L^2 boundedness of Riesz transforms and rectifiability

Xavier Tolsa

(joint work with F. Nazarov and A. Volberg)

24 May

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$$\mathcal{R}_{\mu,\varepsilon} f(x) = \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{n+1}}\, f(y)\, d\mu(y).$$

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Rectifiability

We say that E is rectifiable if it is $\mathcal{H}^1$-a.e. contained in a countable union of curves of finite length.

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Recall that if E is the graph of a Lipschitz function $A : \mathbb{R} \to \mathbb{R}$ and $\mu = \mathcal{H}^1_E$, then $\mathcal{C}_\mu$ is bounded in $L^2(\mu)$
(Calderón, Coifman-McIntosh-Meyer).

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Question: Let $E \subset \mathbb{R}^d$ with $\mathcal{H}^n(E) < \infty$, and $\mu = \mathcal{H}^n_E$. If $\mathcal{R}_\mu : L^2(\mu) \to L^2(\mu)$ is bounded, is then E n-rectifiable?

The Cauchy and transform and rectifiability

Theorem (David-Léger, 1998)

Let $E \subset \mathbb{C}$ with $\mathcal{H}^1(E) < \infty$, and $\mu = \mathcal{H}^1_E$. If $\mathcal{C}_\mu$ is bounded in $L^2(\mu)$, then E is rectifiable.

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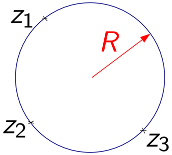

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- It involves ideas from quantitative rectifiability which go back to the Jones' traveling salesman theorem.
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- This is one of the main steps in the proof of Vitushkin's conjecture about the the removability of singularities of bounded analytic functions.

Riesz transforms and rectifiability

Our first main theorem:

Theorem (Nazarov-T.-Volberg, 2012)

Let $E \subset \mathbb{R}^{n+1}$ with $\mathcal{H}^n(E) < \infty$, and $\mu = \mathcal{H}^n_E$. If $\mathcal{R}_\mu : L^2(\mu) \to L^2(\mu)$ is bounded, then E n-rectifiable.

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- Already known that existence of the principal value $\lim_{\varepsilon\to 0} \mathcal{R}_{\mu,\varepsilon} 1(x)$ for μ-a.e. $x \in \mathbb{R}^d$ implies rectifiability (Mattila-Preiss, T.).

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- We first solve the AD-regular case.
  We say that E is AD-regular if $\exists c > 0$ such that

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$$\liminf_{r\to 0} \frac{\mathcal{H}^n(E \cap B(x,r))}{(2r)^n} = 0 \quad \mathcal{H}^n\text{-a.e. on } E$$

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- The theorem above follows by combining this result and the AD regular case.

Removable singularities for Lipschitz harmonic functions

Let $E \subset \mathbb{R}^{n+1}$ be compact. E is removable for Lipschitz harmonic functions if for every open set $\Omega \supset E$, every function $f : \Omega \to \mathbb{R}$ which is Lipschitz in Ω and harmonic in $\Omega \setminus E$ is harmonic in the whole Ω.

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The preceding theorem in combination with a *Tb*-theorem of Nazarov-Treil-Volberg yields:

Corollary

Let $E \subset \mathbb{R}^{n+1}$ with $\mathcal{H}^n(E) < \infty$. Then E is non removable for Lipschitz harmonic functions iff there exists an n-rectifiable subset $F \subset E$ with $\mathcal{H}^n(F) > 0$.

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This is the analog of Vitushkin's conjecture, solved by David (relying on David-Mattila) for $n = 1$.

AD-regular sets and uniform rectifiability

Let $E \subset \mathbb{R}^d$. We say that it is AD-regular (or Ahlfors-David regular) if $\exists c > 0$ such that

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E is uniformly n-rectifiable if it is AD-regular and there are $M, \theta > 0$ such that for all $x \in E$, $0 < r \leq \operatorname{diam}(E)$, there exists a Lipschitz map

$$g : \mathbb{R}^n \supset B_n(0,r) \to \mathbb{R}^d, \qquad \|\nabla g\|_\infty \leq M,$$

such that

$$\mathcal{H}^n\big(E \cap B(x,r) \cap g(B_n(0,r))\big) \geq \theta\, r^n.$$

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Uniform n-rectifiability is a quantitative version of n-rectifiability introduced by David and Semmes.

The David-Semmes problem

Let $E \subset \mathbb{R}^d$ be AD-regular, and $\mu = \mathcal{H}^n_E$.

- David and Semmes showed that E is uniformly n-rectifiable iff all n-dimensional Calderón-Zygmund operators with odd kernel are bounded in $L^2(\mathcal{H}^n_E)$.

