# NONLINEAR BELTRAMI EQUATIONS

## Uniqueness and QC Families

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W4: Quasiconformal Geometry and Elliptic PDEs

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# WITH

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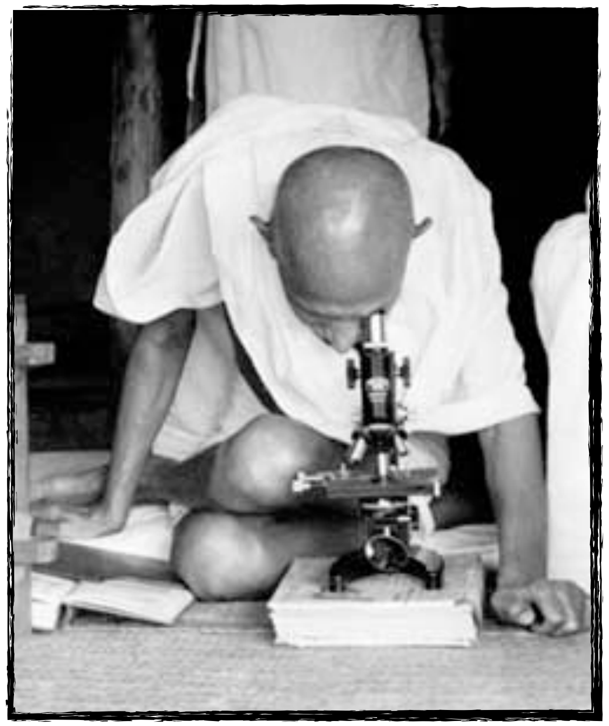
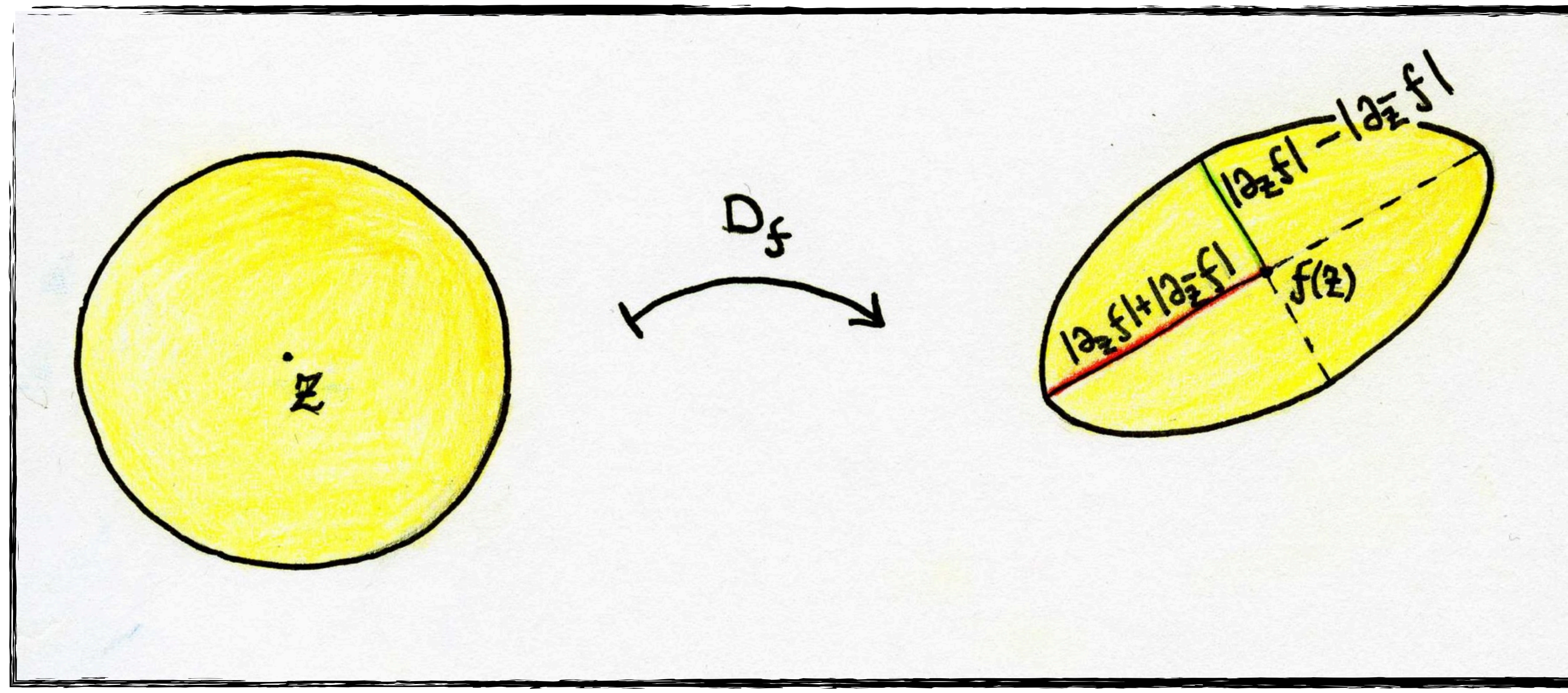
# QUASICONFORMAL MAPPING






