Sobolev (and Bi-Sobolev) homeomorphisms with zero Jacobian a.e.

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Does there exist homeomorphisms with $J_f \equiv 0$?

Consequence: By area formula $\exists N \subset \Omega$, $\mathcal{L}_n(\Omega \setminus N) = \mathcal{L}_n(\Omega)$

$$0 = \int_{\Omega\setminus N} J_f(x) = \int_{f(\Omega\setminus N)} 1 = \mathcal{L}_n\big(f(\Omega \setminus N)\big)$$

$$\Rightarrow \mathcal{L}_n(N) = 0 \quad \text{but} \quad \mathcal{L}_n\big(f(N)\big) = \mathcal{L}_n\big(f(\Omega)\big)$$

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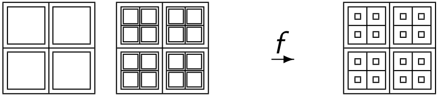
Known: $\exists f$: $J_f = 0$ on a set of positive measure



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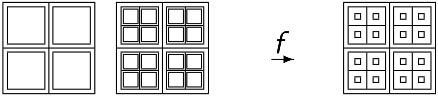

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Theorem

Let $1 \leq p < n$. There is a homeomorphism $f \in W^{1,p}((0,1)^n,(0,1)^n)$ such that $J_f(x) = 0$ a.e.

Reshetnyak: $W^{1,n}$ homeomorphism $\Longrightarrow$ Lusin (N) condition

Sharpness and Motivation

Theorem (R. Černý)

There is a homeomorphism $f \in W^{1,1}((0,1)^n,(0,1)^n)$ *with* $\lim_{\varepsilon\to 0} \varepsilon \int |Df|^{n-\varepsilon} \leq C$ *such that* $J_f(x) = 0$ *a.e.*

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Question (C. Sbordone): Is it possible that also $f^{-1} \in W^{1,1}$?

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Theorem (H., D'Onofrio, Schiattarella)

Let $n \geq 3$. There is a bi-Sobolev homeomorphism $f : (0,1)^n \to (0,1)^n$ such that $J_f(x) = 0$ and $J_{f^{-1}}(y) = 0$ a.e.

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Open problems:

- find optimal p, q such that there is homeomorphism $f \in W^{1,p}$, $f^{-1} \in W^{1,q}$ with $J_f = 0$ and $J_{f^{-1}}$ a.e.

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- find optimal p, q such that there is homeomorphism $f \in W^{1,p}$, $f^{-1} \in W^{1,q}$ with $J_f = 0$ and $J_{f^{-1}}$ a.e.
- $n \geq 4$, $f \in W^{1,1}(\Omega, \mathbf{R}^n)$ homeomorphism, $\overset{?}{\Longrightarrow} J_f \geq 0$ a.e. or $J_f \leq 0$ a.e.

Key ingredients of the construction ($n = 2$ only)

A. f not MFD $J_f(x) = 0 \Longrightarrow |Df(x)| = 0$ - **not symmetric**

Construct F_1: $J_{F_1} = 0$ on C_1, $|C_1| = \frac{1}{2}$
Construct F_2: $F_2 = F_1$ on C_1, $J_{F_2} = 0$ on C_2, $|C_2| = \frac{1}{4}$
Construct F_3: $F_3 = F_1$ on $C_1 \cup C_2$, $J_{F_3} = 0$ on C_3, $|C_3| = \frac{1}{8}$
F_{2k} squeeze C_{2k} in x, F_{2k+1} squeeze C_{2k+1} in y, $F = \lim F_j$

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$$\left\| \begin{pmatrix} d_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & d_2 \end{pmatrix} \right\| \leq \min\{d_1, d_2\}$$

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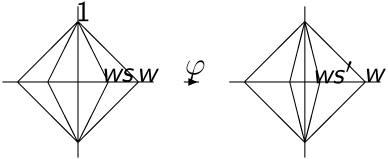

C. Basic building block

$$D\varphi = \begin{pmatrix} \frac{s'}{s} & 0 \\ 0 & 1 \end{pmatrix}, \quad D\varphi = \begin{pmatrix} \left(\frac{1-s'}{1-s}\right) & \pm w \left(1 - \frac{1-s'}{1-s}\right) \\ 0 & 1 \end{pmatrix}$$

$J_{F_1} = 0$ on C_1 and $F_1 \in W^{1,p}$

Parameters $w_k = \frac{k+1}{tk^2-1}$, $s_k = 1 - \frac{1}{tk^2}$ and $s'_k = s_k \frac{k}{k+1}$.

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$C_1 = \bigcap_{k=1}^{\infty}(\bigcup I_D)$, $|C_1| = |Q_0| s_1 s_2 \ldots = |Q_0| \prod_{i=1}^{\infty} s_i > 0$

on C_1, $DF_1 = \prod_k \begin{pmatrix} \frac{s_k'}{s_k} & 0 \\ 0 & 1 \end{pmatrix} = \prod_k \begin{pmatrix} \frac{k}{k+1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

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$$\int_{Q_0 \setminus C_1} |DF_1|^p = (1-s_1)\left(\frac{1-s'_1}{1-s_1}\right)^p + s_1(1-s_2)\left(\frac{1-s'_2}{1-s_2}\right)^p \left(\frac{s'_1}{s_1}\right)^p + \dots$$

$$\leq \sum_k \frac{1}{tk^2}(tk)^p \frac{1}{k^p} = t^{p-1} \sum_k \frac{1}{k^2} < \infty$$

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$$\int_{Q_0 \setminus C_1 \cup C_2} |DF_2 - DF_1|^p \leq$$
$$\leq \sum_{k_2, k_1 = 1}^{\infty} (1 - s_{k_2})(1 - s_{k_1}) \left\| \begin{pmatrix} t & \pm 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pm 1 & t \end{pmatrix} \right\|^p$$
$$\leq \sum_{k_2, k_1 = 1}^{\infty} \frac{1}{tk_1^2} \frac{1}{tk_2^2} C t^p \leq C t^{p-2}$$

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$$\int_{Q_0 \setminus C_1 \cup .. \cup C_{2i}} |DF_{2i} - DF_{2i-1}|^p \leq$$
$$\leq \sum_{k_1,\dots,k_{2i}=1}^{\infty} (1 - s_{k_{2i}}) \dots (1 - s_{k_1}) \left\| D_{2i} D_{2i-1} \right\|^p \dots \left\| D_2 D_1 \right\|^p$$
$$\leq \left(\sum_{k,l=1}^{\infty} \frac{1}{tk^2} \frac{1}{tl^2} Ct^p \right)^i \leq \left(Ct^{p-2} \right)^i \leq \left(\tfrac{1}{2} \right)^i$$

Notes for $n \geq 3$ and $f^{-1} \in W^{1,1}$

$n = 3$ we have to squeeze in 3 directions

$$\left\| \begin{pmatrix} d_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \right\| \leq \min\{d_1, d_2, d_3\}$$

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