On the sharpness of Mockenhaupt's restriction estimate

Izabella Łaba

IPAM, Los Angeles, 2013

Restriction estimates for measures on $\mathbb{R}^n$

Let μ be a finite, nonnegative Borel measure on $\mathbb{R}^n$, $f \in L^1(d\mu)$. Define the Fourier transform

$$\widehat{fd\mu}(\xi) = \int f(x) e^{-2\pi i \xi \cdot x} d\mu(x)$$

Restriction estimates for measures on $\mathbb{R}^n$

Let μ be a finite, nonnegative Borel measure on $\mathbb{R}^n$, $f \in L^1(d\mu)$. Define the Fourier transform

$$\widehat{fd\mu}(\xi) = \int f(x) e^{-2\pi i \xi \cdot x} d\mu(x)$$

When is $f \to \widehat{fd\mu}$ a bounded mapping $L^q(d\mu) \to L^p(\mathbb{R}^n)$?

Restriction estimates for measures on $\mathbb{R}^n$

Let μ be a finite, nonnegative Borel measure on $\mathbb{R}^n$, $f \in L^1(d\mu)$. Define the Fourier transform

$$\widehat{fd\mu}(\xi) = \int f(x) e^{-2\pi i \xi \cdot x} d\mu(x)$$

When is $f \to \widehat{fd\mu}$ a bounded mapping $L^q(d\mu) \to L^p(\mathbb{R}^n)$?

Trivially, we have $\|\widehat{fd\mu}\|_\infty \leq \mu(\mathbb{R}^n) \|f\|_{L^1(d\mu)}$.

Restriction estimates for measures on $\mathbb{R}^n$

Let μ be a finite, nonnegative Borel measure on $\mathbb{R}^n$, $f \in L^1(d\mu)$. Define the Fourier transform

$$\widehat{fd\mu}(\xi) = \int f(x) e^{-2\pi i \xi \cdot x} d\mu(x)$$

When is $f \to \widehat{fd\mu}$ a bounded mapping $L^q(d\mu) \to L^p(\mathbb{R}^n)$?

Trivially, we have $\|\widehat{fd\mu}\|_\infty \leq \mu(\mathbb{R}^n) \|f\|_{L^1(d\mu)}$.

Without additional assumptions on μ, that's all we can say. But for some types of μ, there is much more...

A classic result: Stein-Tomas restriction theorem

Theorem (Stein-Tomas, 1970s.) Let σ be the surface measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Then for all $p \geq \frac{2n+2}{n-1}$,

$$\|\widehat{fd\sigma}\|_{L^p(\mathbb{R}^n)} \leq C(p)\|f\|_{L^2(d\sigma)}$$

for all $f \in L^2(d\sigma)$.

A classic result: Stein-Tomas restriction theorem

Theorem (Stein-Tomas, 1970s.) Let σ be the surface measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Then for all $p \geq \frac{2n+2}{n-1}$,

$$\|\widehat{fd\sigma}\|_{L^p(\mathbb{R}^n)} \leq C(p)\|f\|_{L^2(d\sigma)}$$

for all $f \in L^2(d\sigma)$.

Proof is based on the decay of $\widehat{\sigma}$ at infinity:

$$|\widehat{\sigma}(\xi)| \leq C(1 + |\xi|)^{-(n-1)/2}$$

A classic result: Stein-Tomas restriction theorem

Theorem (Stein-Tomas, 1970s.) Let σ be the surface measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Then for all $p \geq \frac{2n+2}{n-1}$,

$$\|\widehat{fd\sigma}\|_{L^p(\mathbb{R}^n)} \leq C(p)\|f\|_{L^2(d\sigma)}$$

for all $f \in L^2(d\sigma)$.

Proof is based on the decay of $\widehat{\sigma}$ at infinity:

$$|\widehat{\sigma}(\xi)| \leq C(1+|\xi|)^{-(n-1)/2}$$

The range of p is optimal.