Infinitesimally quasiconformal functions map disks into ellipsoids.

Homeomorphism $f : \Omega \to \mathbb{C} \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is *K*-quasiconformal if for almost everywhere the *classical Beltrami equation* holds

$$\partial_{\bar z} f(z) = \mu(z)\, \partial_z f(z), \qquad |\mu(z)| \leqslant k < 1, \quad K = \frac{1+k}{1-k}$$

Cauchy-Riemann

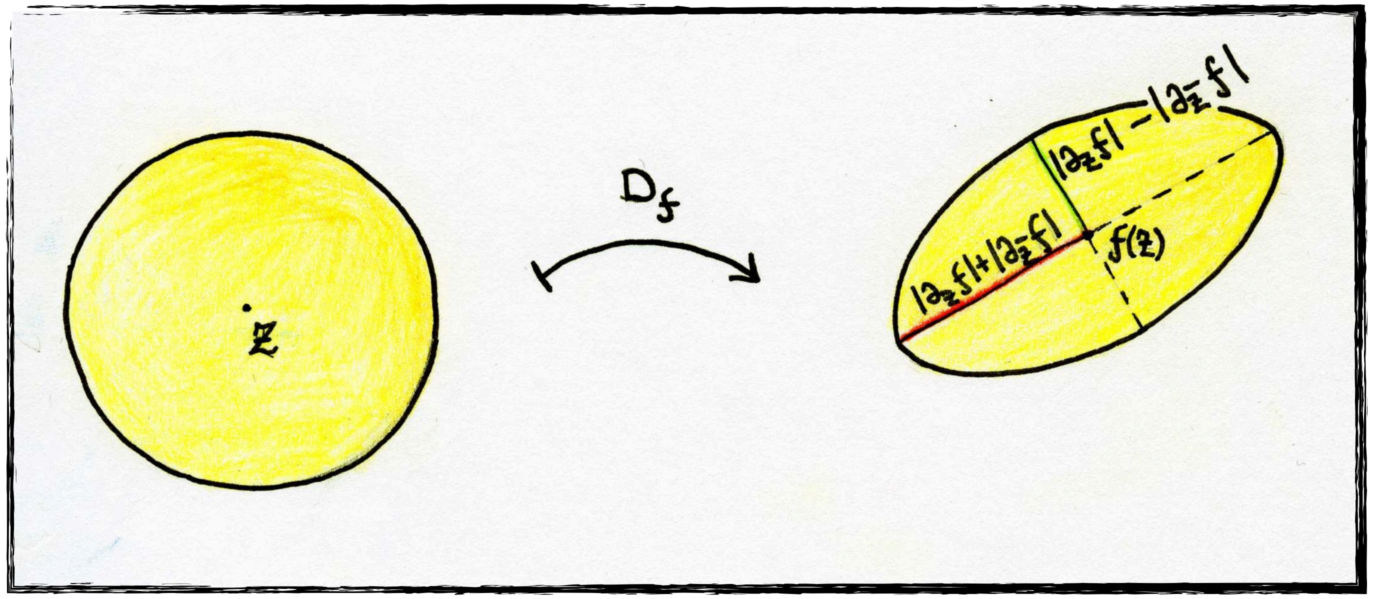
formal adjoint

$$2\partial_{\bar z} f(z) = \partial_x f(z) + i\partial_y f(z), \quad 2\partial_z f(z) = \partial_x f(z) - i\partial_y f(z), \quad z = x + iy$$

$$\partial_{\bar z} f(z) = \mu(z)\,\partial_z f(z), \qquad |\mu(z)| \leqslant k < 1, \quad K = \frac{1+k}{1-k}$$

One can measurably preassign the eccentricity and angle of the ellipses.

$$\frac{\text{major axis}}{\text{minor axis}} = \frac{|\partial_z f| + |\partial_{\bar z} f|}{|\partial_z f| - |\partial_{\bar z} f|} \leq K$$




Every solution $g \in W^{1,2}_{\mathrm{loc}}(\Omega)$ can be factorized as $g = h \circ f$ where $h$ is analytic and $f$ is a homeomorphic solution (Stoïlow factorization).