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- For all $n \in [1, d-1]$: the boundedness of $V_\rho(\mathcal{R}_\mu)$ in $L^2(\mu)$ is equivalent to uniform n-rectifiability (Mas-T.), where

$$V_\rho(\mathcal{R}_\mu)f(x) = \sup_{\{\varepsilon_m\}} \left(\sum_m |\mathcal{R}_{\mu,\varepsilon_m} f(x) - \mathcal{R}_{\mu,\varepsilon_{m+1}} f(x)|^\rho \right)^{1/\rho},$$

for $\varepsilon_m \searrow 0$, $\rho > 2$.

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  for $\varepsilon_m \searrow 0$, $\rho > 2$.
- For $E = \partial\Omega$, where Ω is a domain good enough for harmonic measure, the $L^2(\mu)$ boundedness of $\mathcal{R}_\mu$ implies uniform rectifiability in codimension 1 (Hofmann-Martell-Mayboroda).

The David-Semmes problem in codimension 1

Theorem (Nazarov, T., Volberg, 2012)

Let $E \subset \mathbb{R}^{n+1}$ be AD-regular, and let $\mu = \mathcal{H}^n_E$. Then:
$\mathcal{R}_\mu$ is bounded in $L^2(\mu) \iff E$ is uniformly n-rectifiable.

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- The proof only works in codimension 1, because we use that the kernel is harmonic.
- We use a deep characterization of uniform rectifiability due to David and Semmes: the so called BAUP condition.

The BAUP condition

BAUP= bilateral approximation by unions of planes

For $Q \in \mathcal{D}$, we denote $B_Q = B(x_Q, 2\,\mathrm{diam}\,Q)$, where x_Q is the center of Q.

Let $\delta > 0$. We say that a cube $Q \in \mathcal{D}$ is δ-BAUP if there exists a set L_Q which is formed by a union of n-planes such that

$$\mathrm{dist}_H(B_Q \cap E,\, B_Q \cap L_Q) \leq \delta\, \ell(Q).$$

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Theorem (David-Semmes)

E is uniformly n-rectifiable iff for every $\delta > 0$ the collection of non-δ-BAUP cubes is a Carleson family. That is,

$$\sum_{\substack{Q\in\mathcal{D}:\,Q\subset R\\ Q\text{ non-}\delta\text{-BAUP}}} \mu(Q) \leq c(\delta)\,\mu(R) \qquad \text{for every } R \in \mathcal{D}.$$

Ideas for the proof of our theorem

We show that if the BAUP condition does not hold, then $\mathcal{R}_\mu$ is not bounded in $L^2(\mu)$

By contradiction: Suppose $\mathcal{R}_\mu$ is bounded in $L^2(\mu)$ and the BAUP condition does not hold.

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Then we show that there exist a cube $Q_0 \in \mathcal{D}$, a hyperplane H, and alternating layers of dyadic cubes contained in Q_0, which become smaller as $k \to \infty$: $\mathcal{F}_0$, $\mathcal{B}_0$, $\mathcal{F}_1$, $\mathcal{B}_1$, ..., $\mathcal{F}_M$, $\mathcal{B}_M$ such that:

- $\mathcal{F}_0 = \{Q_0\}$.

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- $\mathcal{F}_0 = \{Q_0\}$.
- The layers $\mathcal{F}_0, \mathcal{B}_0, \mathcal{F}_1, \mathcal{B}_1 \ldots$ form a filtration, and each layer $\mathcal{F}_i$, $\mathcal{B}_i$ covers "almost all Q_0".

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- The layers $\mathcal{F}_0, \mathcal{B}_0, \mathcal{F}_1, \mathcal{B}_1 \ldots$ form a filtration, and each layer $\mathcal{F}_i$, $\mathcal{B}_i$ covers "almost all Q_0".
- In the cubes from $\mathcal{F}_i$, the measure μ is very flat and its support is close to a hyperplane L_Q parallel to H.

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Then we show that there exist a cube $Q_0 \in \mathcal{D}$, a hyperplane H, and alternating layers of dyadic cubes contained in Q_0, which become smaller as $k \to \infty$: $\mathcal{F}_0$, $\mathcal{B}_0$, $\mathcal{F}_1$, $\mathcal{B}_1$, ..., $\mathcal{F}_M$, $\mathcal{B}_M$ such that:

- $\mathcal{F}_0 = \{Q_0\}$.
- The layers $\mathcal{F}_0$, $\mathcal{B}_0$, $\mathcal{F}_1$, $\mathcal{B}_1 \ldots$ form a filtration, and each layer $\mathcal{F}_i$, $\mathcal{B}_i$ covers "almost all Q_0".
- In the cubes from $\mathcal{F}_i$, the measure μ is very flat and its support is close to a hyperplane L_Q parallel to H.
- The cubes from $\mathcal{B}_i$ are non-δ-BAUP.

Ideas for the proof of our theorem

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- In the cubes from $\mathcal{F}_i$, the measure μ is very flat and its support is close to a hyperplane L_Q parallel to H.
- The cubes from $\mathcal{B}_i$ are non-δ-BAUP.
- M can be taken arbitrarily big.

