Reversibility of Backward SLE Lamination

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Joint work with Steffen Rohde

The Schramm-Loewner Evolution (SLE) is a stochastic process of random conformal maps that has received a lot of attention over the last decade. A number of two-dimensional lattice models have been proved to converge to SLE with different parameters, thanks to the work by Schramm, Lawler, Werner, Smirnov, Sheffield, and many others.

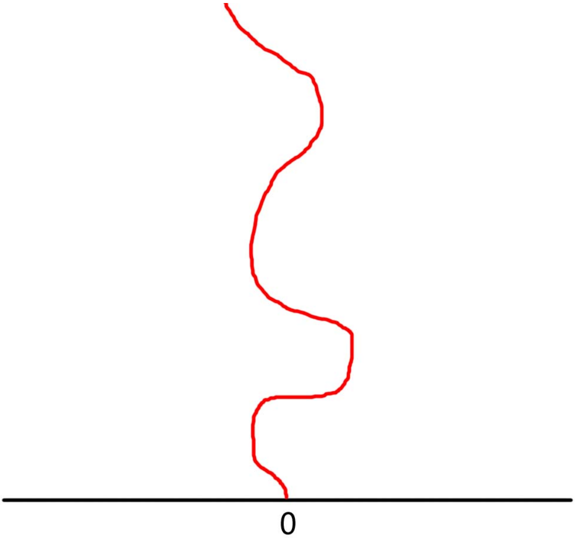
SLE = Loewner's differential equation + random driving function.
We are mainly concerned with the chordal Loewner equation:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \quad g_0(z) = z,$$

where $z \in \mathbb{C}$, and $\lambda \in C([0,T),\mathbb{R})$ is called the driving function. Fix $\kappa > 0$, and let $B(t)$ be a standard Brownian motion. The solution of the chordal Loewner equation with $\lambda(t) = \sqrt{\kappa}B(t)$ is called chordal SLE_κ.

Rohde and Schramm showed that chordal SLE_κ generates a random curve called the trace: $\beta(t)$, $0 \le t < \infty$, in the closure of the upper half plane, which satisfies $\beta(0) = 0$ and $\lim_{t\to\infty} \beta(t) = \infty$.

The simplest case is $\kappa \in (0, 4]$, in which β is a simple curve with $\beta(t) \in \mathbb{H} = \{\operatorname{Im} z > 0\}$ for $t > 0$, and for every $t > 0$, $g_t : \mathbb{H} \setminus \beta(0, t] \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{H}$.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Adding a minus sign to the (forward) chordal Loewner equation, we get the backward chordal Loewner equation:

$$\partial_t f_t(z) = \frac{-2}{f_t(z) - \lambda(t)}, \quad f_0(z) = z.$$

Setting $\lambda(t) = \sqrt{\kappa} B(t)$, we then get the backward chordal SLE$_\kappa$.

The backward and forward Loewner equations are related as follows. Fix T_0 such that λ is defined on $[0, T_0]$. Let $\lambda_{T_0}(t) = \lambda(T_0 - t)$, $0 \le t \le T_0$. It is easy to check

$$f^{\lambda}_{T_0 - t} \circ (f^{\lambda}_{T_0})^{-1} = g_t^{\lambda_{T_0}}, \quad 0 \le t \le T_0.$$

Taking $t = T_0$, we get $(f^{\lambda}_{T_0})^{-1} = g_{T_0}^{\lambda_{T_0}}$.

A half-open simple curve (as a set) in $\mathbb{H}$ is called an $\mathbb{H}$-simple curve, if its open side approaches a single point on $\mathbb{R}$. If β is an SLE$_\kappa$ ($\kappa \le 4$) trace, then $\beta(0, t]$ is an $\mathbb{H}$-simple curve for every t.

Suppose $\kappa \in (0, 4]$ and $\lambda(t) = \sqrt{\kappa}B(t)$. Then $\lambda_{T_0}(t) - \lambda(T_0)$ has the same distribution as $\lambda(t)$, $0 \le t \le T_0$. This together with $(f^\lambda_{T_0})^{-1} = g^{\lambda_{T_0}}_{T_0}$ and the property of the forward SLE$_\kappa$ trace shows that the backward chordal SLE$_\kappa$ generates a family of $\mathbb{H}$-simple curves (β_t) such that, for every t, $f_t : \mathbb{H} \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{H} \setminus \beta_t$.

Fix $t_0 \geq 0$, and let f_{t,t_0}, $t \geq t_0$, be the solution of

$$\partial_t f_{t,t_0}(z) = \frac{-2}{f_{t,t_0}(z) - \lambda(t)}, \quad f_{t_0,t_0}(z) = z.$$

If $t_2 > t_1$, then $f_{t_2,t_1} : \mathbb{H} \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{H} \setminus \beta_{t_2,t_1}$, where β_{t_2,t_1} is an $\mathbb{H}$-simple curve. We have $f_{t_2,t_1} \circ f_{t_1} = f_{t_2}$, and so $\beta_{t_2} = \beta_{t_2,t_1} \cup f_{t_2,t_1}(\beta_{t_1})$.