# QUASICONFORMAL FAMILY

$$\partial_{\bar{z}} f(z) = \mu(z)\, \partial_z f(z)$$

Homeomorphic solution $\Phi : \mathbb{C} \to \mathbb{C} \in W^{1,2}_{\text{loc}}(\mathbb{C})$ is called normalized if $\Phi(0) = 0, \Phi(1) = 1$

by Stoilow, solution = (analytic ∘ homeomorphism)

There is a unique homeomorphic solution that maps $0 \mapsto 0,\ 1 \mapsto a \in \mathbb{C} \setminus \{0\}$; namely, $a\,\Phi(z)$

$\{a\,\Phi(z) : a \in \mathbb{C}\}$ is a $\mathbb{C}$-linear family of quasiconformal maps (and constant $0$)

Conversely, if one has a $\mathbb{C}$-linear family of quasiconformal maps $\{a\, f : a \in \mathbb{C}\}$, one can associate to it a classical Beltrami equation, by setting

$$\mu(z) = \frac{\partial_{\bar{z}} f(z)}{\partial_z f(z)}$$

It is well-defined (and unique), since $\partial_z f(z) \neq 0$ almost everywhere.

the family is generated by one function, $f$, (injectivity)

Families appear in the context of *G*-convergence properties of $\mathbb{R}$-linear Beltrami operators,

$$\partial_{\bar z} - \mu_j(z)\partial_z - \nu_j(z)\overline{\partial_z}, \qquad |\mu_j(z)| + |\nu_j(z)| \leqslant k < 1$$

*Giannetti, Iwaniec, Kovalev, Moscariello,* and *Sbordone* (2004),
*Bojarski, D'Onofrio, Iwaniec,* and *Sbordone* (2005)

Homeomorphic solutions to $\mathbb{R}$-*linear Beltrami equation*

$$\partial_{\bar z} f(z) = \mu(z)\,\partial_z f(z) + \nu(z)\,\overline{\partial_z f(z)}, \qquad |\mu(z)| + |\nu(z)| \leq k < 1$$

form an $\mathbb{R}$-linear family of quasiregular mappings. Is their linear combination injective?

Homeomorphic solutions to $\mathbb{R}$-*linear Beltrami equation*

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form an $\mathbb{R}$-linear family of quasiregular mappings. Is their linear combination injective?

Yes (after normalization): Homeomorphic solution is uniquely defined knowing its values at two distinct points. Moreover, the linear combination is either homeomorphism or constant.

Idea: $\Psi = F \circ \Phi$, where homeomorphism $F$ solves a reduced equation

$$\partial_{\bar z} f(z) = \lambda(z)\,\mathrm{Im}(\partial_z f(z)) \qquad |\lambda(z)| \leqslant 2k/(1+k^2)$$

The only homeomorphic solution to the reduced equation that fixes two points is the identity, *Astala, Iwaniec,* and *Martin* (2009): $z \mapsto \dfrac{f(z) - tz}{1-t}$

If we normalize $\Phi(0) = 0 = \Psi(0)$, the linear independence of $\Phi(1), \Psi(1)$ implies that $\alpha\Phi(z) + \beta\Psi(z),\ \alpha, \beta \in \mathbb{R}$, is *K*-quasiconformal, and we have an $\mathbb{R}$-linear family of quasiconformal mappings.

# $\mathbb{R}$-LINEAR FAMILY OF QC MAPS

$$\partial_{\bar z} f(z) = \mu(z)\,\partial_z f(z) + \nu(z)\,\overline{\partial_z f(z)}$$

Conversely, if we have an $\mathbb{R}$-linear family of quasiconformal mappings

$$\{\alpha\,\Phi(z) + \beta\,\Psi(z) : \alpha, \beta \in \mathbb{R}\}$$

can we define $\mu$ and $\nu$ so that every mapping of the linear family solves the $\mathbb{R}$-linear equation given by $\mu, \nu$?

generated by two mappings (injectivity)

Yes we can!

when matrix of the system of linear equations is invertible

$$\partial_{\bar z}\Phi(z) = \mu(z)\,\partial_z\Phi(z) + \nu(z)\,\overline{\partial_z\Phi(z)}$$
$$\partial_{\bar z}\Psi(z) = \mu(z)\,\partial_z\Psi(z) + \nu(z)\,\overline{\partial_z\Psi(z)}$$

$$\mu(z) = i\,\frac{\Psi_{\bar z}\overline{\Phi_z} - \overline{\Psi_z}\Phi_{\bar z}}{2\mathrm{Im}(\Phi_z\overline{\Psi_z})} \qquad \nu(z) = i\,\frac{\Phi_{\bar z}\Psi_z - \Phi_z\Psi_{\bar z}}{2\mathrm{Im}(\Phi_z\overline{\Psi_z})}$$

On the singular set, we set $\nu \equiv 0$.

