Geometry of quasicircles and Beurling-type operators

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joint work with K. Astala

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1. Introduction

2. Theorems

3. Applications and questions
Let $\rho$ be a quasiconformal mapping on the plane with complex dilatation $\mu$, $\|\mu\|_{\infty} < 1$.

$$\overline{\partial}\rho - \mu \partial\rho = 0$$

A Jordan curve $\Gamma$ passing through $\infty$ is a \textit{quasicircle} if it is the image of the real line $\mathbb{R}$ under a quasiconformal mapping on the plane.

\textbf{Problem:} Understand the geometric properties of $\Gamma$ in terms of $\mu$. 
**Curves**

- **Chord-arc curves:** The length of the arc is comparable to the chord
  \[ l_{\Gamma}(z_1, z_2) \leq c |z_1 - z_2| \]

- **BJ curves:** Curves which have big pieces of rectifiable curves on all scales.

We say that \( \Gamma = \partial \Omega \) is a BJ curve if for any \( z \in \Omega \), there is a chord-arc domain \( \Omega_z \subset \Omega \) containing \( z \) of diameter uniformly comparable to \( d(z, \partial \Omega) \), such that \( H_1(\Gamma \cap \partial \Omega_z) \geq c(\Omega)d(z, \partial \Omega) \).
Known Results

**Theorem (A-Z,Mc,S)**

Γ is a chord-arc curve with **small constant** if and only if there is a quasiconformal mapping ρ with ρ(ℝ) = Γ and with dilatation µ such that |µ|^2/|y| is a Carleson measure with **small norm**.

**Theorem (A-Z,Mc,B-J)**

Let Γ = ∂Ω be a quasicircle, and denote by Φ the Riemann map from ℝ^2_+ onto Ω. The following are equivalent

i) Γ is a BJ curve.

ii) log Φ′ ∈ BMOA(ℝ^2_+).

iii) There exists a quasiconformal mapping ρ such that ρ(ℝ) = Γ, and the dilatation µ satisfies that |µ|^2/|y| is a Carleson measure.
**Motivation:** Open problem of the **connectivity** of the manifold of chord-arc curves. The topology on this manifold is defined by

$$d(\Gamma_1, \Gamma_2) = \| \log |\Phi'_1| - \log |\Phi'_2|\|_{\text{BMO}(\mathbb{R})}$$

**Theorem (A-Z):** The space of BJ curves is connected ($\mu \rightarrow t\mu$ gives a continuous deformation)
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Let $\Gamma = \partial \Omega$ be a quasicircle, and denote by $f$ the Riemann map from $\mathbb{R}^2_+$ onto $\Omega$.

**Theorem (Enríquez Salamanca, G.)**

The q.c. $\Gamma$ is a $C^{1+\alpha}$ curve if and only if $f$ extends to a global quasiconformal map whose dilatation $\mu$ satisfies that $|\mu(z)|^2/|y|^{1+\varepsilon}$ is a Carleson measure, where $\varepsilon = \varepsilon(\|\mu\|_{\infty})$ and $\alpha = \alpha(\varepsilon, \|\mu\|_{\infty})$. 
Chord-arc case: Operators

The Beurling operator $S$ defined by

$$Sf(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{f(w)}{(w-z)^2} \, dm(w)$$

$S$ is bounded on $L^p(\mathbb{C})$, $1 < p < \infty$ and $\|Sf\|_2 = \|f\|_2$.

The Cauchy operator $T$ defined by

$$Tf(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{f(w)}{w-z} \, dm(w)$$

$$\overline{\partial}(Tf) = f$$

$$\partial(Tf) = Sf$$
Given a function $f$ on $\Gamma$, define its Cauchy integral $F(z) = C_\Gamma f(z)$ off $\Gamma$ by

$$F(z) = \frac{1}{2\pi} \int_\Gamma \frac{f(w)}{w-z} \, dw, \quad z \notin \Gamma.$$ 

If $F_+$ and $F_-$ are the restrictions of $F$ to $\Omega_+$ and $\Omega_-$, and if $f_+$ and $f_-$ denote their boundary values, then the classical Plemelj formula states that

$$f_\pm(z) = \pm \frac{1}{2} f(z) + \frac{1}{2\pi} \text{P.V.} \int_\Gamma \frac{f(w)}{w-z} \, dw, \quad z \in \Gamma.$$ 

The singular integral is also called the Cauchy integral.

**Theorem (G. David)**

The Cauchy integral is bounded on $L^2(\Gamma)$ if and only if $\Gamma$ is regular (arclength of $B(z_0, R) \cap \Gamma \leq CR$).
Theorem (Astala-G.)

A quasicircle $\Gamma$ is a **BJ curve** if and only if there exists a quasiconformal mapping $\rho$ with $\rho(\mathbb{R}) = \Gamma$, such that the operator $(I - \mu S)$ is **bounded** in $L^2\left(\frac{dm}{|y|}\right)$.

Theorem (Astala-G.)

