On the sharpness of Mockenhaupt’s restriction estimate

Izabella Łaba

IPAM, Los Angeles, 2013
Let $\mu$ be a finite, nonnegative Borel measure on $\mathbb{R}^n$, $f \in L^1(d\mu)$. Define the Fourier transform

$$\hat{fd\mu}(\xi) = \int f(x)e^{-2\pi i \xi \cdot x} d\mu(x)$$
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Without additional assumptions on $\mu$, that’s all we can say. But for some types of $\mu$, there is much more...
Theorem (Stein-Tomas, 1970s.) Let $\sigma$ be the surface measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Then for all $p \geq \frac{2n+2}{n-1}$,

$$\| \hat{f}d\sigma \|_{L^p(\mathbb{R}^n)} \leq C(p) \| f \|_{L^2(d\sigma)}$$

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The range of $p$ is optimal.
This turned out to be a rich and productive line of research:

- Many restriction estimates are now known for surface measures on manifolds (spheres, cones, curves in $\mathbb{R}^3$, ...)

The availability of such estimates, and range of exponents, depends on geometric properties of the manifold: dimension, curvature (via decay of Fourier transform of $\sigma$), Kakeya-type geometric information.

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Restriction estimates for discrete sets

- Restriction estimates for sets of integers (Bourgain, Green, Green-Tao, Tao-Vu).

Green (2003) and Green-Tao (2004) used restriction estimates in proving Roth-type results on arithmetic progressions in sets of integers.

Finite field analogues (Mockenhaupt-Tao, Iosevich, Koh, A. Lewko, M. Lewko, Shen, ...)

Instead of curvature (makes no sense for discrete sets), proofs use arithmetic information, e.g. counting solutions to equations $a_1 + \cdots + a_n = a_{n+1} + \cdots + a_{2n}$ with $a_i$ in a given set.
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Main contribution: changing the point of view, shifting focus from “smooth” surface measures on manifolds to less regular fractal measures.
Restriction estimates for fractal measures

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- Proof follows the same argument as the Tomas-Stein restriction theorem for the sphere.
- Main contribution: changing the point of view, shifting focus from “smooth” surface measures on manifolds to less regular fractal measures.
- X. Chen (2012): a restriction theorem for fractal measures based on a different mechanism.
Let $\mu$ be a probability measure on $\mathbb{R}^n$ and $0 \leq \alpha \leq n$. We say that $\mu$ obeys the $\alpha$-dimensional ball condition if

$$\mu(B(x, r)) \leq Cr^\alpha \quad \forall x \in \mathbb{R}^n, \ r \in (0, \infty)$$

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$B(x, r)$ ball of radius $r$ centered at $x$.

Let $E \subset \mathbb{R}^n$ closed, $\dim_H(E) =$ Hausdorff dimension of $E$. Then

$$\dim_H(E) = \sup\{\alpha \in [0, n] : \ E \text{ supports a probability measure } \mu = \mu_\alpha \text{ obeying } (*)\}$$
The **Fourier dimension** of $E \subset \mathbb{R}^n$ is defined as

$$\dim_F(E) = \sup\{\beta \in [0, n] : E \text{ supports a probability measure } \mu = \mu_\beta \text{ with }$$

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- $\dim_F(E) \leq \dim_H(E)$, always.
- Inequality can be strict: the $2/3$ Cantor set has Hausdorff dimension $\log 2 / \log 3$, but Fourier dimension $0$.  

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**Background: Fourier dimension**

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- $\mu$ is supported on a set of Hausdorff dimension $\alpha_0$,
- $\mu(B(x, r)) \leq C_\alpha r^\alpha$ for all $\alpha < \alpha_0$ (Salem’s construction has $\alpha = \alpha_0$).
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- $|\hat{\mu}(\xi)| \leq C_\beta (1 + |\xi|)^{-\beta/2}$ for all $\beta < \alpha_0$. 