*Giannetti, Iwaniec, Kovalev, Moscariello,* and *Sbordone* (2004),

*Bojarski, D'Onofrio, Iwaniec,* and *Sbordone* (2005)

# $\mathbb{R}$-LINEAR FAMILY OF QC MAPS

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Conversely, if we have an $\mathbb{R}$-linear family of quasiconformal mappings

$$\{\alpha\,\Phi(z) + \beta\,\Psi(z) : \alpha, \beta \in \mathbb{R}\}$$

we define

$$\mu(z) = i\,\frac{\Psi_{\bar z}\overline{\Phi_z} - \overline{\Psi_z}\Phi_{\bar z}}{2\mathrm{Im}(\Phi_z\overline{\Psi_z})} \qquad \nu(z) = i\,\frac{\Phi_{\bar z}\Psi_z - \Phi_z\Psi_{\bar z}}{2\mathrm{Im}(\Phi_z\overline{\Psi_z})}$$

Unique? Yes, by a Wronsky-type theorem, *Alessandrini* and *Nesi* (2009), *Astala* and *Jääskeläinen* (2009); *Bojarski, D'Onofrio, Iwaniec,* and *Sbordone* (2005) $k < 1/2$

**Theorem.** Suppose $\Phi, \Psi \in W^{1,2}_{\mathrm{loc}}(\Omega)$ are homeomorphic solutions to

$$\partial_{\bar z} f(z) = \mu(z)\,\partial_z f(z) + \nu(z)\,\overline{\partial_z f(z)}, \qquad |\mu(z)| + |\nu(z)| \leqslant k < 1,$$

for almost every $z \in \Omega$. Solutions $\Phi$ and $\Psi$ are $\mathbb{R}$-linearly independent if and only if complex gradients $\partial_z\Phi$ and $\partial_z\Psi$ are pointwise independent almost everywhere, i.e.,

$$\mathrm{Im}(\partial_z\Phi\,\overline{\partial_z\Psi}) \neq 0$$ does not change sign, BDIS(2005)

# $\mathbb{R}$-LINEAR FAMILY OF QR MAPS

$$\partial_{\bar z} f(z) = \mu(z)\, \partial_z f(z) + \nu(z)\, \overline{\partial_z f(z)}$$

Wronsky-type theorem, *Alessandrini* and *Nesi* (2009), *Astala* and *Jääskeläinen* (2009); *Bojarski, D'Onofrio, Iwaniec,* and *Sbordone* (2005) $k < 1/2$

**Theorem.** Suppose $\Phi, \Psi \in W^{1,2}_{\text{loc}}(\Omega)$ are ~~homeomorphic~~ solutions to

$$\partial_{\bar z} f(z) = \mu(z)\, \partial_z f(z) + \nu(z)\, \overline{\partial_z f(z)}, \qquad |\mu(z)| + |\nu(z)| \leqslant k < 1,$$

for almost every $z \in \Omega$. Solutions $\Phi$ and $\Psi$ are $\mathbb{R}$-linearly independent if and only if complex gradients $\partial_z \Phi$ and $\partial_z \Psi$ are pointwise independent almost everywhere, i.e.,

$$\operatorname{Im}(\partial_z \Phi\, \overline{\partial_z \Psi}) \neq 0$$

Jääskeläinen (2012)

# BELTRAMI EQUATIONS

ℂ-linear | ℝ-linear | Nonlinear

$$f_{\bar z} = \mu(z)\, f_z$$

$$f_{\bar z} = \mu(z)\, f_z + \nu(z)\,\overline{f_z}$$

$$f_{\bar z} = \mathcal{H}(z, f_z)$$

$$\mathcal{H}(z,w) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$$

$z \mapsto \mathcal{H}(z,w)$ measurable

$w \mapsto \mathcal{H}(z,w)$ $k$-Lipschitz

$\mathcal{H}(z,0) \equiv 0$

Difference of two solutions is $K$-quasiregular

$$|\partial_{\bar z} f(z) - \partial_{\bar z} g(z)| = |\mathcal{H}(z, \partial_z f(z)) - \mathcal{H}(z, \partial_z g(z))| \leqslant k |\partial_z f(z) - \partial_z g(z)|$$

