Geometry of quasicircles and Beurling-type operators

María José González

joint work with K. Astala

IPAM 2013

Outline

Contenido

Problem

- Let ρ be a quasiconformal mapping on the plane with complex dilatation μ, $\|\mu\|_\infty < 1$..

$$\overline{\partial}\rho - \mu\partial\rho = 0$$

- A Jordan curve Γ passing through ∞ is a *quasicircle* if it is the image of the real line $\mathbb{R}$ under a quasiconformal mapping on the plane.

PROBLEM: Understand the geometric properties of Γ in terms of μ.

Curves

- **Chord-arc curves**:The length of the arc is comparable to the chord
$$l_\Gamma(z_1, z_2) \leq c\,|z_1 - z_2|$$
- **BJ curves:** Curves which have big pieces of rectifiable curves on all scales.

  We say that $\Gamma = \partial\Omega$ is a *BJ* curve if for any $z \in \Omega$, there is a chord-arc domain $\Omega_z \subset \Omega$ containing z of diameter uniformly comparable to $d(z, \partial\Omega)$, such that $H_1(\Gamma \cap \partial\Omega_z) \geq c(\Omega)d(z, \partial\Omega)$.

Known Results

Theorem (A-Z,Mc,S)

Γ is a chord-arc curve with **small constant** if and only if there is a quasiconformal mapping ρ with $\rho(\mathbb{R}) = \Gamma$ and with dilatation μ such that $|\mu|^2/|y|$ is a Carleson measure with **small norm.**

Theorem (A-Z,Mc,B-J)

Let $\Gamma = \partial\Omega$ be a quasicircle, and denote by Φ the Riemann map from $\mathbb{R}^2_+$ onto Ω. The following are equivalent

i) Γ is a *BJ* curve.

ii) $\log \Phi' \in \text{BMOA}(\mathbb{R}^2_+)$.

iii) There exists a quasiconformal mapping ρ such that $\rho(\mathbb{R}) = \Gamma$, and the dilatation μ satisfies that $|\mu|^2/|y|$ is a Carleson measure.

MOTIVATION: Open problem of the **connectivity** of the manifold of chord-arc curves. The topology on this manifold is defined by $d(\Gamma_1, \Gamma_2) = \| \log |\Phi_1'| - \log |\Phi_2'| \|_{\mathrm{BMO}(\mathbb{R})}$

Theorem (A-Z): The space of BJ curves is connected ($\mu \to t\mu$ gives a continuous deformation)

Contenido

Smooth curves

Let $\Gamma = \partial\Omega$ be a quasicircle, and denote by f the Riemann map from $\mathbb{R}^2_+$ onto Ω.

Theorem (Enríquez Salamanca, G.)

The q.c. Γ is a $C^{1+\alpha}$ curve if and only if f extends to a global quasiconformal map whose dilatation μ satisfies that $|\mu(z)|^2/|y|^{1+\varepsilon}$ is a Carleson measure, where $\varepsilon = \varepsilon(\|\mu\|_\infty)$ and $\alpha = \alpha(\varepsilon, \|\mu\|_\infty)$.

Chord-arc case:Operators

The Beurling operator S defined by

$$Sf(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(w)}{(w-z)^2}\, dm(w)$$

S is bounded on $L^p(\mathbb{C})$, $1 < p < \infty$ and $\|Sf\|_2 = \|f\|_2$.

The Cauchy operator T defined by

$$Tf(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(w)}{w-z}\, dm(w)$$

$$\bar{\partial}(Tf) = f$$

$$\partial(Tf) = Sf$$

Cauchy Integral

Given a function f on Γ, define its Cauchy integral $F(z) = C_\Gamma f(z)$ off Γ by

$$F(z) = \frac{1}{2\pi} \int_\Gamma \frac{f(w)}{w-z}\, dw, \qquad z \notin \Gamma.$$

If F_+ and F_- are the restrictions of F to Ω_+ and Ω_-, and if f_+ and f_- denote their boundary values, then the classical Plemelj formula states that

$$f_\pm(z) = \pm\frac{1}{2} f(z) + \frac{1}{2\pi}\ \text{P.V.} \int_\Gamma \frac{f(w)}{w-z}\, dw, \qquad z \in \Gamma.$$

The singular integral is also called the Cauchy integral.

Theorem (G. David)

The Cauchy integral is bounded on $L^2(\Gamma)$ if and only if Γ is regular (arclength of $B(z_0, R) \cap \Gamma \leq CR$)

Results

Theorem (Astala-G.)

A quasicircle Γ is a **BJ curve** if and only if there exists a quasiconformal mapping ρ with $\rho(\mathbb{R}) = \Gamma$, such that the operator $(I - \mu S)$ is **bounded** in $L^2\left(\frac{dm}{|y|}\right)$.

Theorem (Astala-G.)