This turned out to be a rich and productive line of research:

- Many restriction estimates are now known for surface measures on manifolds (spheres, cones, curves in $\mathbb{R}^3$, ...)

Restriction estimates in classical harmonic analysis

This turned out to be a rich and productive line of research:

- Many restriction estimates are now known for surface measures on manifolds (spheres, cones, curves in $\mathbb{R}^3$, ...)
- The availability of such estimates, and range of exponents, depends on geometric properties of the manifold: dimension, curvature (via decay of Fourier transform of σ), Kakeya-type geometric information.

Restriction estimates in classical harmonic analysis

This turned out to be a rich and productive line of research:

- Many restriction estimates are now known for surface measures on manifolds (spheres, cones, curves in $\mathbb{R}^3$, ...)
- The availability of such estimates, and range of exponents, depends on geometric properties of the manifold: dimension, curvature (via decay of Fourier transform of σ), Kakeya-type geometric information.
- Many important open problems remain, e.g. Stein's restriction conjecture for the sphere (partial results: Bourgain, Wolff, Tao, Bourgain-Guth.)

Restriction estimates for discrete sets

- Restriction estimates for sets of integers (Bourgain, Green, Green-Tao, Tao-Vu).

Restriction estimates for discrete sets

- Restriction estimates for sets of integers (Bourgain, Green, Green-Tao, Tao-Vu).
- Green (2003) and Green-Tao (2004) used restriction estimates in proving Roth-type results on arithmetic progressions in sets of integers.

Restriction estimates for discrete sets

- Restriction estimates for sets of integers (Bourgain, Green, Green-Tao, Tao-Vu).
- Green (2003) and Green-Tao (2004) used restriction estimates in proving Roth-type results on arithmetic progressions in sets of integers.
- Finite field analogues (Mockenhaupt-Tao, Iosevich, Koh, A. Lewko, M. Lewko, Shen, ...)

Restriction estimates for discrete sets

- Restriction estimates for sets of integers (Bourgain, Green, Green-Tao, Tao-Vu).
- Green (2003) and Green-Tao (2004) used restriction estimates in proving Roth-type results on arithmetic progressions in sets of integers.
- Finite field analogues (Mockenhaupt-Tao, Iosevich, Koh, A. Lewko, M. Lewko, Shen, ...)
- Instead of curvature (makes no sense for discrete sets), proofs use arithmetic information, e.g. counting solutions to equations $a_1 + \cdots + a_n = a_{n+1} + \cdots + a_{2n}$ with a_i in a given set.

Restriction estimates for fractal measures

- Mockenhaupt (2000): restriction theorem for "Salem sets." (Definitions and precise statements will follow shortly.)

Restriction estimates for fractal measures

- Mockenhaupt (2000): restriction theorem for "Salem sets." (Definitions and precise statements will follow shortly.)
- Proof follows the same argument as the Tomas-Stein restriction theorem for the sphere.

Restriction estimates for fractal measures

- Mockenhaupt (2000): restriction theorem for "Salem sets." (Definitions and precise statements will follow shortly.)
- Proof follows the same argument as the Tomas-Stein restriction theorem for the sphere.
- Main contribution: changing the point of view, shifting focus from "smooth" surface measures on manifolds to less regular fractal measures.

Restriction estimates for fractal measures

- Mockenhaupt (2000): restriction theorem for "Salem sets." (Definitions and precise statements will follow shortly.)
- Proof follows the same argument as the Tomas-Stein restriction theorem for the sphere.
- Main contribution: changing the point of view, shifting focus from "smooth" surface measures on manifolds to less regular fractal measures.
- X. Chen (2012): a restriction theorem for fractal measures based on a different mechanism.

Background: dimensionality of measures

- Let μ be a probability measure on $\mathbb{R}^n$ and $0 \leq \alpha \leq n$. We say that μ obeys the α-**dimensional ball condition** if

$$\mu(B(x,r)) \leq Cr^{\alpha} \quad \forall x \in \mathbb{R}^n, \; r \in (0,\infty) \qquad (*)$$

$B(x,r)$ ball of radius r centered at x.