Constants are solutions.

| $\mathbb{C}$-linear | $\mathbb{R}$-linear | Nonlinear |
| --- | --- | --- |
| $f_{\bar{z}} = \mu(z)\, f_z$ | $f_{\bar{z}} = \mu(z)\, f_z + \nu(z)\, \overline{f_z}$ | $f_{\bar{z}} = \mathcal{H}(z, f_z)$ |
| There is a unique homeomorphic solution $\Phi$ such that $\Phi(0) = 0, \Phi(1) = 1$ | There is a unique homeomorphic solution $\Phi$ such that $\Phi(0) = 0, \Phi(1) = 1$ | Not unique in general, *Astala, Clop, Faraco, Jääskeläinen,* and *Székelyhidi Jr.* (2012) |

$z \mapsto \mathcal{H}(z, w)$ measurable  $w \mapsto \mathcal{H}(z, w)$ $k$-Lipschitz  $\mathcal{H}(z, 0) \equiv 0$

**Theorem.** If $\limsup_{|z|\to\infty} k(z) < 3 - 2\sqrt{2} = 0.17157...,$ then the nonlinear equation

$$\partial_{\bar{z}} f(z) = \mathcal{H}(z, \partial_z f(z))$$

admits a unique homeomorphic solution $\Phi : \mathbb{C} \to \mathbb{C} \in W^{1,2}_{\text{loc}}(\mathbb{C})$ normalized by $\Phi(0) = 0, \Phi(1) = 1.$

Furthermore, the bound on $k$ is sharp.

# COUNTEREXAMPLES

*Astala, Clop, Faraco, Jääskeläinen, and Székelyhidi Jr.* (2012)

$z \mapsto \mathcal{H}(z,w)$ measurable $\qquad w \mapsto \mathcal{H}(z,w)$ $k$-Lipschitz $\qquad \mathcal{H}(z,0) \equiv 0$

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admits a unique homeomorphic solution $\Phi : \mathbb{C} \to \mathbb{C} \in W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ normalized by $\Phi(0) = 0, \Phi(1) = 1.$

$$K = \frac{1+k}{1-k} = \sqrt{2} \Leftrightarrow k = 3 - 2\sqrt{2}$$

Furthermore, the bound on $k$ is sharp.

$$f_t(z) = \begin{cases} (1+t)\, z|z|^{\sqrt{2}-1} - t(z|z|^{1/\sqrt{2}-1})^2, & \text{for } |z| > 1, \\ (1+t)\, z - tz^2, & \text{for } |z| \leq 1, \end{cases}$$

$$g_t(z) = \begin{cases} (1+t)\, z|z|^{\sqrt{2}-1} - tz|z|^{1/\sqrt{2}-1}, & \text{for } |z| > 1, \\ z, & \text{for } |z| \leq 1. \end{cases}$$