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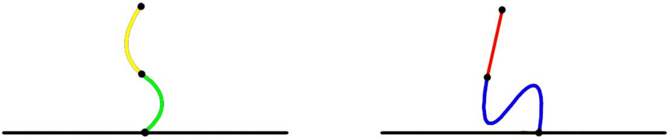
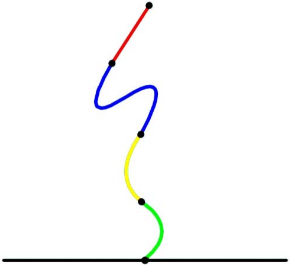
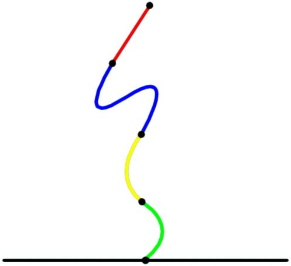
When a forward Loewner process and a backward Loewner process both generate $\mathbb{H}$-simple curves, they look very similar at any fixed time. However, if we let time evolve, the difference will be clear.







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Every f_t has a continuous extension from $\mathbb{H}$ to $\overline{\mathbb{H}}$, which maps two real intervals with common end point 0 onto the two sides of β_t. If $f_t(x_1) = f_t(x_2) \in \beta_t$, then we write $x_1 \sim_t x_2$. If $t_1 < t_2$, from $f_{t_2,t_1} \circ f_{t_1} = f_{t_2}$ we see that $x_1 \sim_{t_1} x_2$ implies that $x_1 \sim_{t_2} x_2$. Thus, we may define a global relation: $x_1 \sim x_2$ if there exists $t > 0$ such that $x_1 \sim_t x_2$. In fact, $x_1 \sim x_2$ iff that the solutions $f_t(x_1)$ and $f_t(x_2)$ blow up at the same time, i.e., $\tau(x_1) = \tau(x_2)$.

It holds that almost surely, $\tau(x) < \infty$ for every $x \in \mathbb{R}$. So we get a random self-homeomorphism ϕ of $\mathbb{R}$ such that $\phi(0) = 0$, $\phi(\pm\infty) = \mp\infty$, and $y = \phi(x)$ implies $x \sim y$. We call such ϕ a backward chordal SLE_κ lamination.

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A fundamental property of the forward chordal SLE is *reversibility*. For $\kappa \le 8$, the law of the SLE_κ trace is invariant under the automorphism $z \mapsto -1/z$ of $\mathbb{H}$, modulo time parametrization. This was first proved for $\kappa \le 4$ (Z, 2007), and later for $4 \le \kappa \le 8$ (Miller and Sheffield, 2012). It is false for $\kappa > 8$.

Our main theorem is that the backward chordal SLE$_\kappa$ lamination has the following reversibility property.

Theorem

Let $\kappa \in (0, 4]$, and ϕ be a backward chordal SLE$_\kappa$ lamination. Then $\psi(x) := -1/\phi^{-1}(-1/x)$ has the same distribution as ϕ.

Sheffield recently proved that, for $\kappa \in (0,4)$, there is a coupling of a backward chordal SLE_κ with a free boundary Gaussian free field in $\mathbb{H}$, such that the GFF determines the backward SLE and a quantum length on $\mathbb{R}$, and for $x < 0 < y$, $\phi(x) = y$ iff $[x, 0]$ and $[0, y]$ have the same quantum length.

Sheffield's theorem seems to be closely related to our main theorem. However, so far we have not found a way to connect these two results. Instead, the proof of our theorem uses an idea in the proof of the reversibility of forward chordal SLE_κ for $\kappa \in (0,4]$.

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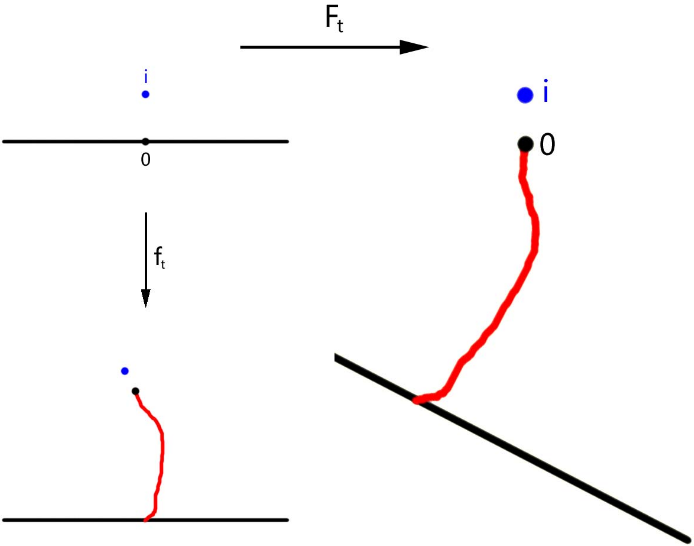
Let $\kappa \in (0, 4]$. Although a backward chordal SLE_κ process does not naturally generate a single trace, we may still define a normalized global backward SLE_κ trace as follows.