Background: dimensionality of measures

- Let μ be a probability measure on $\mathbb{R}^n$ and $0 \leq \alpha \leq n$. We say that μ obeys the α-**dimensional ball condition** if

$$\mu(B(x,r)) \leq Cr^\alpha \quad \forall x \in \mathbb{R}^n, \ r \in (0,\infty) \qquad (*)$$

  $B(x,r)$ ball of radius r centered at x.
- Let $E \subset \mathbb{R}^n$ closed, $\dim_H(E) =$ Hausdorff dimension of E. Then

  $\dim_H(E) = \sup\{\alpha \in [0,n] : \ E$ supports a probability measure $\mu = \mu_\alpha$ obeying (*)$\}$

Background: Fourier dimension

The **Fourier dimension** of $E \subset \mathbb{R}^n$ is defined as

$\dim_F(E) = \sup\{\beta \in [0,n] :$ E supports a probability measure $\mu = \mu_\beta$ with

$$|\widehat{\mu}(\xi)| \leq C(1+|\xi|)^{-\beta/2}\}.$$

Background: Fourier dimension

The **Fourier dimension** of $E \subset \mathbb{R}^n$ is defined as

$\dim_F(E) = \sup\{\beta \in [0, n] :\ E$ supports a probability measure $\mu = \mu_\beta$ with

$$|\widehat{\mu}(\xi)| \leq C(1 + |\xi|)^{-\beta/2}\}.$$

- $\dim_F(E) \leq \dim_H(E)$, always.

Background: Fourier dimension

The **Fourier dimension** of $E \subset \mathbb{R}^n$ is defined as

$\dim_F(E) = \sup\{\beta \in [0,n] :$ E supports a probability measure $\mu = \mu_\beta$ with

$$|\widehat{\mu}(\xi)| \leq C(1+|\xi|)^{-\beta/2}\}.$$

- $\dim_F(E) \leq \dim_H(E)$, always.
- Inequality can be strict: the $2/3$ Cantor set has Hausdorff dimension $\log 2/\log 3$, but Fourier dimension 0.

Salem sets

Salem sets: sets with $\dim_F(E) = \dim_H(E)$. Difficult to construct!

Salem sets

Salem sets: sets with $\dim_F(E) = \dim_H(E)$. Difficult to construct!

- Salem 1950: first examples (randomized Cantor-type sets)

Salem sets

Salem sets: sets with $\dim_F(E) = \dim_H(E)$. Difficult to construct!

- Salem 1950: first examples (randomized Cantor-type sets)
- Kahane 1985: Brownian images of compact sets are almost surely Salem.

Salem sets

Salem sets: sets with $\dim_F(E) = \dim_H(E)$. Difficult to construct!

- Salem 1950: first examples (randomized Cantor-type sets)
- Kahane 1985: Brownian images of compact sets are almost surely Salem.
- Kaufman 1981: deterministic example of a Salem set (diophantine Cantor-type construction)

Salem sets

Salem sets: sets with $\dim_F(E) = \dim_H(E)$. Difficult to construct!

- Salem 1950: first examples (randomized Cantor-type sets)
- Kahane 1985: Brownian images of compact sets are almost surely Salem.
- Kaufman 1981: deterministic example of a Salem set (diophantine Cantor-type construction)
- Further examples: Bluhm 1996-98, Ł.-Pramanik 2008.