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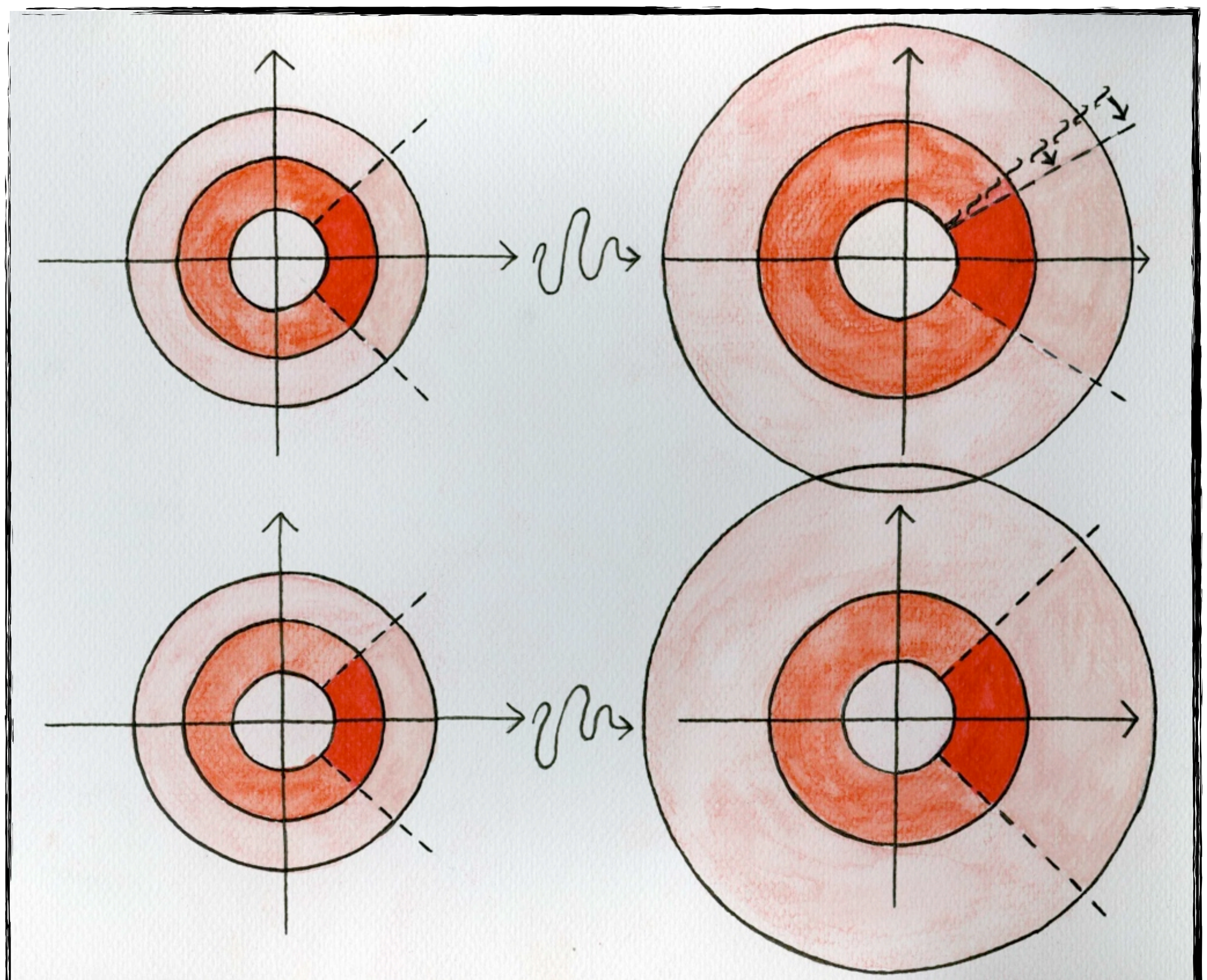

ℂ-linear | ℝ-linear | Nonlinear

**ℂ-linear**

$$f_{\bar z} = \mu(z)\, f_z$$

There is a unique homeomorphic solution $\Phi$ such that $\Phi(0) = 0, \Phi(1) = 1$

**ℝ-linear**

$$f_{\bar z} = \mu(z)\, f_z + \nu(z)\, \overline{f_z}$$

There is a unique homeomorphic solution $\Phi$ such that $\Phi(0) = 0, \Phi(1) = 1$

**Nonlinear**

$$f_{\bar z} = \mathcal{H}(z, f_z)$$

There is a unique homeomorphic solution $\Phi$ such that $\Phi(0) = 0, \Phi(1) = 1$ when near the infinity

$$k(z) < 3 - 2\sqrt{2}$$

Homeomorphic solution is uniquely defined by its values at two distinct points. Difference is homeomorphism or constant.

$$\{\Phi_a : \Phi_a\ K - \mathrm{qc}, 0 \mapsto 0, 1 \mapsto a\}$$

ℂ-linear family of quasiconformal mappings

ℝ-linear family of quasiconformal mappings

family of quasiconformal mappings

# FROM FAMILY TO EQUATION

$$\{\Phi_a : \Phi_a \; K - \text{qc}, 0 \mapsto 0, 1 \mapsto a\}$$

$\mathbb{C}$-linear family of quasiconformal mappings

$$\{a\,\Phi(z) : a \in \mathbb{C}\}$$

$$\Phi(0) = 0, \Phi(1) = 1$$

$\mathbb{R}$-linear family of quasiconformal mappings

$$\{\alpha\,\Phi(z) + \beta\,\Psi(z) : \alpha, \beta \in \mathbb{R}\}$$

$$\Phi(0) = 0, \Phi(1) = 1$$

$$\Psi(0) = 0, \Psi(1) = i$$

family of quasiconformal mappings

unique $\mu$ and $\nu$ s.t. every mapping of the family solves the Beltrami equation (Wronsky-type theorem)

$$f_{\bar z} = \mu(z)\, f_z$$

$$f_{\bar z} = \mu(z)\, f_z + \nu(z)\, \overline{f_z}$$

$$¿\; f_{\bar z} = \mathcal{H}(z, f_z)\, ?$$

linearly independent, thus their complex gradients are linearly independent

# HOW TO DEFINE EQUATION?

We have a family of quasiconformal mappings $\{\Phi_a : \Phi_a \ K - \mathrm{qc}, 0 \mapsto 0, 1 \mapsto a\}$,
$\Phi_a(z) - \Phi_b(z)$ is *K*-quasiconformal.

