}}{\rightarrow} \mathbb{C} \setminus S_K$, where $S_K \subset \mathbb{T}$ is compact, called the support of $K$. Let the $\mathbb{D}$-capacity of $K$ be $\text{dcap}(K) = \ln g_K'(0) \geq 0$. Let $f_K = g_K^{-1}$ be defined on $\mathbb{C} \setminus S_K$ or its subset $\mathbb{D}$. 
Every $\mathbb{D}$-simple curve is a $\mathbb{D}$-hull, whose support is an arc on $\mathbb{T}$. A $\mathbb{D}$-simple curve $\beta$ induces a lamination $\phi_\beta$, which is a self-homeomorphism of $S_\beta$ swapping its two end points, such that $y = \phi_\beta(x)$ implies that $f_\beta(x) = f_\beta(y)$. Note that $f_\beta$ maps the two end points of $S_\beta$ to the base of $\beta$: $\overline{\beta} \cap \mathbb{T}$, and maps the only fixed point of $\phi_\beta$ to the tip of $\beta$.

Suppose $(\beta_t)$ are the $\mathbb{D}$-simple curves generated by a backward radial Loewner equation. Then $f_t = f_{\beta_t}$ for every $t$. 
Let $K$ and $L$ be two $\mathbb{D}$-hulls. If $L \subset K$, we define another $\mathbb{D}$-hull: $K/L = g_L(K \setminus L)$, call it a quotient hull of $K$, and write $K/L \prec K$. If $M \prec K$, then $\text{dcap}(M) \leq \text{dcap}(K)$ and $S_M \subset S_K$. 
Let $K$ and $L$ be two $\mathbb{D}$-hulls. If $L \subset K$, we define another $\mathbb{D}$-hull: 
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**Proposition**

Let $(\beta_t)$ be a family of $\mathbb{D}$-simple curves. Then they are generated by a backward radial Loewner equation if and only if

(i) $t_1 < t_2$ implies that $\beta_{t_1} \prec \beta_{t_2}$;

(ii) $(\beta_t)$ is normalized such that $\text{dcap}(\beta_t) = t$ for each $t$.

Moreover, if (i) holds, then $\phi_{\beta_{t_2}}$ extends $\phi_{\beta_{t_1}}$ if $t_2 > t_1$, so $(\beta_t)$ induces a lamination $\phi$, which is a self-homeomorphism of $\bigcup S_{\beta_t}$, and satisfies that $\phi|_{S_{\beta_t}} = \phi_{\beta_t}$ for each $t$. 
Theorem

Let $\mathcal{W}$ be a conformal map with domain $\Omega$. Suppose $\Omega$ and $\mathcal{W}$ are symmetric in the sense that $I_T(\Omega) = \Omega$ and $\mathcal{W} \circ I_T = I_T \circ \mathcal{W}$. Let $K$ be a $\mathbb{D}$-hull such that $S_K \subset \Omega$. Then there is a unique symmetric conformal map $\mathcal{W}^K$ defined on $\Omega^K := f_K(\Omega \setminus S_K) \cup K^{\text{doub}}$ such that $\mathcal{W}^K \circ f_K = f_{\mathcal{W}^K(K)} \circ \mathcal{W}$ holds in $\Omega \setminus S_K$, and $S_{\mathcal{W}^K(K)} = \mathcal{W}(S_K)$. Moreover, if $K_1 \prec K_2$ and $S_{K_2} \subset \Omega$, then $\mathcal{W}^{K_1}(K_1) \prec \mathcal{W}^{K_2}(K_2)$.

We use $\mathcal{W}^*(K)$ to denote $\mathcal{W}^K(K)$, which is also a $\mathbb{D}$-hull.
If $K = \beta$ is an $\mathbb{D}$-simple curve, then so is $W^*(\beta)$. Now $\beta$ and $W^*(\beta)$ induce laminations $\phi_\beta$ and $\phi_{W^*(\beta)}$, which are self-homeomorphisms of $S_\beta$ and $S_{W^*(\beta)} = W(S_\beta)$, respectively. From $W^\beta \circ f_\beta = f_{W^*(\beta)} \circ W$ we see that $\phi_{W^*(\beta)} = W \circ \phi_\beta \circ W^{-1}$. 
If $K = \beta$ is an $\mathbb{D}$-simple curve, then so is $W^*(\beta)$. Now $\beta$ and $W^*(\beta)$ induce laminations $\phi_{\beta}$ and $\phi_{W^*}(\beta)$, which are self-homeomorphisms of $S_\beta$ and $S_{W^*}(\beta) = W(S_\beta)$, respectively. From $W^\beta \circ f_{\beta} = f_{W^*}(\beta) \circ W$ we see that $\phi_{W^*}(\beta) = W \circ \phi_{\beta} \circ W^{-1}$.