Salem measures

These constructions provide, for any $\alpha_0 \in (0, n)$, probability measures μ on $\mathbb{R}^n$ (**Salem measures**) such that

Salem measures

These constructions provide, for any $\alpha_0 \in (0, n)$, probability measures μ on $\mathbb{R}^n$ (**Salem measures**) such that

- μ is supported on a set of Hausdorff dimension α_0,

Salem measures

These constructions provide, for any $\alpha_0 \in (0, n)$, probability measures μ on $\mathbb{R}^n$ (**Salem measures**) such that

- μ is supported on a set of Hausdorff dimension α_0,
- $\mu(B(x, r)) \leq C_\alpha r^\alpha$ for all $\alpha < \alpha_0$ (Salem's construction has $\alpha = \alpha_0$),

Salem measures

These constructions provide, for any $\alpha_0 \in (0, n)$, probability measures μ on $\mathbb{R}^n$ (**Salem measures**) such that

- μ is supported on a set of Hausdorff dimension α_0,
- $\mu(B(x, r)) \leq C_\alpha r^\alpha$ for all $\alpha < \alpha_0$ (Salem's construction has $\alpha = \alpha_0$),
- $|\widehat{\mu}(\xi)| \leq C_\beta (1 + |\xi|)^{-\beta/2}$ for all $\beta < \alpha_0$.

Mockenhaupt's restriction theorem for fractal measures

Theorem (Mockenhaupt 2000; endpoint Bak-Seeger 2011)
Let μ be a compactly supported positive measure on $\mathbb{R}^n$ such that for some $\alpha, \beta \in (0, n)$

- $\mu(B(x, r)) \leq C_1 r^\alpha$ for all $x \in \mathbb{R}^n$, $r > 0$,
- $|\widehat{\mu}(\xi)| \leq C_2(1 + |\xi|)^{-\beta/2}$

Then for all $p \geq p_{n,\alpha,\beta} := \frac{2(2n-2\alpha+\beta)}{\beta}$,

$$\|\widehat{fd\mu}\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^2(d\mu)}$$

for all $f \in L^2(d\mu)$.

Range of exponents

We are interested in the range of exponents: where does it come from, and is it sharp?

Range of exponents

We are interested in the range of exponents: where does it come from, and is it sharp?

- If $\mu = \sigma$ is the surface measure on the sphere $S^{n-1} \subset \mathbb{R}^n$, Mockenhaupt's theorem recovers the classical Tomas-Stein theorem. In this case, the range of exponents is known to be optimal. (More on next slide.)

Range of exponents

We are interested in the range of exponents: where does it come from, and is it sharp?

- If $\mu = \sigma$ is the surface measure on the sphere $S^{n-1} \subset \mathbb{R}^n$, Mockenhaupt's theorem recovers the classical Tomas-Stein theorem. In this case, the range of exponents is known to be optimal. (More on next slide.)
- For fractal measures, the question was open. **This is what we address here.**

Range of exponents: the Stein-Tomas theorem

Stein-Tomas: if $\mu = \sigma$ is the surface measure on the sphere $S^{n-1} \subset \mathbb{R}^n$, the $L^2(d\mu) \to L^p(\mathbb{R}^n)$ restriction estimate holds for $p \geq \frac{2n+2}{n-1}$.

Range of exponents: the Stein-Tomas theorem

Stein-Tomas: if $\mu = \sigma$ is the surface measure on the sphere $S^{n-1} \subset \mathbb{R}^n$, the $L^2(d\mu) \to L^p(\mathbb{R}^n)$ restriction estimate holds for $p \geq \frac{2n+2}{n-1}$.

- The range of p is optimal.

Range of exponents: the Stein-Tomas theorem

Stein-Tomas: if $\mu = \sigma$ is the surface measure on the sphere $S^{n-1} \subset \mathbb{R}^n$, the $L^2(d\mu) \to L^p(\mathbb{R}^n)$ restriction estimate holds for $p \geq \frac{2n+2}{n-1}$.

- The range of p is optimal.
- **Knapp example**: Let f_δ be the characteristic function of a spherical cap of diameter δ. Then

$$|\widehat{fd\sigma}| \geq C\|f_\delta\|_{L^1(d\mu)}$$

  on a "dual" cylinder. It follows that there are no uniform (in δ) $L^2 \to L^p$ estimates for $p > \frac{2n+2}{n-1}$.

