Sobolev (and Bi-Sobolev) homeomorphisms with zero Jacobian a.e.

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Consequence: By area formula $\exists N \subset \Omega, \mathcal{L}_n(\Omega \setminus N) = \mathcal{L}_n(\Omega)$

$0 = \int_{\Omega \setminus N} J_f(x) = \int_{f(\Omega \setminus N)} 1 = \mathcal{L}_n(f(\Omega \setminus N))$

$\Rightarrow \mathcal{L}_n(N) = 0 \quad \text{but} \quad \mathcal{L}_n(f(N)) = \mathcal{L}_n(f(\Omega))$
Does there exist homeomorphisms with $J_f \equiv 0$?

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**Known:** $\exists f : J_f = 0$ on a set of positive measure

Iteration does not work. Cannot have $J_f = 0 \Rightarrow |Df(x)| = 0$
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**Theorem**

Let $1 \leq p < n$. There is a homeomorphism $f \in W^{1,p}((0,1)^n, (0,1)^n)$ such that $J_f(x) = 0$ a.e.

Reshetnyak: $W^{1,n}$ homeomorphism $\Rightarrow$ Lusin ($N$) condition
**Theorem (R. Černý)**

There is a homeomorphism \( f \in W^{1,1}((0, 1)^n, (0, 1)^n) \) with \( \lim_{\varepsilon \to 0} \varepsilon \int |Df|^{n-\varepsilon} \leq C \) such that \( J_f(x) = 0 \) a.e.

Malý, Koskela, Kauhanen: \( \lim_{\varepsilon \to 0} \varepsilon \int |Df|^{n-\varepsilon} = 0 \implies (N) \)
Sharpness and Motivation

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**Motivation:**

- Nonlinear elasticity, MFD, far from \( W^{1,n} \) no results
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  $n = 2$ or $n = 3$ sense-preserving in $W^{1,1} \implies J_f \geq 0$
- Distributinal $J_f$ can be purely singular even for homeomorphism
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Question (C. Sbordone): Is it possible that also \( f^{-1} \in W^{1,1} \)?
Theorem (H., D’Onofrio, Schiattarella)

Let $n \geq 3$. There is a bi-Sobolev homeomorphism $f : (0, 1)^n \rightarrow (0, 1)^n$ such that $J_f(x) = 0$ and $J_{f^{-1}}(y) = 0$ a.e.

Impossible if $f \in W^{1,n-1}$, especially if $n = 2$
Bi-Sobolev homeomorphism with $J_f \equiv 0$

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R. Černý: $f, f^{-1} \in W^{1,p}$ for $p < \frac{n}{2}$
also minors of $k$-th order can be zero a.e.
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**Open problems:**

- find optimal $p, q$ such that there is homeomorphism $f \in W^{1,p}, f^{-1} \in W^{1,q}$ with $J_f = 0$ and $J_{f^{-1}}$ a.e.
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Theorem (H., D’Onofrio, Schiattarella)

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Open problems:

- find optimal $p, q$ such that there is homeomorphism $f \in W^{1,p}, f^{-1} \in W^{1,q}$ with $J_f = 0$ and $J_{f^{-1}}$ a.e.
- $n \geq 4$, $f \in W^{1,1}(\Omega, \mathbb{R}^n)$ homeomorphism,
  $\Rightarrow J_f \geq 0$ a.e. or $J_f \leq 0$ a.e.
A. $f$ not MFD $J_f(x) = 0 \implies |Df(x)| = 0$ - not symmetric

Construct $F_1$: $J_{F_1} = 0$ on $C_1$, $|C_1| = \frac{1}{2}$

Construct $F_2$: $F_2 = F_1$ on $C_1$, $J_{F_2} = 0$ on $C_2$, $|C_2| = \frac{1}{4}$

Construct $F_3$: $F_3 = F_1$ on $C_1 \cup C_2$, $J_{F_3} = 0$ on $C_3$, $|C_3| = \frac{1}{8}$

$F_{2k}$ squeeze $C_{2k}$ in $x$, $F_{2k+1}$ squeeze $C_{2k+1}$ in $y$, $F = \lim F_j$
Key ingredients of the construction \((n = 2\) only)