We want nonlinear equation $\partial_{\bar{z}} f(z) = \mathcal{H}(z, \partial_z f(z))$

<u>Define</u> pointwise $\partial_{\bar{z}} \Phi_a(z) = \mathcal{H}(z, \partial_z \Phi_a(z))$

$z \mapsto \mathcal{H}(z, w)$ measurable
$w \mapsto \mathcal{H}(z, w)$ $k$-Lipschitz
$\mathcal{H}(z, 0) \equiv 0$

Not overdetermined:

$$|\partial_{\bar{z}} \Phi_a(z) - \partial_{\bar{z}} \Phi_b(z)| \leqslant k |\partial_z \Phi_a(z) - \partial_z \Phi_b(z)|$$

One can <u>extend</u> $w \mapsto \mathcal{H}(z, w)$ to whole plane as a Lipschitz map by Kirszbraun extension theorem. Hence there exists a nonlinear Beltrami equation.

Unique, when one has a full range $\{\partial_z \Phi_a(z) : a \in \mathbb{C}\} = \mathbb{C}$ for almost every $z$.

In the case of linear families $\{a\, \partial_z \Phi(z)\}, \{\alpha\, \partial_z \Phi(z) + \beta\, \partial_z \Psi(z)\}$

complex gradients are linearly independent (Wronsky-type theorem)

# PROPERTIES OF THE FAMILY

*Astala, Clop, Faraco,* and *Jääskeläinen*

We have a family of quasiconformal mappings $\{\Phi_a : \Phi_a \ K - \mathrm{qc}, 0 \mapsto 0, 1 \mapsto a\}$, $\Phi_a(z) - \Phi_b(z)$ is *K*-quasiconformal.

What other properties does the family have? For instance, when do we have a full range $\{\partial_z \Phi_a(z) : a \in \mathbb{C}\} = \mathbb{C}$ for almost every $z$?

It turns out that $a \mapsto \partial_a \Phi_a(z)$ exists for almost every $a$ (exceptional set might depend on $z$; and this causes difficulties). Note that $z \mapsto \partial_z \Phi_a(z)$ exists for almost every $z$ (by quasiconformality). The exceptional set depends on $a$.

We need some relation between $a$ and $z$.

What more can we say about the family, if we know more about the nonlinear Beltrami equation $\partial_{\bar{z}} f(z) = \mathcal{H}(z, \partial_z f(z))$?

$$\partial_{\bar z} f(z) = \mathcal{H}(z, \partial_z f(z))$$

$z \mapsto \mathcal{H}(z,w)$ measurable

$w \mapsto \mathcal{H}(z,w)$ $C^1$

$k$-Lipschitz, $k(z) < 3 - 2\sqrt{2}$ near the infinity

$\mathcal{H}(z,0) \equiv 0$

*Astala, Clop, Faraco, and Jääskeläinen*

$$\{\Phi_a : \Phi_a \ K - \mathrm{qc}, 0 \mapsto 0, 1 \mapsto a\}$$

$\Phi_a(z) - \Phi_b(z)$ is $K$-quasiconformal.

*takes care of the first exceptional set*

**Theorem.** For each fixed $z \in \mathbb{C}$, the mapping $a \mapsto \Phi_a(z)$ is continuously differentiable. Further, the convergence of derivatives $\partial_a \Phi_a(z)$ is locally uniform in $z$ .