Recall that, for each t, $f_t : \mathbb{H} \overset{\mathrm{Conf}}{\twoheadrightarrow} \mathbb{H} \setminus \beta_t$ and $f_t(0)$ is the tip of β_t. We may find $a_t, b_t \in \mathbb{C}$ such that $F_t = a_t f_t + b_t$ fixes both 0 and i. As $t \to \infty$, F_t converges to a conformal map F_∞ defined on $\mathbb{H}$, which also fixes 0 and i. It turns out that $F_\infty(\mathbb{H}) = \mathbb{C} \setminus \beta$, where β is a simple curve, which joins 0 with ∞, and avoids i, and F_∞ is a realization of the lamination ϕ in the sense that $y = \phi(x)$ implies that $F_\infty(x) = F_\infty(y) \in \beta$. We call this β a normalized global backward SLE_κ trace. We have the following reversibility of β


F_t
i
0
f_t
i
0

Theorem

Let $\kappa \in (0, 4)$, and β be a normalized global backward chordal SLE_κ trace. Let $h(z) = -1/z$. Then $h(\beta)$ has the same distribution as β.

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This theorem follows from the main theorem and the fact that the SLE_κ trace is conformally removable, thank to the work by Jones-Smirnov (a Hölder curve is conformally removable) and Rohde-Schramm (an SLE_κ trace is a Hölder curve).

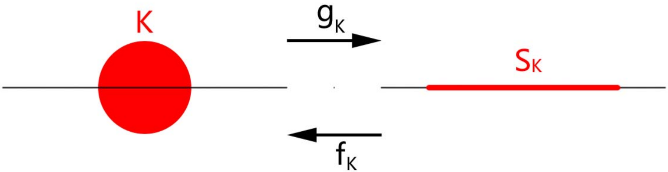
We now define the conformal transformation of a backward Loewner process via a conformal map with suitable conditions. For this purpose, we introduce some definitions.

- A relatively closed subset K of $\mathbb{H}$ is called an $\mathbb{H}$-hull, if K is bounded and $\mathbb{H} \setminus K$ is simply connected.

  Now assume K is an $\mathbb{H}$-hull. Let $I_{\mathbb{R}}(z) = \bar{z}$ be the reflection about $\mathbb{R}$.

- The base of K: $B_K = \overline{K} \cap \mathbb{R}$.
- The double of K: $K^{\text{doub}} = K \cup I_{\mathbb{R}}(K) \cup B_K$.

- g_K is the unique $g_K : \mathbb{H} \setminus K \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{H}$ such that $g_K(z) = z + o(1/z)$ as $\mathbb{H} \ni z \to \infty$.
- g_K extends to a conformal map defined on $\mathbb{C} \setminus K^{\text{doub}}$.
- The support of K: $S_K = \mathbb{C} \setminus g_K(\mathbb{C} \setminus K^{\text{doub}}) \subset \mathbb{R}$.
- The $\mathbb{H}$-capacity of K: $\text{hcap}(K) = \lim_{z\to\infty} z(g_K(z) - z) \geq 0$.
- $f_K = g_K^{-1}$ is defined on $\mathbb{C} \setminus S_K$ or its subset $\mathbb{H}$.


K
g_K
S_K
f_K

Every $\mathbb{H}$-simple curve is an $\mathbb{H}$-hull, whose base is a single point, and whose support is a real interval. An $\mathbb{H}$-simple curve β induces a lamination ϕ_β, which is a self-homeomorphism of S_β swapping its two end points, such that $y = \phi_\beta(x)$ implies that $f_\beta(x) = f_\beta(y)$. Note that f_β maps the two end points of S_β to the base of β: $\overline{\beta} \cap \mathbb{R}$, and maps the only fixed point of ϕ_β to the tip of β.

Let $\kappa \in (0, 4]$, and (β_t) be the $\mathbb{H}$-simple curves generated by a backward SLE$_\kappa$ process. Then $f_t = f_{\beta_t}$ for every t and $\bigcup S_{\beta_t} = \mathbb{R}$. The SLE$_\kappa$ lamination ϕ satisfies $\phi|_{S_{\beta_t}} = \phi_{\beta_t}$ for each t.

Let K and L be two $\mathbb{H}$-hulls. If $L \subset K$, we define another $\mathbb{H}$-hull: $K/L = g_L(K \setminus L)$, call it a quotient hull of K, and write $K/L \prec K$.

Fact: If $M \prec K$, then $\text{hcap}(M) \le \text{hcap}(K)$ and $S_M \subset S_K$.