Range of exponents: Mockenhaupt's theorem for Salem measures on the line

We focus on fractal Salem measures on the line:

- $\dim_H(\text{supp}\mu) = \alpha_0 \in (0, 1)$,

Range of exponents: Mockenhaupt's theorem for Salem measures on the line

We focus on fractal Salem measures on the line:

- $\dim_H(\text{supp}\mu) = \alpha_0 \in (0,1)$,
- ball condition and Fourier decay with α, β arbitrarily close to α_0.

Range of exponents: Mockenhaupt's theorem for Salem measures on the line

We focus on fractal Salem measures on the line:

- $\dim_H(\text{supp}\mu) = \alpha_0 \in (0,1)$,
- ball condition and Fourier decay with α, β arbitrarily close to α_0.

Mockenhaupt: the $L^2(d\mu) \to L^p(\mathbb{R}^n)$ restriction estimate holds for $p \geq \frac{4-2\alpha_0}{\alpha_0}$.

Range of exponents: Mockenhaupt's theorem for Salem measures on the line

We focus on fractal Salem measures on the line:

- $\dim_H(\text{supp}\mu) = \alpha_0 \in (0,1)$,
- ball condition and Fourier decay with α, β arbitrarily close to α_0.

Mockenhaupt: the $L^2(d\mu) \to L^p(\mathbb{R}^n)$ restriction estimate holds for $p \geq \frac{4-2\alpha_0}{\alpha_0}$.

Easy argument via energy integrals: no such estimates if $p < 2/\alpha_0$.

Range of exponents: Mockenhaupt's theorem, cont.

This leaves the intermediate range

$$\frac{2}{\alpha_0} \leq p < \frac{4 - 2\alpha_0}{\alpha_0}$$

Range of exponents: Mockenhaupt's theorem, cont.

This leaves the intermediate range

$$\frac{2}{\alpha_0} \leq p < \frac{4 - 2\alpha_0}{\alpha_0}$$

- Is there an analogue of Knapp's example for fractal sets?

Range of exponents: Mockenhaupt's theorem, cont.

This leaves the intermediate range

$$\frac{2}{\alpha_0} \leq p < \frac{4 - 2\alpha_0}{\alpha_0}$$

- Is there an analogue of Knapp's example for fractal sets?
- Mockenhaupt: cannot exclude possibility that restriction estimates for Salem measures of fractional dimension hold in the intermediate range of exponents.

Range of exponents: Mockenhaupt's theorem, cont.

This leaves the intermediate range

$$\frac{2}{\alpha_0} \leq p < \frac{4-2\alpha_0}{\alpha_0}$$

- Is there an analogue of Knapp's example for fractal sets?
- Mockenhaupt: cannot exclude possibility that restriction estimates for Salem measures of fractional dimension hold in the intermediate range of exponents.
- Chen (2012): there is a measure μ supported on a set of Hausdorff dimension α in $\mathbb{R}$ for which a restriction estimate holds for all $p \geq 2/\alpha_0$. (Based on a probabilistic construction by Körner. The measure μ need not be Salem.)

Main result

Theorem (Hambrook-Łaba 2012)
Let $\alpha = \frac{\log(t_0)}{\log(N_0)}$ with $t_0, N_0 \in \mathbb{N}$. Let $1 \leq p < \frac{4-2\alpha}{\alpha}$. Then there exist a probability measure μ and functions $\{f_\ell\}_{j\in\mathbb{N}}$ on $[0,1]$ such that

- μ is supported on a set E of dimension α,
- μ obeys ball condition with the given value of α,
- $|\widehat{\mu}(\xi)| \leq C_\beta(1+|\xi|)^{-\beta/2}$ for all $\beta < \alpha$,
- the restriction estimate fails:

$$\frac{\|\widehat{f_\ell d\mu}\|_{L^p(\mathbb{R})}}{\|f_\ell\|_{L^2(d\mu)}} \to \infty \quad \text{as} \quad \ell \to \infty.$$

Main result: comments

- This proves that range of exponents in Mockenhaupt's theorem, in its stated generality, is optimal.