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Construct \(F_1\): \(J_{F_1} = 0\) on \(C_1, |C_1| = \frac{1}{2}\)

Construct \(F_2\): \(F_2 = F_1\) on \(C_1, J_{F_2} = 0\) on \(C_2, |C_2| = \frac{1}{4}\)

Construct \(F_3\): \(F_3 = F_1\) on \(C_1 \cup C_2, J_{F_3} = 0\) on \(C_3, |C_3| = \frac{1}{8}\)

\(F_{2k}\) squeeze \(C_{2k}\) in \(x\), \(F_{2k+1}\) squeeze \(C_{2k+1}\) in \(y\), \(F = \lim F_j\)

B. Matrices almost diagonal - key estimate

\[
\left\| \begin{pmatrix} d_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & d_2 \end{pmatrix} \right\| \leq \min\{d_1, d_2\}
\]
Key ingredients of the construction \((n = 2\) only)

A. \(f\) not MFD \(J_f(x) = 0 \implies |Df(x)| = 0\) - not symmetric

Construct \(F_1: J_{F_1} = 0\) on \(C_1, |C_1| = \frac{1}{2}\)

Construct \(F_2: F_2 = F_1\) on \(C_1, J_{F_2} = 0\) on \(C_2, |C_2| = \frac{1}{4}\)

Construct \(F_3: F_3 = F_1\) on \(C_1 \cup C_2, J_{F_3} = 0\) on \(C_3, |C_3| = \frac{1}{8}\)

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C. Basic building block

\[
D\varphi = \begin{pmatrix} \frac{s'}{s} & 0 \\ 0 & 1 \end{pmatrix}, \ D\varphi = \begin{pmatrix} \frac{1-s'}{1-s} & \pm w \left(1 - \frac{1-s'}{1-s}\right) \\ 0 & 1 \end{pmatrix}
\]
\[ J_{F_1} = 0 \text{ on } C_1 \text{ and } F_1 \in W^{1,p} \]

Parameters \( w_k = \frac{k+1}{tk^2-1} \), \( s_k = 1 - \frac{1}{tk^2} \) and \( s'_k = s_k \frac{k}{k+1} \).

\[
D\varphi = \begin{pmatrix} \frac{s'_k}{s_k} & 0 \\ \frac{s'_k}{s_k} & 0 \\ 0 & 1 \end{pmatrix}, \quad D\varphi = \begin{pmatrix} \left(1 - \frac{s'_k}{s_k}\right) & \pm w_k \left(1 - \frac{1-s'_k}{1-s_k}\right) \\ \left(1 - \frac{s'_k}{s_k}\right) & 0 \\ 0 & 1 \end{pmatrix}
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\[
D\varphi = \left( \begin{array}{cc} \frac{k}{k+1} & 0 \\ 0 & 1 \end{array} \right), \quad D\varphi = \left( \begin{array}{ccc} \frac{tk^2+k}{k+1} & (\leq tk) & \pm1 \\ 0 & 1 \end{array} \right)
\]
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\[ C_1 = \bigcap_{k=1}^{\infty} (\bigcup I_D), \quad |C_1| = |Q_0| s_1 s_2 \ldots = |Q_0| \prod_{i=1}^{\infty} s_i > 0 \]

on \( C_1 \), \( D F_1 = \prod_k \begin{pmatrix} s'_k & 0 \\ s_k & 1 \end{pmatrix} = \prod_k \begin{pmatrix} \frac{k}{k+1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \).
$J_{F_1} = 0$ on $C_1$ and $F_1 \in W^{1, p}$

Parameters $w_k = \frac{k+1}{tk^2-1}$, $s_k = 1 - \frac{1}{tk^2}$ and $s'_k = s_k \frac{k}{k+1}$.