A quasicircle $\Gamma$ is a **chord-arc** curve if and only if there exists a quasiconformal mapping $\rho$ with $\rho(\mathbb{R}) = \Gamma$ such that the operator $(I - \mu S)$ is **invertible** in $L^2\left(\frac{dm}{|y|}\right)$. 
Theorem (Astala-G.)

Let $\rho : \mathbb{C} \to \mathbb{C}$ be a quasiconformal mapping, analytic at $\infty$, such that $\int_{\mathbb{C}} \frac{|\bar{\partial} \rho|^2}{|y|} \, dm < \infty$. Then $\Gamma = \rho(\mathbb{R})$ is rectifiable and $\rho'|_{\mathbb{R}} \in L^2_{loc}$. 

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Apply the boundedness assumption to an appropriate function \( f \).

Let \( x_0 \in \mathbb{R}, r > 0 \) and \( B_0 = B(x_0, r) \). Consider the ball \( \tilde{B}_0(x_0 + i2r, r) \) and let \( f(z) = \chi_{\tilde{B}_0}(z) \), then

\[
(Sf)(z) = \frac{r^2}{(z - z_0)^2} \chi_{\mathbb{C} \setminus \tilde{B}_0}(z)
\]

Thus,

\[
r \succsim \int_{\tilde{B}_0} \frac{1}{|y|} \, dm(z) \succsim \int_{\overline{\mathbb{C}}} \left| \frac{\mu(z)}{|y|} \right|^2 |Sf(z)|^2 \, dm(z)
\]

\[
\geq \int_{B_0} \frac{|\mu(z)|^2}{|y|} \frac{r^2}{|z - z_0|^2} \, dm(z) \simeq \int_{B_0} \frac{|\mu(z)|^2}{|y|} \, dm(z).
\]
\[ \| (\mu S)f \|_{L^2 \left( \frac{dm}{|y|} \right)}^2 = \frac{1}{\pi} \int_{\mathbb{R}^2_+} \frac{\left| \mu(z) \right|^2}{y} \left| \int_{\mathbb{C}} \frac{f(w)}{(w-z)^2} \, dm(w) \right| \, dm(z) \]

\[ \lesssim \int_{\mathbb{R}^2_+} \frac{\left| \mu(z) \right|^2}{y} \left| \int_{w \in \mathbb{R}^2_+} \frac{f(w)}{(w-z)^2} \, dm(w) \right|^2 \, dm(z) \]

\[ + \int_{\mathbb{R}^2_+} \frac{\left| \mu(z) \right|^2}{y} \left| \int_{w \in \mathbb{R}^2_+ \setminus Q^*_k(z)} \frac{f(w)}{(w-z)^2} \, dm(w) \right|^2 \, dm(z) \]

\[ + \int_{\mathbb{R}^2_+} \frac{\left| \mu(z) \right|^2}{y} \left| \int_{w \in Q^*_k(z)} \frac{f(w)}{(w-z)^2} \, dm(w) \right|^2 \, dm(z) \]

\[ = I_1 + I_2 + I_3 \lesssim \int_{\mathbb{R}^2_+} \frac{|f(z)|^2}{y} \, dm(z). \]
Chord-Arc $\Rightarrow$ Invertivility

- Given $\Gamma$ chord-arc, we can find a bilipschitz map $\rho : \mathbb{C} \to \mathbb{C}$ such that $\rho(\mathbb{R}) = \Gamma$ and $|\mu_{\rho}|^2 / |y|$ is a Carleson measure. (Semmes)

- Need to solve $(I - \mu S)h = \Phi$ with bounds. Set $\overline{\partial}H = h$ and $\partial H = Sh$. Thus, $H$ satisfies the equation

$$\overline{\partial}H - \mu \partial H = \Phi.$$ 

Apply a quasiconformal change of variables as in (AIS). Set $u = H \circ \rho^{-1}$, then $H = u \circ \rho$

- To show invertivility we need to proof that the operator $(\tilde{\mu} S)$ is bounded on $L^2 \left( \frac{dm(w)}{\text{dist}(w, \Gamma)} \right)$ when the measure $\frac{|\tilde{\mu}(w)|^2}{\text{dist}(w, \Gamma)} dm(w)$ is Carleson with respect to $\Gamma$. 

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**Proposition** (Semmes setting): If the operator \((I - \mu S)\) is invertible in \(L^2 \left( \frac{dm}{|y|} \right)\) then the following holds:

If \(f \in L^2(\mathbb{R})\) and \(H \in W^{1,2}_{\text{loc}}(\mathbb{C})\) satisfies:

\[
\overline{\partial} H - \mu \frac{\partial}{\partial \bar{z}} H = \mu C'_f \quad \text{a.a. } z \in \mathbb{C}
\]

then the boundary values \(H|_\mathbb{R}\) belong to \(L^2(\mathbb{R})\) and

\[
\|H|_\mathbb{R}\|_2 \leq c \|f\|_2
\]

where \(c\) is a positive constant \(c = c(\mu)\).
Duality argument: Let $h \in L^2(\mathbb{R})$ with $\|h\|_{L^2(\mathbb{R})} = 1$

$$\int_{\mathbb{R}} H(x)h(x) \, dx = 2i \int_{\mathbb{C}} \overline{\partial}H(z)C_h(z) \, dm(z)$$

$$= 2i \int_{\mathbb{C}} (I - \mu S)^{-1}(\mu C'_f)(z)C_h(z) \, dm(z)$$

$$= 2i \int_{\mathbb{C}} \mu(z)(I - S\mu)^{-1}(C'_f)(z)C_h(z) \, dm(z)$$

$$= 2i \int_{\mathbb{C}} C'_f(z)(I - \mu S)^{-1}(\mu C_h)(z) \, dm(z).$$
We can use the assumption on the invertibility of the operator \((I - \mu S)\) on \(L^2 \left( \frac{dm}{|y|} \right)\) to obtain:

\[
\left| \int_{\mathbb{R}} H(x)h(x) \, dx \right|
\leq 2 \left( \int_{\mathbb{C}} |C'_f(z)|^2 |y| \, dm(z) \right)^{1/2} \left( \int_{\mathbb{C}} \frac{|(I - \mu S)^{-1}(\mu C_h)(z)|^2}{|y|} \, dm(z) \right)^{1/2}
\leq c(\mu) \left( \int_{\mathbb{C}} |y| |C'_f(z)|^2 \, dm(z) \right)^{1/2} \leq c(\mu) \|C_f(x)\|_2 \leq c(\mu) \|f\|_2
\]
Proposition $\Rightarrow$ Chord-Arc

Let $f \in L^2(\mathbb{R})$ and let $F$ be the pull-back of the Cauchy integral on $\Gamma$ via $\rho$ of the function $f \circ \rho^{-1}$. Set $H = F - C_f$. Then

$$\bar{\partial}H - \mu \partial H = \mu C'_f.$$ 

The boundary values of $F$ are given by

$$F(x) = \pm \frac{1}{2} f(x) + \frac{1}{2\pi i} \text{P.V.} \int_{\mathbb{R}} \frac{f(y)}{\rho(y) - \rho(x)} \, d\rho(y), \quad x \in \mathbb{R}.$$ 

The $L^2$-estimate on $H|_{\mathbb{R}}$ and the boundedness of $C_f$ on $L^2(\mathbb{R})$ imply that

$$Kf(x) = \text{P.V.} \int_{\mathbb{R}} \frac{f(y)}{\rho(y) - \rho(x)} \rho'(y) \, dy, \quad x \in \mathbb{R},$$

defines a bounded operator on $L^2(\mathbb{R})$. Semmes argument shows that the curve is chord arc.
**Open Problem:** Characterize the quasiconformal mappings whose dilatation satisfy that \((I - \mu S)\) is invertible in \(L^2 \left( \frac{dm}{|y|} \right)\).

As part of the proof of the Theorem we obtain:

**Corollary**

If \(\rho\) is bilipschitz and its dilatation \(\mu\) satisfies that \(|\mu|^2/|y|\) is a Carleson measure then the operator \((I - \mu S)\) is invertible in \(L^2 \left( \frac{dm}{|y|} \right)\).

**BUT!!!**

**Theorem**

There exists a quasiconformal mapping \(\rho\) which is not bilipschitz and satisfies that \(|\mu|^2/|y|\) is a Carleson measure and \((I - \mu S)\) is invertible in \(L^2 \left( \frac{dm}{|y|} \right)\).
Corollary

If $\Gamma$ is a chord-arc curve, the Cauchy integral on $\Gamma$ is a bounded operator in $L^2(\Gamma)$.

Proof:
Let $\rho$ be the bilipschitz map associated to $\Gamma$.
Given $g \in L^2(\Gamma)$, let $G$ be the Cauchy integral on $\Gamma$ of $g$. Since bilipschitz mappings preserve $L^2$, the function $f = g \circ \rho$ belongs to $L^2(\mathbb{R})$. As before, if $F = G \circ \rho$ we get that $H = F - C_f$, satisfies

$$\overline{\partial}H - \mu \partial H = \mu C'_f.$$ 

By the Proposition $H\big|_{\mathbb{R}}$ is bounded in $L^2(\mathbb{R})$, and so is $C_f$. Thus

$$\|F_{\pm}\|_{L^2(\mathbb{R})} \leq c \|f\|_{L^2(\mathbb{R})}.$$ 

Using again that $\rho$ is bilipschitz we obtain that

$$\|G_{\pm}\|_{L^2(\Gamma)} \leq c \|g\|_{L^2(\Gamma)}.$$ 

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Since the set of invertible operators is an open set, we get the following:

**Corollary**

Chord-arc curves are an open subset of $BJ$ curves.

We also recover Semmes result:

**Corollary**

If a quasiconformal mapping $\rho$ satisfies that $|\mu_\rho|^2/|y|$ is a Carleson measure with small norm then $\rho(\mathbb{R}) = \Gamma$ is chord-arc.

Connectivity????