Main result: comments

- This proves that range of exponents in Mockenhaupt's theorem, in its stated generality, is optimal.
- Chen's example: there are measures for which restriction estimates hold with exponents beyond Mockenhaupt's range. We do not know whether this can happen for Salem measures.

Main result: comments

- This proves that range of exponents in Mockenhaupt's theorem, in its stated generality, is optimal.
- Chen's example: there are measures for which restriction estimates hold with exponents beyond Mockenhaupt's range. We do not know whether this can happen for Salem measures.
- It should be possible to modify the construction to allow arbitrary $\alpha \in (0, 1)$, but a dense set of α is still enough to get our conclusion.

Proof: construction of μ

Construct $E = \bigcap E_j$ via Cantor iteration:

- Let $N = N_0^{2n_0}$, $t = t_0^{2n_0}$ so that $\log t / \log N = \alpha$. (We will want n_0 to be sufficiently large.)

Proof: construction of μ

Construct $E = \bigcap E_j$ via Cantor iteration:

- Let $N = N_0^{2n_0}$, $t = t_0^{2n_0}$ so that $\log t / \log N = \alpha$. (We will want n_0 to be sufficiently large.)
- Divide $[0, 1]$ into N intervals of equal length, choose t of them. (The choice will be specified shortly.) This is E_1.

Proof: construction of μ

Construct $E = \bigcap E_j$ via Cantor iteration:

- Let $N = N_0^{2n_0}$, $t = t_0^{2n_0}$ so that $\log t / \log N = \alpha$. (We will want n_0 to be sufficiently large.)
- Divide $[0, 1]$ into N intervals of equal length, choose t of them. (The choice will be specified shortly.) This is E_1.
- Suppose E_j has been constructed as a union of t^j intervals of length N^{-j}. For each such interval, subdivide it into N subintervals of length N^{-j-1}, then choose t of them, for a total of t^{j+1} subintervals. (The choices may be different for different intervals of E_j.)

Proof: construction of μ

Construct $E = \bigcap E_j$ via Cantor iteration:

- Let $N = N_0^{2n_0}$, $t = t_0^{2n_0}$ so that $\log t / \log N = \alpha$. (We will want n_0 to be sufficiently large.)
- Divide $[0, 1]$ into N intervals of equal length, choose t of them. (The choice will be specified shortly.) This is E_1.
- Suppose E_j has been constructed as a union of t^j intervals of length N^{-j}. For each such interval, subdivide it into N subintervals of length N^{-j-1}, then choose t of them, for a total of t^{j+1} subintervals. (The choices may be different for different intervals of E_j.)
- This produces $E_1 \supset E_2 \supset \dots$, with E_j as described above.

Construction of μ, cont.

- Let $\mu_j = \frac{1}{|E_j|}\mathbf{1}_{E_j}$, then μ_j converge weakly to μ, a probability measure on $E = \bigcap E_j$.

Construction of μ, cont.

- Let $\mu_j = \frac{1}{|E_j|}\mathbf{1}_{E_j}$, then μ_j converge weakly to μ, a probability measure on $E = \bigcap E_j$.
- For *any* choice of subintervals in the construction, E has dimension α, and $\mu(B(x, r)) \leq Cr^\alpha$ for all $x \in \mathbb{R}$, $r > 0$.

Construction of μ, cont.

- Let $\mu_j = \frac{1}{|E_j|}\mathbf{1}_{E_j}$, then μ_j converge weakly to μ, a probability measure on $E = \bigcap E_j$.
- For *any* choice of subintervals in the construction, E has dimension α, and $\mu(B(x,r)) \leq Cr^{\alpha}$ for all $x \in \mathbb{R}$, $r > 0$.
- (Łaba-Pramanik 2008) There is a randomized choice of subintervals such that μ is a Salem measure.

Subsets with arithmetic structure

- The sets E_j are "random", but can they contain much smaller subsets that are arithmetically structured?