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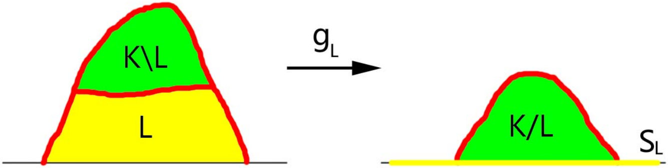

Proposition

Let (β_t) be a family of $\mathbb{H}$-simple curves. Then they are generated by a backward chordal Loewner equation if and only if

(i) $t_1 < t_2$ implies that $\beta_{t_1} \prec \beta_{t_2}$;

(ii) (β_t) is normalized such that $\mathrm{hcap}(\beta_t) = 2t$ for each t.

Moreover, if (i) holds, then $\phi_{\beta_{t_2}}$ extends $\phi_{\beta_{t_1}}$ if $t_2 > t_1$, so (β_t) induces a lamination ϕ, which is a self-homeomorphism of $\bigcup S_{\beta_t}$, and satisfies that $\phi|_{S_{\beta_t}} = \phi_{\beta_t}$ for each t.

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Remark. From $\beta_{t_2} = \beta_{t_2,t_1} \cup f_{t_2,t_1}(\beta_{t_1})$, we get $\beta_{t_1} = \beta_{t_2}/\beta_{t_2,t_1}$.

Theorem

Let W be a conformal map with domain Ω. Suppose Ω and W are symmetric in the sense that $I_{\mathbb{R}}(\Omega) = \Omega$ and $W \circ I_{\mathbb{R}} = I_{\mathbb{R}} \circ W$. Let K be an $\mathbb{H}$-hull such that $S_K \subset \Omega$. Then there is a unique symmetric conformal map W^K defined on $\Omega^K := f_K(\Omega \setminus S_K) \cup K^{\mathrm{doub}}$ such that $W^K \circ f_K = f_{W^K(K)} \circ W$ holds in $\Omega \setminus S_K$, and $S_{W^K(K)} = W(S_K)$. Moreover, if $K_1 \prec K_2$ and $S_{K_2} \subset \Omega$, then $W^{K_1}(K_1) \prec W^{K_2}(K_2)$.

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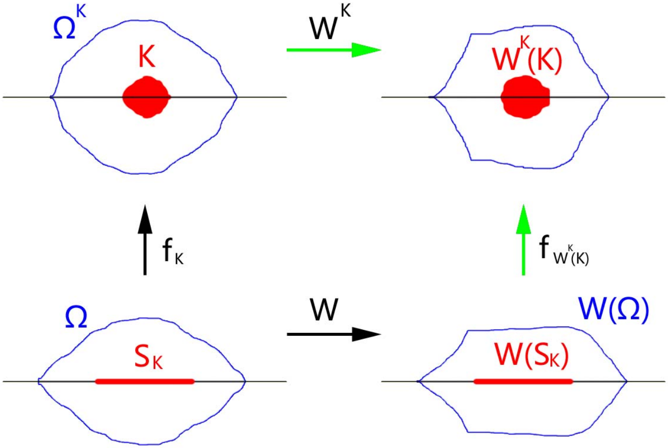
We use $W^*(K)$ to denote $W^K(K)$, which is also an $\mathbb{H}$-hull.


Ω^K
K
W^K
W^K(K)
f_K
f_{W^K(K)}
Ω
S_K
W
W(Ω)
W(S_K)

To prove the theorem, we first consider the case that K is an analytic $\mathbb{H}$-simple curve. Some result on conformal welding is used in this case. Then we use analytic $\mathbb{H}$-simple curves to approximate a general $\mathbb{H}$-hull in the Carathéodory topology.

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Now we explain why the theorem is useful. If $K = \beta$ is an $\mathbb{H}$-simple curve, then so is $W^*(\beta)$. Now β and $W^*(\beta)$ induce laminations ϕ_β and $\phi_{W^*(\beta)}$, which are self-homeomorphisms of S_β and $S_{W^*(\beta)} = W(S_\beta)$, respectively. From $W^\beta \circ f_\beta = f_{W^*(\beta)} \circ W$ we get $\phi_{W^*(\beta)} = W \circ \phi_\beta \circ W^{-1}$.

Suppose (β_t) are generated by a backward Loewner equation such that $S_{\beta_t} \subset \Omega$ for every t. If $t_1 < t_2$, then $\beta_{t_1} \prec \beta_{t_2}$, so $W^*(\beta_{t_1}) \prec W^*(\beta_{t_2})$. But $(W^*(\beta_t))$ may not be normalized by $\text{hcap}(W^*(\beta_t)) = 2t$. This can be handled with a time-change. Let $u(t) = \text{hcap}(W^*(\beta_t))/2$. Then u is continuous and increasing with $u(0) = 0$, and $(W^*(\beta_{u^{-1}(t)}))$ is normalized, and so are generated by a backward Loewner equation. We call $(W^*(\beta_{u^{-1}(t)}))$ the conformal transformation of (β_t) via W.