Suppose $(\beta_t)$ is generated by a backward radial Loewner equation, then $W^*(\beta_{t_1}) \prec W^*(\beta_{t_2})$ if $t_1 < t_2$. Let $u(t) = \text{dcap}(W^*(\beta_t))$. Then $(W^*(\beta_{u^{-1}(t)}))$ is normalized, and so is generated by a backward radial Loewner process. We call this process the conformal transformation of $(\beta_t)$ via $W$. Let $\phi$ and $\phi_W$ be the laminations induced by $(\beta_t)$ and $(W^*(\beta_{u^{-1}(t)}))$, respectively. Then they are self-homeomorphisms of $S := \bigcup S_{\beta_t}$ and $S_W = \bigcup S_{W^*}(\beta_t)$, respectively, and we have $S_W = W(S)$ and $\phi_W = W \circ \phi \circ W^{-1}$. 
Let $\rho \in \mathbb{R}$. Let $x, y \in \mathbb{R}$ be such that $e^{ix} \neq e^{iy}$. Suppose $\lambda(t)$ and $p(t)$ solve the equations

$$
\begin{cases}
  d\lambda(t) = \sqrt{\kappa} dB(t) - \frac{\rho}{2} \cot((\lambda(t) - p(t))/2) dt, & \lambda(0) = x; \\
  dp(t) = -\cot((p(t) - \lambda(t))/2) dt, & p(0) = y.
\end{cases}
$$

Then we call the backward radial Loewner process driven by $\lambda$ the backward radial SLE($\kappa; \rho$) process started from $(e^{ix}; e^{iy})$. 
If $W$ is a conformal map from $\mathbb{H}$ onto $\mathbb{D}$, then we may similarly define the conformal transformation of a backward chordal Loewner process via $W$, and get a backward radial Loewner process. The theorem below also resembles Schramm-Wilson's result.
If $W$ is a conformal map from $\mathbb{H}$ onto $\mathbb{D}$, then we may similarly define the conformal transformation of a backward chordal Loewner process via $W$, and get a backward radial Loewner process. The theorem below also resembles Schramm-Wilson’s result.

**Proposition**

Suppose $W$ maps $\mathbb{H}$ conformally onto $\mathbb{D}$. Let $\kappa \in (0, 4]$ and $(\beta_t)$ be backward chordal $\text{SLE}_\kappa$ traces. Then the conformal transformation of $(\beta_t)$ via $W$ is a backward radial $\text{SLE}(\kappa; -\kappa - 6)$ process started from $(W(0); W(\infty))$. 
The above backward radial SLE($\kappa; -\kappa - 6$) process started from $(W(0); W(\infty))$ induces a lamination $\phi_W$, which is a self-homeomorphism of $\mathbb{T} \setminus \{W(\infty)\}$ with one fixed point: $W(0)$. If $\phi$ is the lamination induced by $(\beta_t)$, then $\phi_W = W \circ \phi \circ W^{-1}$. We may extend $\phi_W$ to a self-homeomorphism of $\mathbb{T}$, which has two fixed points: $W(0)$ and $W(\infty)$.
The above backward radial SLE($\kappa; -\kappa - 6$) process started from $(W(0); W(\infty))$ induces a lamination $\phi_W$, which is a self-homeomorphism of $\mathbb{T} \setminus \{W(\infty)\}$ with one fixed point: $W(0)$. If $\phi$ is the lamination induced by $(\beta_t)$, then $\phi_W = W \circ \phi \circ W^{-1}$. We may extend $\phi_W$ to a self-homeomorphism of $\mathbb{T}$, which has two fixed points: $W(0)$ and $W(\infty)$.

Fix $z_1 \neq z_2 \in \mathbb{T}$. To prove the main theorem, it suffices to show that, we may couple a backward radial SLE($\kappa; -\kappa - 6$) process started from $(z_1; z_2)$ with a backward radial SLE($\kappa; -\kappa - 6$) process started from $(z_2; z_1)$, such that the two processes induce the same lamination.
Theorem

Let \( \kappa \in (0, 4] \) and \( z_1 \neq z_2 \in \mathbb{T} \). There exists a coupling of two families of \( \mathbb{D} \)-simple curves \((\beta^1_t)\) and \((\beta^2_t)\) such that the following hold.

(i) For \( j = 1, 2 \), \((\beta^j_t)\) is a backward radial SLE\((\kappa; -\kappa - 6)\) process started from \((z_j; z_{3-j})\);

(ii) Let \( t_2 < \infty \) be a stopping time for \((\beta^2_t)\), \( f^2_{t_2} = f^2_{\beta^2_{t_2}} \), and \( T_1(t_2) \) be the first time such that \( S_{\beta^1_{t_2}} \) intersects \( S_{\beta^2_{t_2}} \). Then the transformation of \((\beta^1_t)_{0 \leq t < T_1(t_2)}\) via \( f^2_{t_2} \) is a backward radial SLE\((\kappa; -\kappa - 6)\) process started from \((f^2_{t_2}(z_1); B_{\beta^2_{t_2}})\). A similar result holds if the indices "1" and "2" are switched.
Note that the transformation of \((\beta^1_t)_{0 \leq t < T_1(t_2)}\) via \(f^2_{t_2}\) is well defined because for \(t < T_1(t_2)\), \(S_{\beta^1_t}\) is contained in \(\mathbb{C} \setminus S_{\beta^2_{t_2}}\), which is the domain of \(f^2_{t_2}\).
Note that the transformation of $(\beta^1_t)_{0 \leq t < T_1(t_2)}$ via $f^2_{t_2}$ is well defined because for $t < T_1(t_2)$, $S_{\beta^1_t}$ is contained in $\mathbb{C} \setminus S_{\beta^2_{t_2}}$, which is the domain of $f^2_{t_2}$.