We show that $\|\mathcal{R}_\mu \chi_{Q_0}\|_{L^2(\mu)} \to \infty$ as $M \to \infty$ as follows:

For each $Q \in \mathcal{F}_i$, we set

$$f_Q(x) = \sum_{P \in \mathcal{F}_{i+1}: P \subset Q} \chi_P(x) \int_{\mathrm{diam}(P) < |x-y| \leq \mathrm{diam}(Q)} K(x-y)\, \chi_{Q_0}(y)\, d\mu(y)$$

(or a suitable variant). Then, (with $M = \infty$)

$$\chi_{Q_0}\, \mathcal{R}_\mu \chi_{Q_0} = \sum_i \sum_{Q \in F_i} f_Q.$$

Therefore, setting $\mathcal{F} = \bigcup_i \mathcal{F}_i$,

$$\|\chi_{Q_0} \mathcal{R}_\mu \chi_{Q_0}\|^2_{L^2(\mu)} = \sum_{Q \in \mathcal{F}} \|f_Q\|^2_{L^2(\mu)} + \sum_{Q, R \in \mathcal{F}: Q \neq R} \langle f_Q, f_R \rangle_{L^2(\mu)}.$$

We show that $\|f_Q\|^2_{L^2(\mu)} \geq c(\delta)\mu(Q)$ and thus

$$\sum_{Q\in\mathcal{F}} \|f_Q\|^2_{L^2(\mu)} = \sum_{i=0}^{M-1} \sum_{Q\in\mathcal{F}_i} \|f_Q\|^2_{L^2(\mu)} \geq \sum_{i=0}^{M-1} \sum_{Q\in\mathcal{F}_i} c(\delta)\,\mu(Q)$$
$$= (M-1)\,c(\delta)\,\mu(Q_0) \to \infty$$

as $M \to \infty$.
Also, we prove that

$$\Bigg|\sum_{Q,R\in\mathcal{F}:Q\neq R} \langle f_Q, f_R\rangle_{L^2(\mu)}\Bigg| \ll \sum_{Q\in\mathcal{F}} \|f_Q\|^2_{L^2(\mu)}.$$

Thus

$$\|\chi_{Q_0}\mathcal{R}_\mu\chi_{Q_0}\|^2_{L^2(\mu)} \geq \sum_{Q\in\mathcal{F}} \|f_Q\|^2_{L^2(\mu)} - \Bigg|\sum_{Q,R\in\mathcal{F}:Q\neq R} \langle f_Q, f_R\rangle_{L^2(\mu)}\Bigg| \to \infty$$

as $M \to \infty$.

Why can we find many layers $\mathcal{B}_i$ of non-δ-BAUP cubes?

Because we are assuming that the BAUP condition does not hold.

That is, there is $\delta > 0$ and Q_0 such that

$$\sum_{\substack{Q\in\mathcal{D}:\, Q\subset Q_0\\ Q \text{ non-}\delta\text{-BAUP}}} \mu(Q) \gg M\,\mu(Q_0).$$

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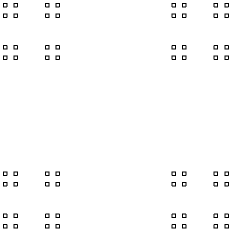

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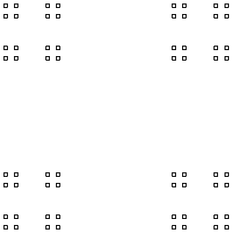

This case is well understood. By "touching point arguments", one shows that the Riesz transform cannnot be bounded.

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To get quantitative estimates we rely on estimates which essentially use a Wasserstein distance from mass transport.

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Suppose $H = [e_1]$ and that

$$\|\mathcal{R}^1\sigma\|^2_{L^2(\sigma)} \leq \varepsilon\, \sigma(Q) \qquad \text{with } \varepsilon \ll \delta,$$

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Since $g \equiv 1$ is admissible,

$$\inf_g F(g) \leq 2\varepsilon\,\sigma(Q).$$

Let g_0 be minimizer of $F(g)$. Then

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Consider a function $\varphi = \sum_{R\in\mathcal{B}_i} \varphi_R$ such that $\int \varphi\, d(g_0\sigma) \approx \delta\sigma(Q)$, and (using Fourier transform) take ψ such that $\mathcal{R}^1(\psi\, dm) = \varphi$.

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This contradicts (1).

The case of codimension > 1

Open problem:
In $\mathbb{R}^d$, suppose that $n < d - 1$. Let

$$\mathcal{R}_\mu f(x) = \int \frac{x - y}{|x - y|^{n+1}} f(y)\, d\mu(y).$$

Let $E \subset \mathbb{R}^d$ AD-regular (n-dimensional), and set $\mu = \mathcal{H}^n_E$.
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Difficulty:
To estimate $\|f_Q\|_{L^2(\mu)}$ from below we need a maximum principle for the variational argument, which seems to fail for $\mathcal{R}_\mu$ in codimension > 1.

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- $\dfrac{z^3}{|z|^4}$ does not imply rectifiability (Nazarov).

Congratulations, Kari.