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Let ϕ and ϕ_W be the laminations induced by (β_t) and $(W^*(\beta_{u^{-1}(t)}))$, respectively. Then they are self-homeomorphisms of $S := \bigcup S_{\beta_t}$ and $S_W = \bigcup S_{W^*(\beta_t)}$, respectively, and we have $S_W = W(S)$ and $\phi_W = W \circ \phi \circ W^{-1}$.

Now we define backward chordal SLE$(\kappa;\rho)$ process, where $\rho \in \mathbb{R}$. Let $x \neq y \in \mathbb{R}$. Suppose $\lambda(t)$ and $p(t)$ solve the equations

$$\begin{cases} d\lambda(t) = \sqrt{\kappa} dB(t) + \frac{-\rho}{\lambda(t) - p(t)} dt, & \lambda(0) = x; \\ dp(t) = \frac{-2}{p(t) - \lambda(t)} dt, & p(0) = y. \end{cases}$$

Then we call the backward chordal Loewner process driven by λ the backward chordal SLE$(\kappa;\rho)$ process started from $(x;y)$.

Proposition

Let W be a conformal automorphism of $\mathbb{H}$ such that $W(0) \neq \infty$. Let $\kappa \in (0,4]$ and (β_t) be backward chordal SLE$_\kappa$ traces. Suppose $W^{-1}(\infty) \notin S_{\beta_t}$ for $0 \le t < T$. Then the conformal transformation of $(\beta_t)_{0\le t<T}$ via W is a backward chordal SLE$(\kappa; -\kappa-6)$ process.

This theorem is similar to the work by Schramm and Wilson, who showed that the image of a forward chordal SLE_κ process under a conformal automorphism of $\mathbb{H}$ is an $\text{SLE}(\kappa; \kappa - 6)$ process. The resemblance makes us to believe that the backward SLE_κ can be understood as SLE with negative parameter $-\kappa$. It is known that the central charge of SLE_κ is $\frac{(8-3\kappa)(\kappa-6)}{2\kappa} \in (-\infty, 1]$, so we guess that backward SLE_κ has central charge

$$\frac{(8-3(-\kappa))(-\kappa-6)}{2(-\kappa)} = \frac{(8+3\kappa)(\kappa+6)}{2\kappa} \in [25, \infty).$$

Radial SLE is another important version of SLE. For radial SLE, the unit disc $\mathbb{D} = \{|z| < 1\}$ plays the role of $\mathbb{H}$, the center 0 plays the role of ∞, and the unit circle $\mathbb{T} = \{|z| = 1\}$ plays the role of $\mathbb{R}$. We have a very similar theory.

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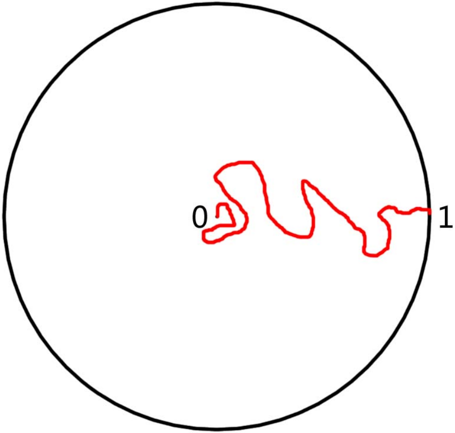
The forward radial Loewner equation is

$$\partial_t g_t(z) = g_t(z) \cdot \frac{e^{i\lambda(t)} + g_t(z)}{e^{i\lambda(t)} - g_t(z)}, \quad g_0(z) = z.$$

If $\lambda(t) = \sqrt{\kappa} B(t)$, we get the radial SLE$_\kappa$ process. In the case $\kappa \in (0, 4]$, there is a random simple curve β, called the radial SLE$_\kappa$ trace, with $\beta(0) = 1$, $\beta(t) \in \mathbb{D} \setminus \{0\}$ for $t > 0$, and $\lim_{t\to\infty} \beta(t) = 0$, such that for every t, $g_t : \mathbb{D} \setminus \beta(0, t] \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{D}$.


0
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Adding a minus sign, we get the backward radial Loewner equation

$$\partial_t f_t(z) = -f_t(z) \cdot \frac{e^{i\lambda(t)} + f_t(z)}{e^{i\lambda(t)} - f_t(z)}, \quad f_0(z) = z.$$

If $\lambda(t) = \sqrt{\kappa} B(t)$, we get the backward radial SLE$_\kappa$ process. In the case $\kappa \in (0, 4]$, the process generates a family of $\mathbb{D}$-simple curves (β_t) such that for each t, $f_t : \mathbb{D} \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{D} \setminus \beta_t$. Here a $\mathbb{D}$-simple curve is a half-open simple curve in $\mathbb{D} \setminus \{0\}$, whose open end approaches a single point on $\mathbb{T}$.