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**Theorem (Mockenhaupt 2000; endpoint Bak-Seeger 2011)**

Let $\mu$ be a compactly supported positive measure on $\mathbb{R}^n$ such that for some $\alpha, \beta \in (0, n)$

- $\mu(B(x, r)) \leq C_1 r^\alpha$ for all $x \in \mathbb{R}^n$, $r > 0$,
- $|\hat{\mu}(\xi)| \leq C_2 (1 + |\xi|)^{-\beta/2}$

Then for all $p \geq p_{n, \alpha, \beta} := \frac{2(2n-2\alpha+\beta)}{\beta}$,

$$\left\| f d\mu \right\|_{L^p(\mathbb{R}^n)} \leq C_p \left\| f \right\|_{L^2(d\mu)}$$

for all $f \in L^2(d\mu)$. 

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We are interested in the range of exponents: where does it come from, and is it sharp?
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- If $\mu = \sigma$ is the surface measure on the sphere $S^{n-1} \subset \mathbb{R}^n$, Mockenhaupt’s theorem recovers the classical Tomas-Stein theorem. In this case, the range of exponents is known to be optimal. (More on next slide.)
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- If $\mu = \sigma$ is the surface measure on the sphere $S^{n-1} \subset \mathbb{R}^n$, Mockenhaupt’s theorem recovers the classical Tomas-Stein theorem. In this case, the range of exponents is known to be optimal. (More on next slide.)

- For fractal measures, the question was open. **This is what we address here.**
Stein-Tomas: if $\mu = \sigma$ is the surface measure on the sphere $S^{n-1} \subset \mathbb{R}^n$, the $L^2(d\mu) \rightarrow L^p(\mathbb{R}^n)$ restriction estimate holds for $p \geq \frac{2n+2}{n-1}$. 

The range of $p$ is optimal.

Knapp example: Let $f_\delta$ be the characteristic function of a spherical cap of diameter $\delta$. Then $|\hat{f}\sigma| \geq C \|f_\delta\|_{L^1(d\mu)}$ on a "dual" cylinder. It follows that there are no uniform (in $\delta$) $L^2 \rightarrow L^p$ estimates for $p > \frac{2n+2}{n-1}$. 

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We focus on fractal Salem measures on the line:

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▶ $\dim_H(\text{supp} \mu) = \alpha_0 \in (0, 1)$,
▶ ball condition and Fourier decay with $\alpha, \beta$ arbitrarily close to $\alpha_0$. 

Mockenhaupt: the $L^2(\mathbb{R}) \rightarrow L^p(\mathbb{R}^n)$ restriction estimate holds for $p \geq 4 - \frac{2}{\alpha_0}$.

Easy argument via energy integrals: no such estimates if $p < \frac{2}{\alpha_0}$. 

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Range of exponents: Mockenhaupt’s theorem for Salem measures on the line

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▶ Is there an analogue of Knapp’s example for fractal sets?
▶ Mockenhaupt: cannot exclude possibility that restriction estimates for Salem measures of fractional dimension hold in the intermediate range of exponents.
▶ Chen (2012): there is a measure $\mu$ supported on a set of Hausdorff dimension $\alpha$ in $\mathbb{R}$ for which a restriction estimate holds for all $p \geq 2/\alpha_0$. (Based on a probabilistic construction by Körner. The measure $\mu$ need not be Salem.)
Main result

**Theorem (Hambrook-Łaba 2012)**

Let \( \alpha = \frac{\log(t_0)}{\log(N_0)} \) with \( t_0, N_0 \in \mathbb{N} \). Let \( 1 \leq p < \frac{4-2\alpha}{\alpha} \). Then there exist a probability measure \( \mu \) and functions \( \{f_\ell\}_{\ell \in \mathbb{N}} \) on \([0,1]\) such that

- \( \mu \) is supported on a set \( E \) of dimension \( \alpha \),
- \( \mu \) obeys ball condition with the given value of \( \alpha \),
- \( |\widehat{\mu}(\xi)| \leq C_\beta (1 + |\xi|)^{-\beta/2} \) for all \( \beta < \alpha \),
- the restriction estimate fails:

\[
\frac{\| \hat{f}_\ell \, d\mu \|_{L^p(\mathbb{R})}}{\| f_\ell \|_{L^2(d\mu)}} \to \infty \quad \text{as} \quad \ell \to \infty.
\]
Main result: comments