Subsets with arithmetic structure

- The sets E_j are "random", but can they contain much smaller subsets that are arithmetically structured?
- Idea: modify the random construction so that the set of left endpoints of E_j contains a generalized arithmetic progression

$$P_j = N^{-1}P + N^{-2}P + \cdots + N^{-j}P$$

where $P \subset \{0, 1, \ldots, N-1\}$ is an arithmetic progression of length $\sqrt{t}$.

Subsets with arithmetic structure

- The sets E_j are "random", but can they contain much smaller subsets that are arithmetically structured?
- Idea: modify the random construction so that the set of left endpoints of E_j contains a generalized arithmetic progression

$$P_j = N^{-1}P + N^{-2}P + \cdots + N^{-j}P$$

  where $P \subset \{0, 1, \ldots, N-1\}$ is an arithmetic progression of length $\sqrt{t}$.
- Main challenge: we need to do this without destroying the Fourier decay estimates. Turns out that $\sqrt{t}$ is the largest size of P for which this is possible.

The functions f_j

- Let $F_j \subset E_j$ consist of those N^{-j}-intervals whose left endpoints lie in P_j.

The functions f_j

- Let $F_j \subset E_j$ consist of those N^{-j}-intervals whose left endpoints lie in P_j.
- Let $f_j = \mathbf{1}_{F_j}$, then $\|f_j\|^2_{L^2(d\mu)} = \mu(F_j) = t^{-j/2}$.

The functions f_j

- Let $F_j \subset E_j$ consist of those N^{-j}-intervals whose left endpoints lie in P_j.
- Let $f_j = \mathbf{1}_{F_j}$, then $\|f_j\|^2_{L^2(d\mu)} = \mu(F_j) = t^{-j/2}$.
- The lower bound

$$\|\widehat{f_j d\mu}\|^p_{L^p(\mathbb{R})} \geq C(r) \frac{N^j r^{-j-1}}{t^{jl(p+1)/2}}$$

is based on counting additive $2r$-tuples

$$a_1 + \cdots + a_r = a_{r+1} + \cdots + a_{2r}, \quad a_i \in P_j,$$

with fixed $r \in \mathbb{N}$ large enough (depending on α)

The functions f_j

- Let $F_j \subset E_j$ consist of those N^{-j}-intervals whose left endpoints lie in P_j.
- Let $f_j = \mathbf{1}_{F_j}$, then $\|f_j\|^2_{L^2(d\mu)} = \mu(F_j) = t^{-j/2}$.
- The lower bound

$$\|\widehat{f_j d\mu}\|^p_{L^p(\mathbb{R})} \geq C(r) \frac{N^j r^{-j-1}}{t^{jl(p+1)/2}}$$

is based on counting additive $2r$-tuples

$$a_1 + \cdots + a_r = a_{r+1} + \cdots + a_{2r}, \quad a_i \in P_j,$$

with fixed $r \in \mathbb{N}$ large enough (depending on α)

- Conclusion follows if $p < 4/\alpha - 2$ and n_0 is large enough.

Knapp example revisited

Compare to Knapp example for the sphere:

- Knapp example: the sphere (curved) contains spherical caps (almost flat). Equivalently, the curved sphere is tangent to flat hyperplanes.

Knapp example revisited

Compare to Knapp example for the sphere:

- Knapp example: the sphere (curved) contains spherical caps (almost flat). Equivalently, the curved sphere is tangent to flat hyperplanes.
- The range of exponents in Stein-Tomas restriction theorem reflects the degree of tangency.

Knapp example revisited

Compare to Knapp example for the sphere:

- Knapp example: the sphere (curved) contains spherical caps (almost flat). Equivalently, the curved sphere is tangent to flat hyperplanes.
- The range of exponents in Stein-Tomas restriction theorem reflects the degree of tangency.
- In our example, we used that a Salem set (random) may contain a generalized arithmetic progression (structured). We could say that E is "tangent" to the more structured sets F_j, and that the degree of tangency is reflected in the range of restriction exponents.

Thank you!