In fact, the directional derivatives

$$\partial_e^a \Phi_a(z) := \lim_{t \to 0^+} \frac{\Phi_{a+te}(z) - \Phi_a(z)}{t}, \qquad e \in \mathbb{C},$$

are quasiconformal mappings of $z$ all satisfying the same $\mathbb{R}$-linear Beltrami equation

$$\partial_{\bar z} f(z) = \mu_a(z) \partial_z f(z) + \nu_a(z) \overline{\partial_z f(z)}$$

$$\mu_a(z) = \partial_w \mathcal{H}\big(z, \partial_z \Phi_a(z)\big), \qquad \nu_a(z) = \partial_{\bar w} \mathcal{H}\big(z, \partial_z \Phi_a(z)\big)$$

$\partial_{\bar{z}} f(z) = \mathcal{H}(z, \partial_z f(z))$

$z \mapsto \mathcal{H}(z,w)$ ~~measurable~~ $C^{\alpha}_{\mathrm{loc}}$

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$\mathcal{H}(z,0) \equiv 0$

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$\Phi_a(z) - \Phi_b(z)$ is *K*-quasiconformal.

Schauder estimates: $\Phi_a \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{C})$

takes care of the second exceptional set

Wronsky-type theorem + Theorem about directional derivatives

*Astala, Clop, Faraco,* and *Jääskeläinen:*

Fixing $z$, Jacobian of $a \mapsto \partial_z \Phi_a(z) : \mathbb{C} \to \mathbb{C}$

$$J(a, a \mapsto \partial_z \Phi_a(z)) = \mathrm{Im}(\partial_z[\partial_1^a \Phi_a(z)]\, \overline{\partial_z\, [\partial_i^a \Phi_a(z)]}) \neq 0 \qquad \text{a.e. } z$$

Hence $a \mapsto \partial_z \Phi_a(z)$ is locally injective (locally homeomorphic, by invariance of domain); in particular, an open mapping.

$$\partial_{\bar{z}} f(z) = \mathcal{H}(z, \partial_z f(z))$$

$z \mapsto \mathcal{H}(z,w)$ ~~measurable~~ $C^{\alpha}_{\text{loc}}$

$w \mapsto \mathcal{H}(z,w)$ $C^1$

$k$-Lipschitz, $k(z) < 3 - 2\sqrt{2}$ near the infinity

$\mathcal{H}(z,0) \equiv 0$

$\{\Phi_a : \Phi_a\ K-\text{qc}, 0 \mapsto 0, 1 \mapsto a\}$

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*takes care of the second exceptional set*

*Wronsky-type theorem + Theorem about directional derivatives*

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Hence $a \mapsto \partial_z \Phi_a(z)$ is locally injective (locally homeomorphic, by invariance of domain); in particular, an open mapping. Can be extended as a continuous mapping between Riemann spheres $\hat{\mathbb{C}}$. Thus 'the covering map stuff' gives that $a \mapsto \partial_z \Phi_a(z) : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is actually a homeomorphism for almost every $z$. (We get more than the full range $\{\partial_z \Phi_a(z) : a \in \mathbb{C}\} = \mathbb{C}$.)

$$\partial_{\bar z} f(z) = \mathcal{H}(z, \partial_z f(z))$$

$z \mapsto \mathcal{H}(z,w)$ ~~measurable~~ $C^\alpha_{\text{loc}}$

$w \mapsto \mathcal{H}(z,w)$ $C^1$

$k$-Lipschitz, $k(z) < 3 - 2\sqrt{2}$ near the infinity

$\mathcal{H}(z,0) \equiv 0$

$\{\Phi_a : \Phi_a\ K - \text{qc}, 0 \mapsto 0, 1 \mapsto a\}$

$\Phi_a(z) - \Phi_b(z)$ is $K$-quasiconformal.

$K(z) < \sqrt{2}$ near the infinity

$\Phi_a \in C^{1,\alpha}_{\text{loc}}(\mathbb{C})$

$\{\partial_z \Phi_a(z) : a \in \mathbb{C}\} = \mathbb{C}$

+ some regularity in $a$

$\{\Phi_a : \Phi_a\ K - \text{qc}, 0 \mapsto 0, 1 \mapsto a\}$

$\Phi_a(z) - \Phi_b(z)$ is $K$-quasiconformal.

$\Phi_a \in C^{1,\alpha}_{\text{loc}}(\mathbb{C})$

$a \mapsto \partial_z \Phi_a(z) : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ a homeomorphism

in particular, the full range $\{\partial_z \Phi_a(z) : a \in \mathbb{C}\} = \mathbb{C}$

+ some regularity in $a$

$$\partial_{\bar z} f(z) = \mathcal{H}(z, \partial_z f(z))$$

$z \mapsto \mathcal{H}(z,w)$ ~~measurable~~ $C^\alpha_{\text{loc}}$

$w \mapsto \mathcal{H}(z,w)$ $C^1$

$k$-Lipschitz,

$\mathcal{H}(z,0) \equiv 0$

# THANK YOU!