A quasicircle Γ is a **chord-arc** curve if and only if there exists a quasiconformal mapping ρ with $\rho(\mathbb{R}) = \Gamma$ such that the operator $(I - \mu S)$ is **invertible** in $L^2\left(\frac{dm}{|y|}\right)$.

Results

Theorem (Astala-G.)

Let $\rho\colon \mathbb{C} \to \mathbb{C}$ be a quasiconformal mapping, analytic at ∞, such that $\int_{\mathbb{C}} \frac{|\bar{\partial}\rho|^2}{|y|} dm < \infty$. Then $\Gamma = \rho(\mathbb{R})$ is rectifiable and $\rho'|_{\mathbb{R}} \in L^2_{loc}$.

Boundedness $\Rightarrow$ BJ

Apply the boundedness assumption to an appropriate function f.

Let $x_0 \in \mathbb{R}$, $r > 0$ and $B_0 = B(x_0, r)$. Consider the ball $\tilde{B}_0(x_0 + i2r, r)$ and let $f(z) = \chi_{\tilde{B}_0}(z)$, then

$$(Sf)(z) = \frac{r^2}{(z - z_0)^2} \chi_{\mathbb{C} \setminus \tilde{B}_0}(z)$$

Thus,

$$r \simeq \int_{\tilde{B}_0} \frac{1}{|y|}\, dm(z) \gtrsim \int_{\mathbb{C}} \frac{|\mu(z)|^2}{|y|} |Sf(z)|^2\, dm(z)$$

$$\geq \int_{B_0} \frac{|\mu(z)|^2}{|y|} \frac{r^2}{|z - z_0|^2}\, dm(z) \simeq \int_{B_0} \frac{|\mu(z)|^2}{|y|}\, dm(z).$$

BJ $\Rightarrow$ Boundedness

$$\|(\mu S)f\|^2_{L^2\left(\frac{dm}{|y|}\right)} = \frac{1}{\pi}\int_{\mathbb{R}^2_+} \frac{|\mu(z)|^2}{y} \left| \int_{\mathbb{C}} \frac{f(w)}{(w-z)^2}\, dm(w) \right| dm(z)$$

$$\lesssim \int_{\mathbb{R}^2_+} \frac{|\mu(z)|^2}{y} \left| \int_{w\in\mathbb{R}^2_-} \frac{f(w)}{(w-z)^2}\, dm(w) \right|^2 dm(z)$$

$$+ \int_{\mathbb{R}^2_+} \frac{|\mu(z)|^2}{y} \left| \int_{w\in\mathbb{R}^2_+\setminus Q_k^*(z)} \frac{f(w)}{(w-z)^2}\, dm(w) \right|^2 dm(z)$$

$$+ \int_{\mathbb{R}^2_+} \frac{|\mu(z)|^2}{y} \left| \int_{w\in Q_k^*(z)} \frac{f(w)}{(w-z)^2}\, dm(w) \right|^2 dm(z)$$

$$= I_1 + I_2 + I_3 \lesssim \int_{\mathbb{R}^2_+} \frac{|f(z)|^2}{y}\, dm(z).$$

Chord-Arc $\Rightarrow$ Invertivility

- Given Γ chord-arc, we can find a bilipschitz map $\rho : \mathbb{C} \to \mathbb{C}$ such that $\rho(\mathbb{R}) = \Gamma$ and $|\mu_\rho|^2/|y|$ is a Carleson measure. (Semmes)
- Need to solve $(I - \mu S)h = \Phi$ with bounds. Set $\overline{\partial} H = h$ and $\partial H = Sh$. Thus, H satisfies the equation

  $$\overline{\partial} H - \mu \partial H = \Phi.$$

  Apply a quasiconformal change of variables as in (AIS). Set $u = H \circ \rho^{-1}$, then $H = u \circ \rho$
- To show invertivility we need to proof that the operator $(\tilde{\mu} S)$ is bounded on $L^2\left(\frac{dm(w)}{\mathrm{dist}(w,\Gamma)}\right)$ when the measure $\frac{|\tilde{\mu}(w)|^2}{\mathrm{dist}(w,\Gamma)}\, dm(w)$ is Carleson with respect to Γ.

Invertivility $\Rightarrow$ Chord-Arc

PROPOSITION (Semmes setting): If the operator $(I - \mu S)$ is invertible in $L^2\left(\frac{dm}{|y|}\right)$ then the following holds:
If $f \in L^2(\mathbb{R})$ and $H \in W^{1,2}_{\text{loc}}(\mathbb{C})$ satisfies:

$$\overline{\partial} H - \mu \partial H = \mu C_f' \qquad \text{a.a. } z \in \mathbb{C}$$

then the boundary values $H\big|_{\mathbb{R}}$ belong to $L^2(\mathbb{R})$ and $\|H\big|_{\mathbb{R}}\|_2 \leq c\|f\|_2$ where c is a positive constant $c = c(\mu)$.