For $j = 1, 2$, let $\phi^j$ be the lamination induced by $(\beta^j_t)$. Assume that the above theorem holds true, and the two backward radial SLE$(\kappa; -\kappa - 6)$ processes are coupled according to the theorem. We will show that $\phi^1 = \phi^2$, which then implies the main theorem.
Let $t_2$ be fixed. Since the transformation of $(\beta^1_t)_{0 \leq t < T_1(t_2)}$ via $f^2_{t_2}$ is a backward radial SLE($\kappa; -\kappa - 6$) process started from $(f^2_{t_2}(z_1); B_{\beta^2_{t_2}})$, the union of the supports of $(f^2_{t_2})^*(\beta^1_t)$, $0 \leq t < T_1(t_2)$, is $\mathbb{T} \setminus B_{\beta^2_{t_2}}$. Since $f^2_{t_2}$ maps $\mathbb{T} \setminus S_{\beta^2_{t_2}}$ onto $\mathbb{T} \setminus B_{\beta^2_{t_2}}$, the union of the supports of $\beta^1_t$, $0 \leq t < T_1(t_2)$, is $\mathbb{T} \setminus S_{\beta^2_{t_2}}$. This shows that $S_{\beta^1_{T_1(t_2)}}$ and $S_{\beta^2_{t_2}}$ share two end points. Since both $\phi^1$ and $\phi^2$ flip these two end points, they agree on these two points. Letting $t_2$ vary, we conclude that $\phi^1 = \phi^2$. 
Introduction
Conformal Transformation
Radial SLE
Couplings

Reversibility of Backward SLE Lamination
It remains to prove the above theorem. To construct the coupling, we use the idea in the proof of the reversibility of forward chordal SLE$_{\kappa}$ for $\kappa \in (0, 4]$. First, we construct local couplings. Let $l_1$ and $l_2$ be two closed arcs on $\mathbb{T}$ such that $\text{dist}(l_1, l_2) > 0$ and the interior of $l_j$ contains $z_j$, $j = 1, 2$. We call $(l_1, l_2)$ a disjoint pair. Let $(\beta^j_t)$, $j = 1, 2$, be a backward radial SLE$(\kappa; -\kappa - 6)$ process started from $(z_j; z_{3-j})$. Let $T_j(l_j)$ be the first time that $S_{\beta^j_t}$ is not contained in the interior of $l_j$, $j = 1, 2$. We say that the two processes are well coupled within $(l_1, l_2)$ if the following holds.
If \( t_2 \leq T_2(l_2) \) is a stopping time for \( (\beta^2_t) \), then the transformation of \( (\beta^1_t)_{0 \leq t < T_1(l_1)} \) via \( f^2_{t_2} \) is a stopped backward radial SLE\((\kappa; -\kappa - 6)\) process started from \((f^2_{t_2}(z_1); B_{\beta^2_{t_2}})\). A similar result holds if the indices “1” and “2” are switched.
If \( t_2 \leq T_2(l_2) \) is a stopping time for \((\beta_t^2)\), then the transformation of \((\beta_t^1)_{0 \leq t < T_1(l_1)} \) via \( f_{t_2}^2 \) is a stopped backward radial SLE\((\kappa; -\kappa - 6)\) process started from \((f_{t_2}^2(z_1); B_{\beta_{t_2}^2})\). A similar result holds if the indices “1” and “2” are switched.

Such coupling can be constructed by weighting an independent coupling of two backward radial SLE\((\kappa; -\kappa - 6)\) processes by a suitable Radon-Nikodym derivative, which is obtained by a standard argument on Loewner equations.
Then we are able to show that, for any finitely many disjoint pairs \((I_1^m, I_2^m), 1 \leq m \leq n\), there is a coupling of two backward radial SLE\(\kappa; -\kappa - 6\) processes, such that for any \(m\), the two processes are well coupled within \((I_1^m, I_2^m)\). Such coupling is obtained by weighting an independent coupling of two backward radial SLE\(\kappa; -\kappa - 6\) processes by a RN derivative, which is related with the RN derivatives for a good coupling within each \((I_1^m, I_2^m)\).
Now let \((I^1_m, I^2_m)_{m \in \mathbb{N}}\) be a sequence of disjoint pairs, which is dense in the space of disjoint pairs. For every \(n \in \mathbb{N}\), the above result shows that there is a coupling of two backward radial SLE\((\kappa; -\kappa - 6)\) processes, which are well coupled within \((I^1_m, I^2_m)\), for \(m\) from 1 up to \(n\). Let \(\mu_n\) denote the distribution of such coupling. The sequence \((\mu_n)\) converges in some suitable topology to a measure \(\mu\), which is exactly the coupling of two backward radial SLE\((\kappa; -\kappa - 6)\) processes that we are looking for.
Thank you!