- This proves that range of exponents in Mockenhaupt’s theorem, in its stated generality, is optimal.
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It should be possible to modify the construction to allow arbitrary $\alpha \in (0, 1)$, but a dense set of $\alpha$ is still enough to get our conclusion.
Proof: construction of $\mu$

Construct $E = \bigcap E_j$ via Cantor iteration:

- Let $N = N_0^{2n_0}$, $t = t_0^{2n_0}$ so that $\log t / \log N = \alpha$. (We will want $n_0$ to be sufficiently large.)
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- Suppose $E_j$ has been constructed as a union of $t^j$ intervals of length $N^{-j}$. For each such interval, subdivide it into $N$ subintervals of length $N^{-j-1}$, then choose $t$ of them, for a total of $t^{j+1}$ subintervals. (The choices may be different for different intervals of $E_j$.)
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- This produces \( E_1 \supset E_2 \supset \ldots \), with \( E_j \) as described above.
Let $\mu_j = \frac{1}{|E_j|}1_{E_j}$, then $\mu_j$ converge weakly to $\mu$, a probability measure on $E = \bigcap E_j$. 
Construction of $\mu$, cont.

\begin{itemize}
  \item Let $\mu_j = \frac{1}{|E_j|} \mathbf{1}_{E_j}$, then $\mu_j$ converge weakly to $\mu$, a probability measure on $E = \bigcap E_j$.
  \item For \textit{any} choice of subintervals in the construction, $E$ has dimension $\alpha$, and $\mu(B(x, r)) \leq Cr^\alpha$ for all $x \in \mathbb{R}$, $r > 0$.
\end{itemize}
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(Laba-Pramanik 2008) There is a randomized choice of subintervals such that $\mu$ is a Salem measure.
The sets $E_j$ are “random”, but can they contain much smaller subsets that are arithmetically structured?

Idea: modify the random construction so that the set of left endpoints of $E_j$ contains a generalized arithmetic progression $P_j = N - 1, N - 2, \ldots, N - j$ where $P \subseteq \{0, 1, \ldots, N-1\}$ is an arithmetic progression of length $\sqrt{t}$.

Main challenge: we need to do this without destroying the Fourier decay estimates. Turns out that $\sqrt{t}$ is the largest size of $P$ for which this is possible.
Subsets with arithmetic structure

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Main challenge: we need to do this without destroying the Fourier decay estimates. Turns out that $\sqrt{t}$ is the largest size of $P$ for which this is possible.
The functions $f_j$

- Let $F_j \subset E_j$ consist of those $N^{-j}$-intervals whose left endpoints lie in $P_j$. 

- Let $f_j = \mathbb{1}_{F_j}$, then $\|f_j\|_{L^2(d\mu)} = \mu(F_j) = t - j / 2$.

- The lower bound $\|\hat{f}_j d\mu\|_{L^p(\mathbb{R})} \geq C(r) N^{j - 1} t_j (p + 1) / 2$ is based on counting additive $2r$-tuples $a_1 + \cdots + a_r = a_{r+1} + \cdots + a_{2r}$, $a_i \in P_j$, with fixed $r \in \mathbb{N}$ large enough (depending on $\alpha$).

- Conclusion follows if $p < 4/\alpha - 2$ and $n_0$ is large enough.
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$$\|\hat{f}_j d\mu\|_{L^p(\mathbb{R})}^p \geq C(r) \frac{N^j r^{-j-1} j!}{t^{jl(p+1)/2}}$$

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  \]
  with fixed \( r \in \mathbb{N} \) large enough (depending on \( \alpha \))
- Conclusion follows if \( p < 4/\alpha - 2 \) and \( n_0 \) is large enough.
Knapp example revisited

Compare to Knapp example for the sphere:

- Knapp example: the sphere (curved) contains spherical caps (almost flat). Equivalently, the curved sphere is tangent to flat hyperplanes.
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- Knapp example: the sphere (curved) contains spherical caps (almost flat). Equivalently, the curved sphere is tangent to flat hyperplanes.
- The range of exponents in Stein-Tomas restriction theorem reflects the degree of tangency.
- In our example, we used that a Salem set (random) may contain a generalized arithmetic progression (structured). We could say that $E$ is “tangent” to the more structured sets $F_j$, and that the degree of tangency is reflected in the range of restriction exponents.
Thank you!