A relatively closed subset K of $\mathbb{D}$ is called a $\mathbb{D}$-hull, if $0 \not\in K$ and $\mathbb{D} \setminus K$ is simply connected. Let K be a $\mathbb{D}$-hull. Let $B_K = \overline{K} \cap \mathbb{T}$ be the base of K. Let $K^{\text{doub}} = K \cup I_{\mathbb{T}}(K) \cup B_K$ be the double of K, where $I_{\mathbb{T}}(z) = 1/\overline{z}$ is the reflection of $\mathbb{T}$. There is a unique $g_K : \mathbb{D} \setminus K \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{D}$ such that $g_K(0) = 0$ and $g_K'(0) > 0$, and g_K extends to $g_K : \mathbb{C} \setminus K^{\text{doub}} \overset{\text{Conf}}{\twoheadrightarrow} \mathbb{C} \setminus S_K$, where $S_K \subset \mathbb{T}$ is compact, called the support of K. Let the $\mathbb{D}$-capacity of K be $\text{dcap}(K) = \ln g_K'(0) \geq 0$. Let $f_K = g_K^{-1}$ be defined on $\mathbb{C} \setminus S_K$ or its subset $\mathbb{D}$.

Every $\mathbb{D}$-simple curve is a $\mathbb{D}$-hull, whose support is an arc on $\mathbb{T}$. A $\mathbb{D}$-simple curve β induces a lamination ϕ_β, which is a self-homeomorphism of S_β swapping its two end points, such that $y = \phi_\beta(x)$ implies that $f_\beta(x) = f_\beta(y)$. Note that f_β maps the two end points of S_β to the base of β: $\overline{\beta} \cap \mathbb{T}$, and maps the only fixed point of ϕ_β to the tip of β.

Suppose (β_t) are the $\mathbb{D}$-simple curves generated by a backward radial Loewner equation. Then $f_t = f_{\beta_t}$ for every t.

Let K and L be two $\mathbb{D}$-hulls. If $L \subset K$, we define another $\mathbb{D}$-hull: $K/L = g_L(K \setminus L)$, call it a quotient hull of K, and write $K/L \prec K$. If $M \prec K$, then $\mathrm{dcap}(M) \le \mathrm{dcap}(K)$ and $S_M \subset S_K$.

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Proposition

Let (β_t) be a family of $\mathbb{D}$-simple curves. Then they are generated by a backward radial Loewner equation if and only if

(i) $t_1 < t_2$ implies that $\beta_{t_1} \prec \beta_{t_2}$;

(ii) (β_t) is normalized such that $\text{dcap}(\beta_t) = t$ for each t.

Moreover, if (i) holds, then $\phi_{\beta_{t_2}}$ extends $\phi_{\beta_{t_1}}$ if $t_2 > t_1$, so (β_t) induces a lamination ϕ, which is a self-homeomorphism of $\bigcup S_{\beta_t}$, and satisfies that $\phi|_{S_{\beta_t}} = \phi_{\beta_t}$ for each t.

Theorem

Let W be a conformal map with domain Ω. Suppose Ω and W are symmetric in the sense that $I_{\mathbb{T}}(\Omega) = \Omega$ and $W \circ I_{\mathbb{T}} = I_{\mathbb{T}} \circ W$. Let K be a $\mathbb{D}$-hull such that $S_K \subset \Omega$. Then there is a unique symmetric conformal map W^K defined on $\Omega^K := f_K(\Omega \setminus S_K) \cup K^{\mathrm{doub}}$ such that $W^K \circ f_K = f_{W^K(K)} \circ W$ holds in $\Omega \setminus S_K$, and $S_{W^K(K)} = W(S_K)$. Moreover, if $K_1 \prec K_2$ and $S_{K_2} \subset \Omega$, then $W^{K_1}(K_1) \prec W^{K_2}(K_2)$.

We use $W^*(K)$ to denote $W^K(K)$, which is also a $\mathbb{D}$-hull.

If $K=\beta$ is an $\mathbb{D}$-simple curve, then so is $W^*(\beta)$. Now β and $W^*(\beta)$ induce laminations ϕ_β and $\phi_{W^*(\beta)}$, which are self-homeomorphisms of S_β and $S_{W^*(\beta)}=W(S_\beta)$, respectively. From $W^\beta\circ f_\beta=f_{W^*(\beta)}\circ W$ we see that $\phi_{W^*(\beta)}=W\circ\phi_\beta\circ W^{-1}$.

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Suppose (β_t) is generated by a backward radial Loewner equation, then $W^*(\beta_{t_1})\prec W^*(\beta_{t_2})$ if $t_1<t_2$. Let $u(t)=\mathrm{dcap}(W^*(\beta_t))$. Then $(W^*(\beta_{u^{-1}(t)}))$ is normalized, and so is generated by a backward radial Loewner process. We call this process the conformal transformation of (β_t) via W. Let ϕ and ϕ_W be the laminations induced by (β_t) and $(W^*(\beta_{u^{-1}(t)}))$, respectively. Then they are self-homeomorphisms of $S:=\bigcup S_{\beta_t}$ and $S_W=\bigcup S_{W^*(\beta_t)}$, respectively, and we have $S_W=W(S)$ and $\phi_W=W\circ\phi\circ W^{-1}$.