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on $C_1$, $DF_1 = \prod_k \begin{pmatrix} s'_k \\ s_k \\ 0 \\ 1 \end{pmatrix} = \prod_k \begin{pmatrix} k \\ k+1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$

$\int_{Q_0 \setminus C_1} |DF_1|^p = (1 - s_1) \left( \frac{1 - s'_1}{1 - s_1} \right)^p + s_1 (1 - s_2) \left( \frac{1 - s'_2}{1 - s_2} \right)^p \left( \frac{s'_1}{s_1} \right)^p + \ldots$

$\leq \sum_k \frac{1}{tk^2} (tk)^p \frac{1}{k^p} = t^{p-1} \sum_k \frac{1}{k^2} < \infty$
\( F_2 \in W^{1,p} \) and \( F_{2i} \in W^{1,p} \)

Parameters \( w_k = \frac{k+1}{tk^2-1} \), \( s_k = 1 - \frac{1}{tk^2} \) and \( s'_k = s_k \frac{k}{k+1} \).

After \( k \) squeezing

\[
D\varphi = \left( \begin{array}{cc}
\frac{tk^2+k}{k+1} & \pm 1 \\
0 & 1
\end{array} \right) \left( \begin{array}{cc}
\frac{1}{k} & 0 \\
0 & 1
\end{array} \right) \sim \left( \begin{array}{cc}
t & \pm 1 \\
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\end{array} \right)
\]
$F_2 \in W^{1,p}$ and $F_{2i} \in W^{1,p}$

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\end{pmatrix} \begin{pmatrix} \frac{1}{k} & 0 
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\end{pmatrix} \sim \begin{pmatrix} t & \pm 1 
0 & 1 
\end{pmatrix}$$

$$\int_{Q_0 \setminus C_1 \cup C_2} |DF_2 - DF_1|^p \leq$$

$$\leq \sum_{k_2, k_1=1}^\infty (1 - s_{k_2})(1 - s_{k_1}) \left\| \begin{pmatrix} t & \pm 1 
0 & 1 
\end{pmatrix} \begin{pmatrix} 1 & 0 
\pm 1 & t 
\end{pmatrix} \right\|^p$$

$$\leq \sum_{k_2, k_1=1}^\infty \frac{1}{tk_1^2} \frac{1}{tk_2^2} Ct^p \leq Ct^{p-2}$$
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$$D\varphi = \left(\frac{tk^2+k}{k+1} \begin{pmatrix} tk & \pm 1 \\ 0 & 1 \end{pmatrix} \right) \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \sim \left( t \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} \right)$$

$$\int_{Q_0 \setminus C_1 \cup C_2} |D F_2 - D F_1|^p \leq \sum_{k_1, k_2 = 1}^{\infty} (1 - s_{k_2})(1 - s_{k_1}) \left\| \begin{pmatrix} t \pm 1 \\ 0 \ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pm 1 & t \end{pmatrix} \right\|^p \leq \sum_{k_2, k_1 = 1}^{\infty} \frac{1}{tk_1^2} \frac{1}{tk_2^2} Ct^p \leq Ct^{p-2}$$

$$\int_{Q_0 \setminus C_1 \cup \ldots \cup C_{2i}} |D F_{2i} - D F_{2i-1}|^p \leq \sum_{k_1, \ldots, k_{2i} = 1}^{\infty} (1 - s_{k_{2i}}) \ldots (1 - s_{k_1}) \left\| D_{2i} D_{2i-1} \right\|^p \ldots \left\| D_2 D_1 \right\|^p \leq \left( \sum_{k, l = 1}^{\infty} \frac{1}{tk^2} \frac{1}{tl^2} Ct^p \right)^i \leq \left( C t^{p-2} \right)^i \leq \left( \frac{1}{2} \right)^i$$
Notes for $n \geq 3$ and $f^{-1} \in W^{1,1}$

$n = 3$ we have to squeeze in 3 directions

$$\left\| \begin{pmatrix} d_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \right\| \leq \min\{d_1, d_2, d_3\}$$

Thank you for your attention.

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\begin{pmatrix}
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we rotate diamonds (QR decomposition) - upper diagonal - then $f \in W^{1,1}$ but $f^{-1} \notin W^{1,1}$
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we squeeze three times in domain, then three times in the target, this gives also $f^{-1} \in W^{1,1}$
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