Proof of the Proposition

Duality argument: Let $h \in L^2(\mathbb{R})$ with $\|h\|_{L^2(\mathbb{R})} = 1$

$$\begin{aligned}
\int_{\mathbb{R}} H(x)h(x)\,dx &= 2i \int_{\mathbb{C}} \overline{\partial} H(z) C_h(z)\,dm(z) \\
&= 2i \int_{\mathbb{C}} (I - \mu S)^{-1}(\mu C_f')(z) C_h(z)\,dm(z) \\
&= 2i \int_{\mathbb{C}} \mu(z)(I - S\mu)^{-1}(C_f')(z) C_h(z)\,dm(z) \\
&= 2i \int_{\mathbb{C}} C_f'(z)(I - \mu S)^{-1}(\mu C_h)(z)\,dm(z).
\end{aligned}$$

Proof of the Proposition

We can use the assumption on the invertibility of the operator $(I - \mu S)$ on $L^2\left(\frac{dm}{|y|}\right)$ to obtain:

$$\left|\int_{\mathbb{R}} H(x)h(x)\,dx\right|$$

$$\leq 2\left(\int_{\mathbb{C}} |C_f'(z)|^2 |y|\,dm(z)\right)^{1/2}\left(\int_{\mathbb{C}} \frac{|(I-\mu S)^{-1}(\mu C_h)(z)|^2}{|y|}\,dm(z)\right)^{1/2}$$

$$\leq c(\mu)\left(\int_{\mathbb{C}} |y||C_f'(z)|^2\,dm(z)\right)^{1/2} \leq c(\mu)\|C_f(x)\|_2 \leq c(\mu)\|f\|_2$$

Proposition ⇒ Chord-Arc

Let $f \in L^2(\mathbb{R})$ and let F be the pull-back of the Cauchy integral on Γ via ρ of the function $f \circ \rho^{-1}$. Set $H = F - C_f$. Then

$$\bar{\partial} H - \mu \partial H = \mu C_f'.$$

The boundary values of F are given by

$$F(x) = \pm \frac{1}{2} f(x) + \frac{1}{2\pi i} \text{ P.V.} \int_{\mathbb{R}} \frac{f(y)}{\rho(y) - \rho(x)} \, d\rho(y), \qquad x \in \mathbb{R}.$$

The L^2-estimate on $H\big|_{\mathbb{R}}$ and the boundedness of C_f on $L^2(\mathbb{R})$ imply that

$$Kf(x) = \text{P.V.} \int_{\mathbb{R}} \frac{f(y)}{\rho(y) - \rho(x)} \rho'(y) \, dy, \qquad x \in \mathbb{R},$$

defines a bounded operator on $L^2(\mathbb{R})$. Semmes argument shows that the curve is chord arc.

Contenido

OPEN PROBLEM: Characterize the quasiconformal mappings whose dilatation satisfy that $(I - \mu S)$ is invertible in $L^2\left(\frac{dm}{|y|}\right)$.

As part of the proof of the Theorem we obtain:

Corollary

If ρ is bilipschitz and its dilatation μ satisfies that $|\mu|^2/|y|$ is a Carleson measure then the operator $(I - \mu S)$ is invertible in $L^2\left(\frac{dm}{|y|}\right)$.

BUT!!!

Theorem

There exists a quasiconformal mapping ρ which is not bilipschitz and satisfies that $|\mu|^2/|y|$ is a Carleson measure and $(I - \mu S)$ is invertible in $L^2\left(\frac{dm}{|y|}\right)$.

Corollary

If Γ is a chord-arc curve, the Cauchy integral on Γ is a bounded operator in $L^2(\Gamma)$.

Proof:
Let ρ be the bilipschitz map associated to Γ
Given $g \in L^2(\Gamma)$, let G be the Cauchy integral on Γ of g. Since bilipschitz mappings preserve L^2, the function $f = g \circ \rho$ belongs to $L^2(\mathbb{R})$. As before,if $F = G \circ \rho$ we get that $H = F - C_f$, satisfies

$$\overline{\partial} H - \mu \partial H = \mu C_f'.$$

By the Proposition $H\big|_{\mathbb{R}}$ is bounded in $L^2(\mathbb{R})$, and so is C_f. Thus

$$\|F_{\pm}\big|_{\mathbb{R}}\|_{L^2(\mathbb{R})} \leq c\|f\|_{L^2(\mathbb{R})}.$$

Using again that ρ is bilipschitz we obtain that

$$\|G_{\pm}\|_{L^2(\Gamma)} \leq c\|g\|_{L^2(\Gamma)}$$

Since the set of invertible operators is an open set, we get the following:

Corollary

Chord-arc curves are an open subset of *BJ* curves.

We also recover Semmes result:

Corollary

If a quasiconformal mapping ρ satisfies that $|\mu_\rho|^2/|y|$ is a Carleson measure with small norm then $\rho(\mathbb{R}) = \Gamma$ is chord-arc.

Connectivity????