Let $\rho \in \mathbb{R}$. Let $x, y \in \mathbb{R}$ be such that $e^{ix} \neq e^{iy}$. Suppose $\lambda(t)$ and $p(t)$ solve the equations

$$\begin{cases} d\lambda(t) = \sqrt{\kappa}dB(t) - \frac{\rho}{2}\cot((\lambda(t) - p(t))/2)dt, & \lambda(0) = x; \\ dp(t) = -\cot((p(t) - \lambda(t))/2)dt, & p(0) = y. \end{cases}$$

Then we call the backward radial Loewner process driven by λ the backward radial SLE$(\kappa; \rho)$ process started from $(e^{ix}; e^{iy})$.

If W is a conformal map from $\mathbb{H}$ onto $\mathbb{D}$, then we may similarly define the conformal transformation of a backward chordal Loewner process via W, and get a backward radial Loewner process. The theorem below also resembles Schramm-Wilson's result.

If W is a conformal map from $\mathbb{H}$ onto $\mathbb{D}$, then we may similarly define the conformal transformation of a backward chordal Loewner process via W, and get a backward radial Loewner process. The theorem below also resembles Schramm-Wilson's result.

Proposition

Suppose W maps $\mathbb{H}$ conformally onto $\mathbb{D}$. Let $\kappa \in (0, 4]$ and (β_t) be backward chordal SLE_κ traces. Then the conformal transformation of (β_t) via W is a backward radial $\text{SLE}(\kappa; -\kappa - 6)$ process started from $(W(0); W(\infty))$.

The above backward radial SLE$(\kappa; -\kappa-6)$ process started from $(W(0); W(\infty))$ induces a lamination ϕ_W, which is a self-homeomorphism of $\mathbb{T} \setminus \{W(\infty)\}$ with one fixed point: $W(0)$. If ϕ is the lamination induced by (β_t), then $\phi_W = W \circ \phi \circ W^{-1}$. We may extend ϕ_W to a self-homeomorphism of $\mathbb{T}$, which has two fixed points: $W(0)$ and $W(\infty)$.

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Fix $z_1 \neq z_2 \in \mathbb{T}$. To prove the main theorem, it suffices to show that, we may couple a backward radial SLE$(\kappa; -\kappa-6)$ process started from $(z_1; z_2)$ with a backward radial SLE$(\kappa; -\kappa-6)$ process started from $(z_2; z_1)$, such that the two processes induce the same lamination.

Theorem

Let $\kappa \in (0,4]$ and $z_1 \neq z_2 \in \mathbb{T}$. There exists a coupling of two families of $\mathbb{D}$-simple curves (β^1_t) and (β^2_t) such that the following hold.

(i) For $j = 1, 2$, (β^j_t) is a backward radial SLE$(\kappa; -\kappa - 6)$ process started from $(z_j; z_{3-j})$;

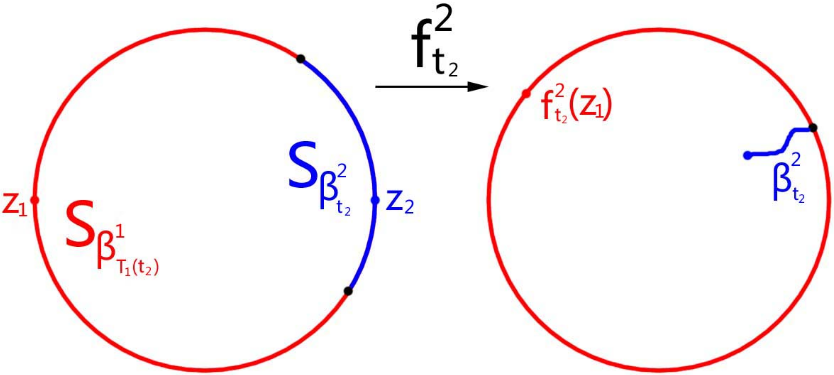
(ii) Let $t_2 < \infty$ be a stopping time for (β^2_t), $f^2_{t_2} = f_{\beta^2_{t_2}}$, and $T_1(t_2)$ be the first time such that $S_{\beta^1_t}$ intersects $S_{\beta^2_{t_2}}$. Then the transformation of $(\beta^1_t)_{0 \le t < T_1(t_2)}$ via $f^2_{t_2}$ is a backward radial SLE$(\kappa; -\kappa - 6)$ process started from $(f^2_{t_2}(z_1); B_{\beta^2_{t_2}})$. A similar result holds if the indices "1" and "2" are switched.

Note that the transformation of $(\beta_t^1)_{0\le t<T_1(t_2)}$ via $f_{t_2}^2$ is well defined because for $t<T_1(t_2)$, $S_{\beta_t^1}$ is contained in $\mathbb{C}\setminus S_{\beta_{t_2}^2}$, which is the domain of $f_{t_2}^2$.

Note that the transformation of $(\beta_t^1)_{0\le t<T_1(t_2)}$ via $f_{t_2}^2$ is well defined because for $t<T_1(t_2)$, $S_{\beta_t^1}$ is contained in $\mathbb{C}\setminus S_{\beta_{t_2}^2}$, which is the domain of $f_{t_2}^2$.

For $j=1,2$, let ϕ^j be the lamination induced by (β_t^j). Assume that the above theorem holds true, and the two backward radial SLE$(\kappa;-\kappa-6)$ processes are coupled according to the theorem. We will show that $\phi^1=\phi^2$, which then implies the main theorem.

Let t_2 be fixed. Since the transformation of $(\beta^1_t)_{0\le t<T_1(t_2)}$ via $f^2_{t_2}$ is a backward radial SLE$(\kappa; -\kappa-6)$ process started from $(f^2_{t_2}(z_1); B_{\beta^2_{t_2}})$, the union of the supports of $(f^2_{t_2})^*(\beta^1_t)$, $0\le t<T_1(t_2)$, is $\mathbb{T}\setminus B_{\beta^2_{t_2}}$. Since $f^2_{t_2}$ maps $\mathbb{T}\setminus S_{\beta^2_{t_2}}$ onto $\mathbb{T}\setminus B_{\beta^2_{t_2}}$, the union of the supports of β^1_t, $0\le t<T_1(t_2)$, is $\mathbb{T}\setminus S_{\beta^2_{t_2}}$. This shows that $S_{\beta^1_{T_1(t_2)}}$ and $S_{\beta^2_{t_2}}$ share two end points. Since both ϕ^1 and ϕ^2 flip these two end points, they agree on these two points. Letting t_2 vary, we conclude that $\phi^1=\phi^2$.


$f^2_{t_2}$
z_1
$S_{\beta^1_{T_1(t_2)}}$
$S_{\beta^2_{t_2}}$
z_2
$f^2_{t_2}(z_1)$
$\beta^2_{t_2}$

It remains to prove the above theorem. To construct the coupling, we use the idea in the proof of the reversibility of forward chordal SLE_κ for $\kappa \in (0,4]$. First, we construct local couplings. Let I_1 and I_2 be two closed arcs on $\mathbb{T}$ such that $\mathrm{dist}(I_1, I_2) > 0$ and the interior of I_j contains z_j, $j = 1, 2$. We call (I_1, I_2) a disjoint pair. Let (β^j_t), $j = 1, 2$, be a backward radial $\mathrm{SLE}(\kappa; -\kappa - 6)$ process started from $(z_j; z_{3-j})$. Let $T_j(I_j)$ be the first time that $S_{\beta^j_t}$ is not contained in the interior of I_j, $j = 1, 2$. We say that the two processes are well coupled within (I_1, I_2) if the following holds.

- If $t_2 \leq T_2(I_2)$ is a stopping time for (β^2_t), then the transformation of $(\beta^1_t)_{0 \leq t < T_1(I_1)}$ via $f^2_{t_2}$ is a stopped backward radial SLE$(\kappa; -\kappa - 6)$ process started from $(f^2_{t_2}(z_1); B_{\beta^2_{t_2}})$. A similar result holds if the indices "1" and "2" are switched.

- If $t_2 \le T_2(I_2)$ is a stopping time for (β_t^2), then the transformation of $(\beta_t^1)_{0 \le t < T_1(I_1)}$ via $f_{t_2}^2$ is a stopped backward radial SLE$(\kappa; -\kappa - 6)$ process started from $(f_{t_2}^2(z_1); B_{\beta_{t_2}^2})$. A similar result holds if the indices "1" and "2" are switched.

Such coupling can be constructed by weighting an independent coupling of two backward radial SLE$(\kappa; -\kappa - 6)$ processes by a suitable Radon-Nikodym derivative, which is obtained by a standard argument on Loewner equations.

Then we are able to show that, for any finitely many disjoint pairs (I_1^m, I_2^m), $1 \le m \le n$, there is a coupling of two backward radial SLE$(\kappa; -\kappa-6)$ processes, such that for any m, the two processes are well coupled within (I_1^m, I_2^m). Such coupling is obtained by weighting an independent coupling of two backward radial SLE$(\kappa; -\kappa-6)$ processes by a RN derivative, which is related with the RN derivatives for a good coupling within each (I_1^m, I_2^m).

Now let $(I_1^m, I_2^m)_{m\in\mathbb{N}}$ be a sequence of disjoint pairs, which is dense in the space of disjoint pairs. For every $n \in \mathbb{N}$, the above result shows that there is a coupling of two backward radial SLE$(\kappa; -\kappa - 6)$ processes, which are well coupled within (I_1^m, I_2^m), for m from 1 up to n. Let μ_n denote the distribution of such coupling. The sequence (μ_n) converges in some suitable topology to a measure μ, which is exactly the coupling of two backward radial SLE$(\kappa; -\kappa - 6)$ processes that we are looking for.

Thank you!